Lecture 6: Intermediate Value Theorem, Limit of a function, Differentiability

In the previous lecture, we discussed two properties of continuous functions which are defined on closed bounded intervals (see Theorems 5.2 and 5.3). In this lecture, we will derive one more property in Theorem 6.2 which has several applications.

Consider a function $f : [a, b] \to \mathbb{R}$ such that f is continuous and satisfies $f(a) < 0$ and $f(b) > 0$. Intuitively, we feel that the graph of f should cross the x-axis between a and b. The following result is formulated based on this observation.

Theorem 6.1. *Let f be continuous on $[a, b]$, and let $f(a) < 0 < f(b)$.Then there exists c such that $a < c < b$ and $f(c) = 0$.*

Proof (*). Let $S = \{x \in [a, b] : f(x) \leq 0\}$. Since $a \in S$, we have $S \neq \emptyset$. Note that S is bounded above by b. Hence S has the least upper bound and we denote it by c. We claim that $f(c) = 0$. Since c is the least upper bound of S, there exists a sequence (x_n) from S such that $x_n \to c$ (see Problem 9 of PP2). Since $x_n \in [a, b]$ for all $n \in \mathbb{N}$, $c \in [a, b]$. By the continuity of f at c, $f(x_n) \to f(c)$. Since $f(x_n) \leq 0$ for all n, we have $f(c) \leq 0$.

We show that $f(c) \geq 0$ which proves the result. First note that $b > c$. Let $y_n = c + (b - c)/n$ for every $n \in \mathbb{N}$. Observe that $y_n \to c$. By the continuity of f, we get $f(y_n) \to f(c)$. Since $f(y_n) > 0$ for all n, $f(c) \geq 0$. □

Theorem 6.1 motivates us to state the following result.

Theorem 6.2 (Intermediate value theorem). *Let $f : [a, b] \to \mathbb{R}$. Suppose α is a real number between $f(a)$ and $f(b)$ (i.e., α is an intermediate value between $f(a)$ and $f(b)$). Then there exists $c \in (a, b)$ such that $f(c) = \alpha$.*

Proof. Define $g(x) = f(x) - \alpha$ for all $x \in [a, b]$. Suppose $f(a) < \alpha < f(b)$. Then $g(a) < 0$ and $g(b) > 0$. Since g is also continuous on $[a, b]$, by Theorem 6.1, there exists $c \in (a, b)$ such that $g(c) = 0$. That is, $f(c) = \alpha$. The proof is similar in case $f(a) > \alpha > f(b)$. □

We now present some applications of the intermediate value theorem. For given $a \in \mathbb{R}$, we let $[a, \infty) = \{x \in \mathbb{R} : x \geq 0\}$, $(a, \infty) = \{x \in \mathbb{R} : x > 0\}$, $(-\infty, a] = \{x \in \mathbb{R} : x \leq a\}$ and $(-\infty, a) = \{x \in \mathbb{R} : x < a\}$.

Application 6.1. Existence of solutions of various equations can be obtained using the intermediate value theorem (in short, IVT). We present a few examples here and some more examples are given in PP6 and PP7.

1. Consider the equation $(1 - x)\cos x = \sin x$. We use the IVT and show that the equation has a soultion in the interval $(0, 1)$. Let $f(x) = (1 - x)\cos x - \sin x$ for $x \in \mathbb{R}$. Then $f(0) = 1$ and $f(1) = -\sin 1 < 0$. By the IVT, applied for f on $[0, 1]$, there is $c \in (0, 1)$ such that $f(c) = 0$. That is, $(1 - c)\cos c = \sin c$.

2. (*Existence of fixed points*). Let $f : [a, b] \to [a, b]$ be continuous. We show that the equation $f(x) = x$ has a solution in $[a, b]$, i.e., there is $c \in [a, b]$ such that $f(c) = c$ (such a point c is called a fixed point of f). As we did in the proceeding application, let $g(x) = f(x) - x$ for $x \in [a, b]$. Then g is continuous, $g(a) \geq 0$ and $g(b) \leq 0$. If $g(a) = 0$ or $g(b) = 0$ then $f(a) = a$ or $f(b) = b$. If $g(a) < 0 < g(b)$, by the IVT, there exists $c \in (a, b)$ such that $g(c) = 0$. That is, $f(c) = c$.

