Lecture 18: Fundamental Theorems of Calculus, Riemann Sum

By looking at the definitions of differentiation and integration (or their geometric interpretations), one may feel that there is no connection between these two notions. In this lecture we will discuss two results, called fundamental theorems of calculus, which say that differentiation and integration are, in a sense, inverse operations.

Theorem 18.1 ***(First Fundamental Theorem of Calculus).*** Let f be integrable on $[a,b]$. For $a \le x \le b$, let $F(x) = \int_a^x f(t)dt$. Then F is continuous on $[a,b]$ and if f is continuous at x_0 then F is differentiable at x_0 and $F'(x_0) = f(x_0)$.

Proof (*). Let f be integrable and $M = \sup\{|f(x)| : x \in [a,b]\}$. Suppose $a \le x < y \le b$. Then

$$|F(y) - F(x)| = \left| \int_x^y f(t)dt \right| \le M(y-x).$$

Thus $|F(x) - F(y)| \le M|x-y|$ for $x, y \in [a,b]$. Hence F is continuous on $[a,b]$.

Suppose f is continuous at x_0. Let $\epsilon > 0$. Then by the continuity of f at x_0, there exists a $\delta > 0$ such that

$$t \in [a,b] \text{ and } |t - x_0| < \delta \ \Rightarrow |f(t) - f(x_0)| < \epsilon. \tag{1}$$

We need to show that $F'(x_0) = f(x_0)$. Now for any $x \in [a,b] \setminus \{x_0\}$,

$$\left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| = \left| \frac{1}{x-x_0} \int_{x_0}^x f(t)dt - \frac{1}{x-x_0} \int_{x_0}^x f(x_0)dt \right| \le \frac{1}{|x-x_0|} \int_c^d |f(t) - f(x_0)|dt$$

where $c = x_0$ and $d = x$ if $x > x_0$ (and $c = x$ and $d = x_0$ if $x < x_0$). Therefore by (1),

$$x \in [a,b] \text{ and } 0 < |x - x_0| < \delta \Rightarrow \left| \frac{F(x) - F(x_0)}{x - x_0} - f(x_0) \right| < \epsilon.$$

By the definitions of the limit and the derivative, $F'(x_0) = f(x_0)$. □

In the previous theorem, in a sense, we obtained f by differentiating the integral of f when f is continuous on $[a,b]$.*A function F satisfying $F'(x) = f(x)$ for all $x \in [a,b]$ is called an antiderivative of f on* $[a,b]$. The existence of an antiderivative for a continuous function on $[a,b]$ follows from the first F.T.C.

If an integrable function f has an antiderivative (and if we can find it), then calculating its integral is very simple. The second F.T.C. which is stated below reveals this.

Theorem 18.2 ***(Second Fundamental Theorem of Calculus).*** Let f be integrable on $[a,b]$. If there is a differentiable function F on $[a,b]$ such that $F' = f$ then $\int_a^b f(x)dx = F(b) - F(a)$.

Proof (*). Let $\epsilon > 0$. Since f is integrable we can find a partition $P = \{x_0, x_1, ..., x_n\}$ of $[a,b]$ such that $U(P,f) - L(P,f) < \epsilon$. By the mean value theorem there exists $c_i \in [x_{i-1}, x_i]$ such that $F(x_i) - F(x_{i-1}) = f(c_i)\Delta x_i$. Hence

$$\sum_{i=1}^n f(c_i)\Delta x_i = F(b) - F(a).$$

We know that

$$L(P,f) \le \int_a^b f dx \le U(P,f)$$

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and

$$L(P,f) \leq \sum_{i=1}^{n} f(c_i)\Delta x_i \leq U(P,f).$$

Therefore $|F(b) - F(a) - \int_a^b f dx| < \epsilon$. This completes the proof. □

The second F.T.C. explains why the indefinite integral of F' is defined to be F.

Remark 18.1. The proof of Theorem 18.2 becomes simpler if we assume f to be continuous on $[a,b]$. In fact, the proof follows from Theorem 18.1 (see Problem .. of PP18).

Exercise 18.1. *Let p be a fixed number and let f be a continuous function on $\mathbb{R}$ that satisfies the equation $f(x+p) = f(x)$ for every $x \in \mathbb{R}$. Show that the integral $\int_a^{a+p} f(t)dt$ has the same value for every real number a.*

Solution. Suppose $a, p > 0$. Then by the first F.T.C.,

$$\frac{d}{da}\left(\int_a^{a+p} f(t)dt\right) = \frac{d}{da}\left(\int_0^{a+p} f(t)dt - \int_0^a f(t)dt\right) = f(a+p) - f(a) = 0.$$

Exercise 18.2. *Let f be a continuous function on $[0, \pi/2]$ and $\int_0^{\pi/2} f(t)dt = 0$. Show that there exists a $c \in (0, \pi/2)$ such that $f(c) = 2\cos 2c$.*

Solution. Define F on $[0, \frac{\pi}{2}]$ such that $F(x) = \int\limits_0^x f(t)dt - \sin 2x$. Note that $F(0) = 0 = F(\frac{\pi}{2})$. By Rolle's theorem there exists $c \in (0, \frac{\pi}{2})$ such that $F'(c) = 0$. By the first FTC, $f(c) - 2\cos 2c = 0$.

