Practice Problems 1: The Real Number System

1. Let $x_0 \in \mathbb{R}$ and $x_0 \geq 0$. If $x_0 < \epsilon$ for every positive real number ϵ, show that $x_0 = 0$.
2. Prove Bernoulli's inequality: for $x > -1$, $(1+x)^n \geq 1 + nx$ for all $n \in \mathbb{N}$.
3. Suppose that α and β are any two real numbers satisfying $\alpha < \beta$. Show that there exists $n \in \mathbb{N}$ such that $\alpha < \alpha + \frac{1}{n} < \beta$. Similarly, show that for any two real numbers s and t satisfying $s < t$, there exists $n \in \mathbb{N}$ such that $s < t - \frac{1}{n} < t$.
4. Let A be a non-empty subset of $\mathbb{R}$ and $\beta \in \mathbb{R}$ be an upper bound of A. Suppose for every $n \in \mathbb{N}$, there exists $a_n \in A$ such that $a_n \geq \beta - \frac{1}{n}$. Show that β is the supremum of A.
5. Find the supremum and infimum of each of the following sets:

   (i) $\left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$ (ii) $\left\{\frac{m}{|m|+n} : n \in \mathbb{N}, m \in \mathbb{Z}\right\}$ (iii) $\left\{\frac{n}{1+2n} : n \in \mathbb{N}\right\}$.
6. Let A be a non-empty bounded above subset of $\mathbb{R}$. If $\beta \in \mathbb{R}$ is an upper bound of A and $\beta \in A$, show that β is the l.u.b. of A.
7. Let A be a non-empty subset of $\mathbb{R}$ and $\beta \in \mathbb{R}$ an upper bound of A. Show that $\beta = \sup A$ if and only if for every $\epsilon > 0$, there is some $a_0 \in A$ such that $\beta - \varepsilon < a_0$.
8. Let $x \in \mathbb{R}$. Show that there exists an integer k such that $k \leq x < k+1$ and an integer l such that $x < l \leq x + 1$.
9. **(*)**
   (a) Let $x \in \mathbb{R}$ and $x > 0$. If $x^2 < 2$, show that there exists $n_0 \in \mathbb{N}$ such that $(x + \frac{1}{n_0})^2 < 2$. Similarly, if $x^2 > 2$, show that there exists $n_1 \in \mathbb{N}$ such that $(x - \frac{1}{n_1})^2 > 2$.
   (b) Let $A = \{x \in \mathbb{R} : x > 0,\ x^2 < 2\}$ and $\beta = \sup A$. Show that $\beta^2 = 2$.
10. **(*)** For a subset A of $\mathbb{R}$, define $-A = \{-x : x \in A\}$. Suppose that S is a nonempty bounded above subset of $\mathbb{R}$.
    (a) Show that $-S$ is bounded below.
    (b) Show that $\inf(-S) = -\sup(S)$.
    (c) From (b) conclude that the l.u.b. property of $\mathbb{R}$ implies the g.l.b. property of $\mathbb{R}$ and vice versa.
11. **(*)** Let k be a positive integer and $x = \sqrt{k}$. Suppose x is rational and $x = \frac{m}{n}$ where $m \in \mathbb{Z}$ and n is the least positive integer such that nx is an integer. Define $n' = n(x - [x])$ where $[x]$ is the integer part of x (see the solution of Problem 8 for the definition of $[x]$).
    (a) Show that $0 \leq n' < n$ and $n'x$ is an integer.
    (b) Show that $n' = 0$.
    (c) From (a) and (b) conclude that $\sqrt{k}$ is either a positive integer or irrational.

Please write to psraj@iitk.ac.in if any typos/mistakes are found in this set of practice problems/solutions/hints.

Practice Problems 1: Hints/Solutions

1. Suppose $x_0 \neq 0$. Then for $\epsilon_0 = \frac{x_0}{2}$, $x_0 > \epsilon_0 > 0$ which is a contradiction.

2. Use Mathematical induction.

3. Since $\beta - \alpha > 0$, by the Archimedean property, there exists $n \in \mathbb{N}$ such that $n > \frac{1}{\beta-\alpha}$.

4. If β is not the supremum then there exists an upper bound α of A such that $\alpha < \beta$. Use Problem 3 and find $n \in \mathbb{N}$ such that $\alpha < \beta - \frac{1}{n}$. Since there exists $a_n \in A$ such that $\beta - \frac{1}{n} < a_n$, α is not an upper bound of A which is a contradiction.

