# Practice Problems 14: Ratio and Root tests

1. Determine the values of $\alpha \in \mathbb{R}$ for which $\sum_{n=1}^{\infty} \left(\frac{\alpha n}{n+1}\right)^n$ converges.

2. Consider $\sum_{n=1}^{\infty} a_n$ where $a_n > 0$ for all $n$. Prove or disprove the following statements.
   (a) If $\frac{a_{n+1}}{a_n} < 1$ for all $n$ then the series converges.
   (b) If $\frac{a_{n+1}}{a_n} > 1$ for all $n$ then the series diverges.
   (c) If $|a_n|^{1/n} < 1$ for all $n$ then the series converges.
   (d) If $|a_n|^{1/n} > 1$ for all $n$ then the series diverges.

3. Consider $\sum_{n=1}^{\infty} a_n$ where $a_1 = 1$ and $a_{n+1} = \frac{a_n(2+\sin n)}{\sqrt{n}}$ for $n \in \mathbb{N}$. Show that $\sum_{n=1}^{\infty} a_n$ converges.

4. Let $(a_n)$ be a sequence such that $|\frac{a_{n+1}}{a_n}| \to \frac{1}{2}$. Show that $\sum_{n=1}^{\infty} n^2 a_n$ converges whereas $\sum_{n=1}^{\infty} 3^n a_n$ diverges.

5. Let $\sum_{n=1}^{\infty} a_n$ be a convergent series. Show that $\sum_{n=1}^{\infty} \frac{3^n + a_n}{4^n + a_n}$, converges.

6. Show that the series $\frac{1}{1^2} + \frac{1}{2^3} + \frac{1}{3^2} + \frac{1}{4^3} + \frac{1}{5^2} + \frac{1}{6^3} + \cdots$ converges and that the root test and ratio test are not applicable.

7. Consider the rearranged geometric series $\frac{1}{2} + 1 + \frac{1}{8} + \frac{1}{4} + \frac{1}{32} + \frac{1}{16} + \frac{1}{128} + \frac{1}{64} + \cdots$. Show that the series converges by the root test and that the ratio test is not applicable.

8. Consider $\sum_{n=1}^{\infty} a_n$ where $a_{2n} = \frac{1}{3^n}$ and $a_{2n-1} = \frac{1}{2^n}$ for all $n$. Show that the ratio test is not applicable. Further, show that $(a_n)^{\frac{1}{n}}$ does not converge and that Theorem 14.3 is applicable.

9. In each of the following cases, discuss the convergence/divergence of the series $\sum_{n=1}^{\infty} a_n$ where $a_n$ equals:
   (a) $\frac{n!}{e^{n^2}}$ (b) $\frac{n^2 2^n}{(2n+1)!}$ (c) $\left(1 - \frac{1}{n}\right)^{n^2}$ (d) $\frac{n^2}{3^n}\left(1 + \frac{1}{n}\right)^{n^2}$
   (e) $(-1)^n \left(n^{\frac{1}{n}} - 1\right)^n$ (f) $\frac{2^n + n^2 - \ln n}{n!}$ (g) $\left(1 + \frac{2}{n}\right)^{n^2 - \sqrt{n}}$ (h) $\frac{n^2(2\pi + (-1)^n)^n}{10^n}$
   (i) $\tan^{-1} e^{-n}$

10. **(*)** Let $a_n \in \mathbb{R}$ and $a_n > 0$ for all $n$.
    (a) If $\frac{a_{n+1}}{a_n} \leq \lambda$ eventually for some $\lambda > 0$ then show that $a_n^{\frac{1}{n}} \leq \lambda + \epsilon$ eventually for every $\epsilon > 0$.
    (b) Show that if Theorem 14.1 (respectively, the ratio test) gives the convergence of a series $\sum_{n=1}^{\infty} a_n$ then Theorem 4.3 (respectively, the root test if $\lim_{n\to\infty} a_n^{\frac{1}{n}}$ exists) also gives the convergence, but the converse is not true (why?).
    (c) If $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \alpha$ and $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \beta$, show that $\alpha \leq \beta$

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Please write to psraj@iitk.ac.in if any typos/mistakes are found in this set of practice problems/solutions/hints.

