Lecture 8: Cauchy Mean Value Theorem, L'Hospital's Rule

In the previous lecture, we discussed how Rolle's theorem and MVT can be used for obtaining certain information about a given function by looking at its derivative. In this lecture, we will discuss L'Hospital's rule which is an useful method for determining limits of some specific types of functions using their derivatives.

We first present a generalization of the MVT, called Cauchy mean value theorem (in short, CMVT), which is needed to prove L'Hospital's rule.

Cauchy mean value theorem

Theorem 8.1 (CMVT). *Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Suppose that $g'(x) \neq 0$ for all $x \in (a, b)$. Then there exists $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

Proof (*). Observe that since $g'(x) \neq 0$ for all $x \in (a, b)$, by Rolle's theorem $g(b) \neq g(a)$. Consider the function $F : [a, b] \to \mathbb{R}$ defined by

$$F(x) = f(x) - f(a) - \frac{f(b) - f(a)}{g(b) - g(a)}(g(x) - g(a)).$$

Note that F is continuous on $[a, b]$, differentiable on (a, b) and $F(a) = F(b) = 0$. By Rolle's theorem there exists $c \in (a, b)$ such that $F'(c) = 0$. This proves the theorem. $\square$

The Cauchy mean value theorem (CMVT) is sometimes called generalized mean value theorem. Because, if we take $g(x) = x$ in CMVT we obtain the MVT. We have seen that the MVT can be used to obtain some inequalities. Since CMVT is a generalization of MVT, CMVT too can also be used for obtaining certain inequalities.

Example 8.1. Using the CMVT, we show that $1 - \frac{x^2}{2!} < \cos x$ for $x \neq 0$. Let $f(x) = 1 - \cos x$ and $g(x) = \frac{x^2}{2}$. By the CMVT there exists c betwen 0 and x such that $\frac{1-\cos x}{x^2/2} = \frac{\sin c}{c} < 1$.

L'Hospital's Rule

There are two forms of L'Hospital's rule. One is called $\frac{0}{0}$ form and the other $\frac{\infty}{\infty}$ form.

$\frac{0}{0}$ form

This form deals with $\lim\limits_{x \to x_0} \frac{f(x)}{g(x)}$, where $\lim\limits_{x \to x_0} f(x) = 0 = \lim\limits_{x \to x_0} g(x)$ and $x_0 \in \mathbb{R}$ or x_0 is $\pm\infty$.

As in the previous lectures, I will denote an interval. Recall that an interval can be any one of the following sets: $\mathbb{R}$, $[a, b]$, (a, b), $(a, b]$, $[a, b)$, (a, ∞), $(-\infty, b)$, $[a, \infty)$, $(-\infty, b]$ where $a < b$.

Theorem 8.2 (L'Hospital's Rule). Suppose $x_0 \in I$ or x_0 is $\pm\infty$. Let

(i) $f, g : I \backslash \{x_0\} \to \mathbb{R}$ be differentiable,

(ii) $g'(x) \neq 0$ and $g(x) \neq 0$ for all $x \in I \backslash \{x_0\}$ and

(iii) $\lim\limits_{x \to x_0} f(x) = 0 = \lim\limits_{x \to x_0} g(x)$

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If $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)} = L$ for some $L \in \mathbb{R}$ or L is $\pm\infty$, then $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)} = \lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = L$.

Proof (*). Case I: Suppose $x_0 \in I$ and $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)} = L$ where $L \in \mathbb{R}$ or L is $\pm\infty$.

We will show that $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = L$. Define $f(x_0) = g(x_0) = 0$ so that f and g are continuous on I. Let (x_n) be any sequence in I such that either $x_n > x_0$ for all $n \in \mathbb{N}$ or $x_n < x_0$ for all $n \in \mathbb{N}$ and $x_n \to x_0$. Then by the CMVT, for every $n \in \mathbb{N}$, there exists c_n between x_n and x_0 such that

$$\frac{f(x_n) - f(x_0)}{g(x_n) - g(x_0)} = \frac{f'(c_n)}{g'(c_n)}.$$

By the sandwich theorem, we get $c_n \to x_0$. Since $f(x_0) = 0 = g(x_0)$ and $\frac{f'(c_n)}{g'(c_n)} \to L$, it follows that $\frac{f(x_n)}{g(x_n)} \to L$. Therefore $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = L$.