5. (i) Let $A = \left\{\frac{m}{m+n} : m, n \in \mathbb{N}\right\}$. First note that $0 < \frac{m}{m+n} < 1$ for all $m, n \in \mathbb{N}$. We guess that $\sup A = 1$, because $\frac{n}{1+n} = \frac{1}{1+1/n} \in A$ for all $n \in \mathbb{N}$ and n can be very large. To show formally that $\sup A = 1$, let $\beta = 1$. We verify below that β satisfies conditions (i) and (ii) of Definition 1.2. Since 1 is an upper bound of A, let us verify (ii). Suppose β does not satisfy (ii). Then there exists an upper bound α of A such that $\alpha < \beta = 1$. Find some $n_0 \in \mathbb{N}$ such that $\frac{n_0}{1+n_0} > \alpha$, i.e., $n_0 > \frac{\alpha}{1-\alpha}$ which is possible because of the Archimedean Property. Note that $\frac{n_0}{1+n_0} \in A$. This contradicts the fact that α is an upper bound of A. Similarly, we can show that $\inf A = 0$.

   (ii) Supremum is 1 and infimum is -1.

   (iii) Supremum is $\frac{1}{2}$ and infimum is $\frac{1}{3}$.

6. If β is not the l.u.b. of A, then there exists an upper bound α of A such that $\alpha < \beta$. But $\beta \in A$ which contradicts the fact that α is an upper bound of A.

7. Suppose $\beta = \sup A$. Then β satisfies (ii) of Definition 1.2. Let $\varepsilon > 0$. If there is no $a \in A$ such that $\beta - \varepsilon < a$, then we have $a \leq \beta - \varepsilon < \beta$ for all $a \in A$. This implies that $\beta - \varepsilon$ is an upper bound of A. This contradicts (ii) of Definition 1.2. To prove the converse, assume that for every $\epsilon > 0$, there is some $a_0 \in A$ such that $\beta - \varepsilon < a_0$. Suppose β does not satisfy (ii). Then there exists an upper bound α of A such that $\alpha < \beta$. This implies that $\alpha < \beta - \frac{\beta-\alpha}{2} < \beta$. By our assumption, there exists $a_0 \in A$ such that $\beta - \frac{\beta-\alpha}{2} < a_0$ which contradicts the fact that α is an upper bound of A.

8. Using the Archimedean property, find $m, n \in \mathbb{N}$ such that $-m < x < n$. Observe that there are only finite number of integers between $-m$ and n. Let k be the largest integer satisfying $-m < k < n$ and $k \leq x$. So, $k \leq x < k + 1$. This implies that $x < k + 1 \leq x + 1$. The integer k satisfying $k \leq x < k + 1$ is called the integer part of x and is denoted by $[x]$. Take $l = [x] + 1$.

9. (a) Suppose $x^2 < 2$. Observe that $(x + \frac{1}{n})^2 < x^2 + \frac{1}{n} + \frac{2x}{n}$ for any $n \in \mathbb{N}$ satisfying $n > 1$. Using the Archimedean property, find some $n_0 \in \mathbb{N}$ such that $x^2 + \frac{1}{n_0} + \frac{2x}{n_0} < 2$. This n_0 will do.

   (b) Using (a), justify that the following cases cannot occur: (i) $\beta^2 < 2$ and (ii) $\beta^2 > 2$.

10. (a) Easy to verify.

    (b) Let $\beta = \sup S$. We claim that $-\beta = \inf(-S)$. Since $\beta = \sup S$, $a \leq \beta$ for all $a \in S$. This implies that $-a \geq -\beta$ for all $a \in S$. Hence $-\beta$ is a lower bound of $-S$. If $-\beta$ is not the g.l.b. of $-S$ then there exists a lower bound α of $-S$ such that $-\beta < \alpha$. Verify that $-\alpha$ is an upper bound of S and $-\alpha < \beta$ which is a contradiction.

    (c) Assume that $\mathbb{R}$ has the l.u.b. property and S is a non empty bounded below subset of $\mathbb{R}$. Then from (b) or the proof of (b), we conclude that $\inf S$ exists and is equal to $-\sup(-S)$.

11. Each part is easy to verify.