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3. (*Existence of n-th roots*). Let $\alpha \in [0,\infty)$ and $n \in \mathbb{N}$. We prove that the equation $x^n = \alpha$ has a solution in $[0,\infty)$. (This was also discussed in Example 1.3). Let $f(x) = x^n - \alpha$ for $x \in [0,\infty)$. Then $f(0) \le 0$ and $f(N) > 0$ for some $N \in \mathbb{N}$. By the IVT, applied for f on $[0, N]$, there exists $\beta \in [0, N]$ such that $\beta^n = \alpha$.

Application 6.2. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Then, either f is a constant function or the range $\{f(x) : x \in [a, b]\}$ is a closed bounded interval.

To prove this, suppose that f is not a constant function. Let $A = \{f(x) : x \in [a, b]\}$. Since f is continuous on $[a, b]$, by Theorem 5.3, there exist $x_0, y_0 \in [a, b]$ such that $f(x_0) = \inf A$ and $f(y_0) = \sup A$. Since f is not a constant function, $x_0 \ne y_0$. Suppose $x_0 < y_0$. Then for every $\alpha \in [\inf A, \sup A]$, by the IVT applied for $[x_0, y_0]$, there exists $c \in [x_0, y_0]$ such that $f(c) = \alpha$. Hence $A = [\inf A, \sup A]$.

Limit of a function

Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. We have seen in Theorem 5.1 that f is continuous at x_0 if $f(x_n) \to f(x_0)$ whenever $x_n \to x_0$. In some cases when f is not continuous at x_0 or f is not even defined at x_0, there may be a number L such that $f(x_n) \to L$ for some $L \in \mathbb{R}$ whenever $x_n \to x_0$ and $x_n \ne x_0$ for all n. In this case we call such a number L the limit of f at x_0. Let us take a simple example to illustrate. Consider the function f defined by $f(x) = x + 2$ for all $x \ne 1$. Observe that f is not defined at 1. In this case, if we take $x_0 = 1$, then $L = 3$. Even if we assign any value for f at x_0 in this example, the value of L does not change. Let us define the limit formally.

We say that $I \subseteq \mathbb{R}$ is an interval if I is any one of the following subsets of $\mathbb{R}$:

$$\mathbb{R}, \quad [a,b], \quad (a,b), \quad (a,b], \quad [a,b), \quad (a,\infty), \quad (-\infty,b), \quad [a,\infty), \quad (-\infty,b]$$

for some $a, b \in \mathbb{R}$ and $a < b$. In this topic and the subsequent lectures, I will denote an interval.

Definition 6.1. Let $x_0 \in I$ and $f : I \setminus \{x_0\} \to \mathbb{R}$ or $f : I \to \mathbb{R}$. We say that a real number L is a limit of f at x_0 if $f(x_n) \to L$ whenever $x_n \in I \setminus \{x_0\}$ for all n and $x_n \to x_0$.

It is clear from Definition 6.1 that a function cannot have more than one limit at a point. If L is the limit of f at x_0, then we write $\lim\limits_{x\to x_0} f(x) = L$ or $f(x) \to L$ as $x \to x_0$. If $\lim\limits_{x\to x_0} f(x) = L$ for some L, then we say that limit of f at x_0 exists.

Example 6.1. 1. Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be given by $f(x) = x \sin(\frac{1}{x})$ for all $x \in \mathbb{R} \setminus \{0\}$. We show that the limit of f at 0 is 0. Since $|f(x)| \le |x|$ for all $x \in \mathbb{R} \setminus \{0\}$, $f(x_n) \to 0$ whenever $x_n \in \mathbb{R} \setminus \{0\}$ for all $n \in \mathbb{N}$ and $x_n \to 0$. Hence $\lim\limits_{x\to 0} f(x) = 0$.