Case II: Suppose that x_0 is $\pm\infty$ and $\lim\limits_{x\to\infty} \frac{f'(x)}{g'(x)} = L$ where $L \in \mathbb{R}$ or L is $\pm\infty$.

Assume that x_0 is ∞. We claim that $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = L$.

In this case, we can and we will assume that there exists some $M > 0$ such that $[M, \infty) \subset I$. Define $F, G : (0, \frac{1}{M}] \to \mathbb{R}$ by $F(x) = f(\frac{1}{x})$ and $G(x) = g(\frac{1}{x})$. Then, for all $x \in (0, \frac{1}{M})$,

$$\frac{F'(x)}{G'(x)} = \frac{f'(\frac{1}{x})(-\frac{1}{x^2})}{g'(\frac{1}{x})(-\frac{1}{x^2})} = \frac{f'(\frac{1}{x})}{g'(\frac{1}{x})}.$$

Set $F(0) = 0, G(0) = 0$, $I = [0, \frac{1}{M}]$ and $x_0 = 0$. Apply Case I (for F and G instead of f and g) to conclude that $\lim\limits_{x\to\infty} \frac{f(x)}{g(x)} = \lim\limits_{y\to 0^+} \frac{F(y)}{G(y)} = \lim\limits_{y\to 0^+} \frac{F'(y)}{G'(y)} = \lim\limits_{y\to 0^+} \frac{f'(\frac{1}{y})}{g'(\frac{1}{y})} = \lim\limits_{x\to\infty} \frac{f'(x)}{g'(x)} = L$.

The proof is similar in case x_0 is $-\infty$. □

Application 8.1. The derivative of a function has some interesting properties. For instance, the derivative of a differentiable function defined on an interval has the intermediate value property (see Problem 18 of PP7). We now derive another property of the derivative using L'Hospital's rule. Let $f : I \to \mathbb{R}$, $x_0 \in I$ and f be continuous at x_0. Suppose that f is differentiable on $I\backslash\{x_0\}$ and $\lim\limits_{x\to x_0} f'(x)$ exists. Then $f'(x_0)$ exists and $f'(x_0) = \lim\limits_{x\to x_0} f'(x)$. This property can be derived as follows. Use Theorem 8.2 to obtain that

$$\lim_{x\to x_0} \frac{f(x) - f(x_0)}{x - x_0} = \lim_{x\to x_0} \frac{f'(x)}{1} = \lim_{x\to x_0} f'(x).$$

This shows that $f'(x_0)$ exists and $f'(x_0) = \lim\limits_{x\to x_0} f'(x)$. The above property can also be derived from the MVT (see Problem 6 in PP8).

Example 8.2. 1. Consider the problem of finding $\lim\limits_{x\to 1} \frac{x-1}{\log x}$. If we take $f(x) = x - 1,\ g(x) = \log x,\ x_0 = 1$ and $I = (0, \infty)$, then by Theorem 8.2, $\lim\limits_{x\to 1} \frac{x-1}{\log x} = 1$.

2. Let us compute $\lim\limits_{x\to 0} \frac{\sin x - x}{2x^3}$. Apply Theorem 8.2 thrice to find the limit as follows

$$\lim_{x\to 0} \frac{\sin x - x}{2x^3} = \lim_{x\to 0} \frac{\cos x - 1}{6x^2} = \lim_{x\to 0} \frac{-\sin x}{12x} = \lim_{x\to 0} \frac{-\cos x}{12} = -\frac{1}{12}.$$

Similar computation shows that $\lim\limits_{x\to 0} \frac{\sin x - x}{x^4} = \infty$.

3. Let us find $\lim\limits_{x\to\infty} \frac{\log(1+\frac{1}{x})}{1/x}$. Applying Theorem 8.2, we obtain that

$$\lim_{x\to\infty} \frac{\log(1+\frac{1}{x})}{1/x} = \lim_{x\to\infty} \frac{(1+\frac{1}{x})^{-1}(-x^{-2})}{-x^{-2}} = \lim_{x\to\infty} \frac{1}{1+1/x} = 1.$$

3. The assumption that $\lim\limits_{x\to x_0} f(x) = 0 = \lim\limits_{x\to x_0} g(x)$ is essential in Theorem 8.2. For example, if $f(x) = 2x+1$ and $g(x) = 3x+5$ for all $x \in \mathbb{R}$, then $\lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{1}{5}$ but $\lim_{x\to 0} \frac{f'(x)}{g'(x)} = \frac{2}{3}$.

