Lecture 9: Taylor's Theorem

In the last two lectures we discussed the mean value theorem (which relates a function and its derivative) and its applications. We now discuss a result called Taylor's Theorem which relates a function, its derivative and its higher derivatives. We will see that Taylor's Theorem is an extension of the mean value theorem. Though Taylor's Theorem has several applications in calculus, it basically deals with approximation of functions by polynomials. The linear approximation or tangent line approximation, which is described below, gives an idea about the approximation mentioned above.

Linear Approximation

Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x_0 \in \mathbb{R}$. Consider the linear polynomial $P_1(x)$ defined by

$$P_1(x) = f(x_0) + f'(x_0)(x - x_0)$$

where $x \in \mathbb{R}$. Observe that $f(x_0) = P_1(x_0)$ and $P_1(x) \to f(x_0)$ as $x \to x_0$. Hence $P_1(x)$ is considered as an approximation of $f(x)$ near x_0. Geometrically, this is clear because we approximate the graph of f near $(x_0, f(x_0))$ by the tangent line at $(x_0, f(x_0))$. The following result provides an estimation of the size of the error $E_1(x) = f(x) - P_1(x)$.

Theorem 9.1(Extended Mean Value Theorem). *Let $f : [a, b] \to \mathbb{R}$ be such that f' is continuous on $[a, b]$ and f'' exists on (a, b). Suppose $x_0 \in [a, b]$. Then, for any $x \in [a, b] \backslash \{x_0\}$, there exists c between x and x_0 such that*

$$f(x) = f(x_0) + f'(x)(x - x_0) + \frac{f''(c)}{2}(x - x_0)^2.$$

We will not present the proof of Theorem 9.1 as this result is a particular case of Taylor's Theorem which is stated and proved below.

Let f, x and x_0 be as in Theorem 9.1. We may assume that $x_0 < x$. Let $M = \sup\{\mid f''(t) \mid : t \in [x_0, x]\} < \infty$. Then

$$\mid E_1(x) \mid = \mid f(x) - P_1(x) \mid \leq \frac{M}{2}(x - x_0)^2.$$

The above estimate gives an idea "how good the approximation $P_1(x)$ is, i.e., how fast the error $E_1(x)$ goes to 0 as $x \to x_0$".

Example 9.1. To illustrate the linear approximation and the error estimation, consider $f(x) = \sqrt{x}$ and $x_0 = 1$. Then $P_1(x) = \frac{x}{2} + \frac{1}{2}$ which is the linear approximation to f near x_0. Note that $P_1(1.1) = 1.05$ which is an approximate value of $\sqrt{1.1}$. However, the actual value of $\sqrt{1.1}$, up to five decimal places, is 1.04880. If we take $x = 1.1$, then $M = \sup\{\mid f''(t) \mid : t \in [1, 1.1]\} \leq \frac{1}{4}$ and $E_1(x) \leq \frac{(0.1)^2}{8} = \frac{1}{800}$.

Naturally, one asks the question: Can we get better estimates for the error if we use approximation by higher order polynomials? Taylor's theorem provides the answer affirmatively.

Taylor's theorem

There might be several ways to approximate a given function by a polynomial of degree greater than or equal to 2. However, Taylor's theorem deals with the polynomial which agrees with f and some of its derivatives at a given point x_0, as $P_1(x)$ does in case of the linear approximation.

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Let $f:\mathbb{R}\to\mathbb{R}$ and $x_0\in\mathbb{R}$ be such that $f^{(n)}(x_0)$ exists where $n\geq 1$ and $f^{(n)}(x_0)$ denotes the $n-$th derivative of f at x_0. The polynomial

$$P_n(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n$$

has the property that $P_n(x_0)=f(x_0)$ and $P_n^{(k)}(x_0)=f^{(k)}(x_0)$ for all $k=1,2,..,n$. The polynomial $P_n(x)$ is called Taylor's polynomial of degree n (with respect to f and x_0).

The following theorem, called Taylor's Theorem, provides an estimate for the error function $E_n(x)=f(x)-P_n(x)$ for a given $n\in\mathbb{N}$.