2. Let $f : \mathbb{R} \to \mathbb{R}$ be given by $f(x) = \sin(1/x)$ for all $x \ne 0$ and $f(0) = 0$. We show that the limit of f at 0 does not exist. Define $x_n = 2/\{\pi(2n+1)\}$ for $n = 1, 2, \ldots$. Then $x_n \to 0$ and $f(x_n) = (-1)^n$ for every $n \in \mathbb{N}$. Note that $(f(x_n))$ does not converge to any element as $n \to \infty$. Hence the limit of f at 0 does not exist

Remark 6.1: 1. Let $x_0 \in I$ and $f : I \to \mathbb{R}$. It is clear from Definition 6.1 that f is continuous at x_0 if and only if $\lim\limits_{x\to x_0} f(x) = f(x_0)$.

2. To define the continuity of a function f at a point x_0, the function f has to be defined at x_0. To define the limit of a function at a point the function need not be defined at that point.

Let us define the one sided limits $\lim\limits_{x\to x_0^+} f(x)$ and $\lim\limits_{x\to x_0^-} f(x)$. Let $x_0 \in I$, $x_0 < y$ for some

$y \in I$ and $f : I \setminus \{x_0\} \to \mathbb{R}$ or $f : I \to \mathbb{R}$. We say that $\lim_{x \to x_0^+} f(x) = L$ if $f(x_n) \to L$ whenever $x_n \in I \setminus \{x_0\}, x_n > x_0$ for all n and $x_n \to x_0$. Similarly, we define $\lim_{x \to x_0^-} f(x)$.

Let us define $\lim_{x \to \infty} f(x)$. Let I be any one of the sets: $\mathbb{R}, [a, \infty)$ or $[a, \infty)$. Let $f : I \to \mathbb{R}$ and $L \in \mathbb{R}$. We say that $\lim_{x \to \infty} f(x) = L$ if $f(x_n) \to L$ whenever $x_n \in I$ for all n and $x_n \to \infty$. In this case we also write $f(x) \to L$ as $x \to \infty$. We define $\lim_{x \to \infty} f(x) = \infty$, $\lim_{x \to x_0} f(x) = \infty$ $\lim_{x \to -\infty} f(x) = L$, $\lim_{x \to -\infty} f(x) = -\infty$, ... similarly.

The proof of the following result is similar to that of Theorem 5.1.

Theorem 6.3. Let $x_0 \in I$, $f : I \setminus \{x_0\} \to \mathbb{R}$ or $f : I \to \mathbb{R}$ and $L \in \mathbb{R}$.

1. $\lim_{x \to x_0} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \in I$ and $0 < |x - x_0| < \delta$.
2. Suppose $x_0 < y$ for some $y \in I$. Then $\lim_{x \to x_0^+} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \in I, x > x_0$ and $0 < |x - x_0| < \delta$.
3. Suppose $y < x_0$ for some $y \in I$. Then $\lim_{x \to x_0^-} f(x) = L$ if and only if for every $\epsilon > 0$ there exists $\delta > 0$ such that $|f(x) - L| < \epsilon$ whenever $x \in I, x < x_0$ and $0 < |x - x_0| < \delta$.

Limits of addition, multiplication, division and compositions of two functions are discussed in Problem 5 in PP6. The relation between the limit and the one sided limit is discussed in Problem 6 in PP6.

Differentiation

At the introductory level, the concept of derivative is generally introduced for finding the tangent line at a point to a graph of a function. We will see that the notion of a derivative has many applications. In particular, information about a given function can be extracted by looking at its derivative if it exists.

Definition 6.2. Let I be an interval and $x_0 \in I$. Let $f : I \to \mathbb{R}$. We say that f is differentiable at x_0 if the limit

$$\lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0} \tag{6.1}$$

exists.

If the above limit exists, it is called the derivative of f at x_0 and is denoted by $f'(x_0)$. If f is differentiable at each $x \in I$, then we say that f is differentiable on I.