$\frac{\infty}{\infty}$ **form**

This form deals with $\lim\limits_{x\to x_0} \frac{f(x)}{g(x)}$, where $\lim\limits_{x\to x_0} f(x) = \infty = \lim\limits_{x\to x_0} g(x)$ and $x_0 \in \mathbb{R}$ or x_0 is $\pm\infty$.

Theorem 8.3 (L'Hospital's Rule). Suppose $x_0 \in I$ or x_0 is $\pm\infty$. Let

(i) $f, g : I\backslash\{x_0\} \to \mathbb{R}$ be differentiable,

(ii) $g'(x) \neq 0$ and $g(x) \neq 0$ for all $x \in I\backslash\{x_0\}$ and

(iii) $\lim\limits_{x\to x_0} f(x) = \infty = \lim\limits_{x\to x_0} g(x)$

If $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)} = L$ for some $L \in \mathbb{R}$ or L is $\pm\infty$, then $\lim\limits_{x\to x_0} \frac{f'(x)}{g'(x)} = \lim\limits_{x\to x_0} \frac{f(x)}{g(x)} = L$.

We will not present the proof of Theorem 8.3 but we will use it.

Application 8.2. Let $f : I \to \mathbb{R}$ be a continuous function at $x_0 \in I$. If we want to verify whether f is differentiable at x_0, we need to verify the existence of $\lim\limits_{x\to x_0} \frac{f(x)-f(x_0)}{x-x_0}$ which is of the form $\frac{0}{0}$ or $\lim\limits_{x\to x_0} \frac{1/(x-x_0)}{1/(f(x)-f(x_0))}$ which is of the form $\frac{\infty}{\infty}$. Therefore, L'Hospital rule can be used for verifying the differentiability and finding the derivative of a function at a point. For instance, let $f(x) = e^{-1/x^2}$ if $x \neq 0$ and $f(0) = 0$. We will use this function later in one of the lectures. We will now show that f is differentiable at 0 and $f'(0) = 0$ using L'Hospital's rule. To show this, we have to evaluate $\lim\limits_{x\to 0^+} \frac{e^{-1/x^2}-0}{x-0}$ which is of $\frac{0}{0}$ form. Applying Theorem 8.2 will lead to a problem of finding a limit which is again of $\frac{0}{0}$ form. Therefore we convert the problem into $\frac{\infty}{\infty}$ form and apply Theorem 8.3. Observe that $\lim\limits_{x\to 0^+} \frac{e^{-1/x^2}}{x} = \lim\limits_{y\to\infty} ye^{-y^2} = \lim\limits_{y\to\infty} \frac{y}{e^{y^2}}$. Therefore, by Theorem 8.3, $\lim\limits_{y\to\infty} \frac{y}{e^{y^2}} = 0$ and hence $f'(0) = 0$.

Example 8.3. 1. Let us compute $\lim\limits_{x\to\infty} \frac{x^p}{q^x}$ where $p > 0$ and $q > 1$. If $p < 1$, then applying Theorem 8.3, we obtain

$$\lim_{x\to\infty} \frac{x^p}{q^x} = \lim_{x\to\infty} \frac{px^{p-1}}{q^x \log q} = 0.$$

If $p > 1$, then we apply Theorem 8.3 more than once to get $\lim\limits_{x\to\infty} \frac{x^p}{q^x} = 0$. This implies that $\lim\limits_{n\to\infty} \frac{n^p}{q^n} = 0$ which was already discussed in Example 2.4. Similar to what was said in Example 2.4, we say that the function q^x goes to infinity "faster" than x^p as $x \to \infty$.

2. Consider the problem of finding $\lim\limits_{x\to\infty} \frac{\log x}{x^p}$ for some $p > 0$. It follows from Theorem 8.3 that $\lim\limits_{x\to\infty} \frac{\log x}{x^p} = 0$. This reveals that the function x^p goes to infinity "faster" than $\log x$ as $x \to \infty$.

3. Using Theorem 8.3, verify that, for any $p > 0$, e^x goes to infinity "faster" than x^p as $x \to \infty$.

4. It is easy to very directly that $\lim_{x\to\infty} \frac{x - sinx}{2x + sinx} = \frac{1}{2}$. However, one cannot apply L'Hospital's Rule to find the limit, because $\lim_{x\to\infty} \frac{1-cosx}{2+cosx}$ does not exist.