Exercise 18.3. *Show that $\lim\limits_{x\to 0} \frac{1}{x^3} \int\limits_0^x \frac{t^2}{1+t^4}dt = \frac{1}{3}$.*

Solution. By the first F.T.C. and the L'Hospital rule, $\lim\limits_{x\to 0} \frac{1}{x^3} \int\limits_0^x \frac{t^2}{1+t^4}dt = \lim\limits_{x\to 0} \frac{\frac{x^2}{1+x^4}}{3x^2} = \frac{1}{3}$.

Two methods, called the integration by parts and the integration by substitution or the change of variables, are mainly used for computing integral of a function. Both the methods can be derived from the fundamental theorems of calculuss. These are dicussed in Problems 23 and 24 in PP 18.

Riemann Sum

We now see an important property of integrable functions.

Definition 18.1. Let $f : [a,b] \to \mathbb{R}$ and let $P = \{x_0, x_1, ..., x_n\}$ be a partition of $[a,b]$. Let $\mathcal{C} = \{c_i \in [x_{i-1}, x_i] : i = 1, 2, ..., n\}$ be a set of intermediate points corresponding to P. Then a *Riemann sum* for f (corresponding to the partition P and the set of intermediate points $\mathcal{C}$) is $S(P, \mathcal{C}, f) = \sum_{i=1}^n f(c_i)\Delta x_i$.

The norm of P is defined by $\|P\| = \max\{\Delta x_i : 1 \leq i \leq n\}$.

The following result is anticipated.

Theorem 18.3. *Let $f : [a,b] \to \mathbb{R}$ be integrable. Then*

$$S(P_n, \mathcal{C}_n, f) \to \int_a^b f(x)dx$$

whenever $\|P_n\| \to 0$ and $\mathcal{C}_n$ is any set of intermediate points corresponding to the partition P_n.

We will not present the proof of this theorem. However, if we assume f to be continuous in the statement of Theorem 18.3, then the proof is relatively easier.

Proof of Theorem 18.3 with the assumption that f is continuous (*): Suppose f is continuous on $[a, b]$. Let $\epsilon > 0$. As shown in the proof of Theorem 17.2, there exists a $\delta > 0$ such that

$$U(P, f) - L(P, f) < \epsilon(b - a) \tag{2}$$

whenever P is such that $\|P\| < \delta$. Let $\|P_n\| \to 0$ and $\mathcal{C}_n$ be any set of intermediate points corresponding to the partition P_n. Observe that for every n,

$$L(P_n, f) \leq \int_a^b f(x)dx \leq U(P_n, f) \tag{3}$$

and

$$L(P_n, f) \leq S(P_n, \mathcal{C}_n, f) \leq U(P_n, f). \tag{4}$$

Since $\|P_n\| \to 0$, find $N \in \mathbb{N}$ such that $||P_n\| < \delta$ for all $n \geq N$. Therefore by (2),

$$U(P_n, f) - L(P_n, f) < \epsilon(b - a)$$

whenever $n \geq N$. It follows from (3) and (4) that

$$|S(P_n, f) - \int_a^b f(x)dx| < \epsilon(b - a)$$

for all $n \geq N$. This proves the result. $\square$

We will use Theorem 18.3 later when we discuss the applications of integration. This result can also be used to find limits of certain type of sequences.

Example 18.1. Let us evaluate $\lim\limits_{n\to\infty} x_n$ where $x_n = \frac{1}{n} + \frac{1}{n+1} + ... + \frac{1}{2n-1}$ using Theorem 18.3. Basically we have to write x_n as a Riemann sum of some function on some interval. Note that

$$x_n = \frac{1}{n}(\frac{1}{1} + \frac{1}{1 + \frac{1}{n}} + \frac{1}{1 + \frac{2}{n}} + ... + \frac{1}{1 + \frac{n-1}{n}}). \tag{5}$$

Let $f(x) = \frac{1}{x}$ for $x \in [1, 2]$, $P_n = \{1, 1 + \frac{1}{n}, 1 + \frac{2}{n}, ..., 1 + \frac{n}{n}\}$ and $\mathcal{C}_n = \{1 + \frac{i-1}{n} : 1 \leq i \leq n\}$. Then it follows from (5) that $x_n = S(P_n, \mathcal{C}_n, f)$ for all n. Note that $\|P_n\| \to 0$. Therefore, by Theorem 18.3, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty}(P_n, \mathcal{C}_n, f) = \int_1^2 \frac{1}{x}dx = \ln 2$.

Note that if we take $f(x) = \frac{1}{1+x}$ on $[0, 1]$, $P_n = \{0, \frac{1}{n}, \frac{2}{n}, ..., \frac{n}{n}\}$ and $\mathcal{C}_n = \{\frac{i-1}{n} : 1 \leq i \leq n\}$, then $x_n = S(P_n, \mathcal{C}_n, f)$ for all n. In this case, $\lim\limits_{n\to\infty} x_n = \lim\limits_{n\to\infty}(P_n, \mathcal{C}_n, f) = \int_0^1 \frac{1}{1+x}dx = \ln 2$.