Remark 6.2. 1. If x_0 is an end point of I, for instance, x_0 is the left end point of I, then we only consider $x > x_0$ in (6.1) (see Theorem 6.3).

2. If we use the variable h in place of $x - x_0$ in (6.1) (see Problem 3 of PP6), we obtain that f is differentiable at $x_0 \in I$ if and only if $\lim_{h \to 0} \frac{f(x_0+h) - f(x_0)}{h}$ exists.

We now prove that differentiability implies continuity.

Theorem 6.4. *Let $f : I \to \mathbb{R}$. If f is differentiable at a point $x_0 \in I$, then it is continuous at x_0.*

Proof: Let $x \in I$ and $x \neq x_0$. Then $f(x) - f(x_0) = \frac{f(x)-f(x_0)}{x-x_0}(x - x_0)$. Hence $\lim\limits_{x\to x_0}(f(x) - f(x_0)) = f'(x_0) \cdot 0 = 0$. Thus $\lim\limits_{x\to x_0} f(x) = f(x_0)$. Therefore, by Remark 6.1, f is continuous at x_0. $\square$

Example 6.2. 1. Let $f(x) = \sin\frac{1}{x}$, if $x \neq 0$ and $f(0) = 0$. By Example 5.3, f is not continuous at 0 and hence it is not differentiable at 0.

2. Let $f(x) = x\sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. Then by Example 6.1, $\lim\limits_{x\to 0}\frac{f(x)-0}{x-0}$ does not exist. Hence f is not differentiable at 0. We have seen in Example 5.2 that f is continuous at 0.

3. Let $f(x) = x^2\sin\frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$. By Example 6.1, $\lim\limits_{x\to 0}\frac{x^2 sin\frac{1}{x}}{x} = 0$. Hence f is differentiable at 0 and $f'(0) = 0$.

The following two results enable us to evaluate the derivatives of certain combinations of functions.

Theorem 6.5. *Let $f, g : I \to \mathbb{R}$ be differentiable at $x_0 \in I$. Then*

(i) $f + g$ is differentiable and x_0 and $(f + g)'(x_0) = f'(x_0 + g'(x_0)$;

(ii) fg is differentiable at x_0 and $(fg)(x_0) = f'(x_0)g(x_0) + f(x_0)g'(x_0)$;

(iii) if $f(x_0) \neq 0$, then $\frac{1}{f}$ (see Problem 10 of PP5) is differentiable at x_0 and $(\frac{1}{f})'(x_0) = -\frac{f'(x_0)}{f(x_0)^2}$.

Theorem 6.6 (Chain Rule). *Let I and J be intervals. Suppose $g : I \to \mathbb{R}$ and $f : J \to \mathbb{R}$. Let $x_0 \in J$ and $f(J) \subseteq I$. If f is differentiable at x_0 and g is differentiable at $f(x_0)$ then $(g \circ f)$ is differentiable at x_0 and $(g \circ f)'(x_0) = g'(f(x_0))f'(x_0)$.*

Example 6.3. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2\sin\frac{1}{x}$ if $x \neq 0$ and $f(0) = 0$. It is already shown in Example 6.2 that f is differentiable at 0. Since $f = g(h \circ p)$ where $g(x) = x^2, h(x) = \sin x$ and $p(x) = \frac{1}{x}$ for all $x \in \mathbb{R}\backslash\{0\}$, by Theorems 6.5 and 6.6, f is differentiable on $\mathbb{R}\backslash\{0\}$ and $f'(x) = 2x\sin\frac{1}{x} - \cos\frac{1}{x}$ for all $x \in \mathbb{R}\backslash\{0\}$. Observe that the derivative f' is continues on $\mathbb{R}\backslash\{0\}$ but it not continuous at 0 which is verified as follows. Since $\lim\limits_{x\to 0}\cos\frac{1}{x}$ does not exist and $\lim\limits_{x\to 0} 2x\sin\frac{1}{x}$ exists, $\lim\limits_{x\to 0} f'(x)$ does not exist. Hence f' is not continuous at 0.