Other forms

The forms such as $0 \cdot \infty$ and $\infty - \infty$ can be reduced to $\frac{0}{0}$ form by algebraic manipulations. Such manipulations are illustrated in the following examples.

Example 8.4. 1. Let $I = (0, \frac{\pi}{2})$ and consider $\lim\limits_{x\to\frac{\pi}{2}^-} (\tan x - \sec x)$ which is of the form $\infty - \infty$. Since $\tan x - \sec x = \frac{\sin x - 1}{\cos x}$, by Theorem 8.2, $\lim\limits_{x\to\pi/2^-} (\tan x - \sec x) = 0$.

2. Let $I = (0, \infty)$ and consider $\lim\limits_{x\to 0^+} (x \log x)$ which is of the form $0 \cdot (-\infty)$. Since $x \log x = \frac{\log x}{x^{-1}}$, by Theorem 8.2, $\lim\limits_{x\to 0^+} (x \log x) = \lim\limits_{x\to 0^+} \frac{1/x}{-x^{-2}} = 0$.

The forms such as $1^\infty, 0^0, \infty^0$ can be reduced to $0 \cdot \infty$ by involving the logarithmic function. This is illustrated in the following examples.

Example 8.5. 1. Consider $\lim\limits_{x\to 0^+} x^x$ which is of the form 0^0. Observe that $\log(x^x) = x \log x$. By Example 8.4, $\lim\limits_{x\to 0^+} (x \log x) = 0$. Since $x^x = e^{x \log x}$, by the continuity of the exponential function, $\lim\limits_{x\to 0^+} x^x = \lim\limits_{x\to 0^+} e^{x \log x} = e^0 = 1$.

2. Consider $\lim\limits_{x\to\infty} (1 + \frac{1}{x})^x$ which is of the form 1^∞. Here we can take $I = (1, \infty)$. Note that $\log(1 + \frac{1}{x})^x = x \log(1 + \frac{1}{x})$. By Example 8.2, $\lim\limits_{x\to\infty} x \log(1 + \frac{1}{x}) = 1$. Since $(1 + \frac{1}{x})^x = e^{x \log(1 + \frac{1}{x})}$, $\lim\limits_{x\to\infty} (1 + \frac{1}{x})^x = \lim\limits_{x\to\infty} e^{x \log(1+\frac{1}{x})} = e$. In fact, $\lim\limits_{x\to\infty} (1 + \frac{a}{x})^x = e^a$ for all $a \in \mathbb{R}$.

3. Consider $\lim\limits_{x\to\infty} x^{\frac{1}{x}}$ which is of the form ∞^0. Since $\log(x^{\frac{1}{x}}) = \frac{\log x}{x}$, by Example 8.3, $\lim\limits_{x\to\infty} \log(x^{\frac{1}{x}}) = 0$. Therefore, $\lim\limits_{x\to\infty} x^{\frac{1}{x}} = \lim\limits_{x\to\infty} e^{\log(x^{\frac{1}{x}})} = e^0 = 1$.

Remark 8.1. 1. L'Hospital's Rule cannot be applied if the form is neither $\frac{0}{0}$ nor $\frac{\infty}{\infty}$. For instance, consider $\lim\limits_{x\to 0} \frac{x+1}{x}$. It is easy to see that $\lim\limits_{x\to 0^+} \frac{x+1}{x} = \infty$. If we take $f(x) = 1 + x$ and $g(x) = x$ then $\lim\limits_{x\to 0} \frac{f'(x)}{g'(x)} = 1$.

2. In some cases, L'Hospital's Rule may not lead to the desired limit. Consider $\lim\limits_{x\to\infty} \frac{\sqrt{1+x^2}}{x}$. In this case, it is easy to see directly that the limit is 1. However, if we apply L'Hosiptal's rule we obtain

$$\lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x} = \lim_{x\to\infty} \frac{x/\sqrt{1+x^2}}{1} = \lim_{x\to\infty} \frac{x}{\sqrt{1+x^2}} = \lim_{x\to\infty} \frac{1}{x/\sqrt{1+x^2}} = \lim_{x\to\infty} \frac{\sqrt{1+x^2}}{x}.$$