Theorem 9.2. *Let $f:[a,b]\to\mathbb{R}$ be such that $f^{(n)}$ is continuous on $[a,b]$ and $f^{(n+1)}$ exists on (a,b). Suppose $x_0\in[a,b]$. Then, for any $x\in[a,b]\backslash\{x_0\}$, there exists c between x and x_0 such that*

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{f^{(n+1)}(c)}{(n+1)!}(x-x_0)^{n+1}.$$

Proof (*). We will construct a new function g (out of f) satisfying $g(x)=g(x_0)$ and apply Rolle's theorem. Fix $x_0\in[a,b]$ and $x\in[a,b]\backslash\{x_0\}$. Then, choose M satisfying the following equation:

$$f(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2!}(x-x_0)^2+\cdots+\frac{f^{(n)}(x_0)}{n!}(x-x_0)^n+\frac{M}{(n+1)!}(x-x_0)^{n+1}.$$

For $t\in\mathbb{R}$, define

$$g(t)=f(t)+f'(t)(x-t)+\frac{f''(t)}{2!}(x-t)^2+\cdots+\frac{f^{(n)}(t)}{n!}(x-t)^n+\frac{M}{(n+1)!}(x-t)^{n+1}.$$

Observe that $g(x)=f(x)=g(x_0)$. Hence by Rolle's theorem, there exists c between x and x_0 such that

$$g'(c)=\frac{f^{(n+1)}(c)}{n!}(x-c)^n-\frac{M}{n!}(x-c)^n=0.$$

This implies that $M=f^{(n+1)}(c)$ which proves the result. $\square$

Let us see some consequences of Taylor's theorem.

Application 9.1. 1 **(Quadratic approximation).** Let $f:\mathbb{R}\to\mathbb{R}$ be twice differentiable at $x_0\in\mathbb{R}$. Consider $P_2(x)$ defined by

$$P_2(x)=f(x_0)+f'(x_0)(x-x_0)+\frac{f''(x_0)}{2}(x-x_0)^2$$

which is the Taylor's polynomial of degree 2 and is called the quadratic approximation to f near x_0. If f,x and x_0 are considered as in Taylor's theorem with $n=2$ and $M=\sup\{\mid f^{(3)}(t)\mid : t\in[x_0,x]\}<\infty$, then the error function $|E_2(x)|\leq\frac{M}{6}|x-x_0|^3$. In case $f(t)=\sqrt{t}, x_0=1$ and $x=1.1$, then $|E_2(x)|\leq\frac{3}{8\times 6}(0.1)^3=\frac{1}{16000}$. Verify that $P_2(1.1)=1.04875$ and compare this value with $P_1(1.1)$.

2. Certain inequalities can be derived using Taylor's theorem. One example is illustrated below. For any $k\in\mathbb{N}$ and for all $x>0$, we show that

$$x-\frac{1}{2}x^2+\cdots-\frac{1}{2k}x^{2k}\ <\ \log(1+x)\ <\ x-\frac{1}{2}x^2+\cdots+\frac{1}{2k+1}x^{2k+1}$$

as follows. By Taylor's theorem, there exists $c \in (0, x)$ such that

$$\log(1+x) = x - \frac{1}{2}x^2 + ... + \frac{(-1)^{n-1}}{n}x^n + \frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}}.$$

Note that, for any $x > 0$, $\frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}} > 0$ if $n = 2k$ and $\frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}} < 0$ if $n = 2k+1$.

3. We now illustrate that using certain properties of the second derivative of a given function we can get certain properties of the function. Let $f : [a, b] \to \mathbb{R}$ be such that $f''(x) \geq 0$ for all $x \in [a, b]$. Fix $x_0 \in [a, b]$. Using the extended mean value theorem (in short, EMVT) we show the following property of f. For any $x \in [a, b]$,

$$f(x) \geq f(x_0) + f'(x_0)(x - x_0)$$

i.e., the graph of f lies above the tangent line to the graph at $(x_0, f(x_0))$. This is verified as follows. Let $x \in [a, b]\backslash\{x_0\}$. Then by the EMVT, there exists c between x_0 and x such that $f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{(x-x_0)^2}{2}f''(c)$. This implies the required inequality. With the same assumption on f (i.e., $f''(x) \geq 0$ for all $x \in [a, b]$), another geometric property of f is derived in Problem 9 in PP 9 The above mentioned properties of a function f are useful for sketching the graph of f which will be discussed later.