# Practice Problems 14: Hints/Solutions

1. Since $|\frac{\alpha n}{n+1}| \to \alpha$, by the root test, the series converges for $|\alpha| < 1$ and diverges for $|\alpha| > 1$. For $|\alpha| = 1$, the series diverges because $(\frac{n}{n+1})^n \to \frac{1}{e} \neq 0$.

2. (a) For $a_n = \frac{1}{n}$, $\frac{a_{n+1}}{a_n} < 1$ for all $n$ but $\sum_{n=1}^{\infty} \frac{1}{n}$ diverges.

   (b) If $\frac{a_{n+1}}{a_n} > 1$ then $a_n \nrightarrow 0$. Hence $\sum_{n=1}^{\infty} a_n$ diverges.

   (c) Let $a_n = (1 - \frac{1}{n})^n$. Then $|a_n|^{1/n} < 1$ for all $n$. Since $a_n \to e^{-1}$, $\sum_{n=1}^{\infty} a_n$ diverges.

   (d) If $|a_n|^{1/n} > 1$ for all $n$ then $a_n \nrightarrow 0$. Hence $\sum_{n=1}^{\infty} a_n$ diverges.

3. Observe that $a_n \neq 0$ for all $n$ and $|\frac{a_{n+1}}{a_n}| \to 0$. Apply the ratio test.

4. Apply the ratio test.

5. Let $c_n = \frac{3^n + a_n}{4^n + a_n}$. Observe first that since $\sum_{n=1}^{\infty} a_n$ converges, $a_n \to 0$ and hence $c_n > 0$ eventually. Verify that $\frac{c_{n+1}}{c_n} \to \frac{3}{4}$ and apply the ratio test. Convergence of $\sum_{n=1}^{\infty} c_n$ can also be shown using the LCT. Observe that $\frac{c_n}{b_n} \to 1$ where $b_n = (\frac{3}{4})^n$ for all $n$.

6. By the comparison test (with $\frac{1}{n^2}$), the series converges.

7. The $n$th term $a_n$ is $\frac{1}{2^n}$ if $n$ is odd and $\frac{1}{2^{n-2}}$ if $n$ is even. Since the consecutive ratio alternate in value between $\frac{1}{8}$ and 2, the ratio test is not applicable. However $a_n^{\frac{1}{n}} \to \frac{1}{2}$.

8. Observe that $\frac{a_{2n+1}}{a_{2n}} = (\frac{3}{2})^n \frac{1}{2} \to \infty$ and $\frac{a_{2n}}{a_{2n-1}} = (\frac{2}{3})^n \to 0$. Therefore, the ratio test is not applicable. Since $a_{2n}^{1/2n} \to \frac{1}{\sqrt{3}}$ and $a_{2n-1}^{1/2n-1} \to \frac{1}{\sqrt{2}}$, we have $a_n^{\frac{1}{n}} < L$ eventually for some $L$ satisfying $\frac{1}{\sqrt{2}} < L < 1$. Hence Theorem 14.3 is applicable and the series converges.

9. (a) Converges by the Ratio test.

   (b) Converges by the Ratio test.

   (c) Converges by the Root test: $(1 - \frac{1}{n})^n \to \frac{1}{e}$

   (d) Converges by the Root test: $a_n^{\frac{1}{n}} \to \frac{e}{3} < 1$.

   (e) Converges absolutely by the Root test.

   (f) Converges: By the LCT test with $\frac{2^n}{n!}$ and then the Ratio test for $\sum_{n-1}^{\infty} \frac{2^n}{n!}$.

   (g) Diverges because $\left(1 + \frac{2}{n}\right)^{n^2 - \sqrt{n}} \nrightarrow 0$ as $(1 + \frac{2}{n}) > 1$.

   (h) Converges absolutely: Use $|a_n| \leq \frac{n^2 (2\pi+1)^n}{10^n}$ and then the Ratio test.

   (i) Note that $\lim_{x\to\infty} \frac{\tan^{-1} e^{-(x+1)}}{\tan^{-1} e^{-x}} = \lim_{x\to\infty} \frac{-e^{-(x+1)}/1+e^{-2(x+1)}}{-e^{-x}/1+e^{-2x}} = e^{-1}$.

10. (a) Suppose $\frac{a_{n+1}}{a_n} \leq \lambda$ for all $n \geq N$ for some $N$. Then for all $n \geq N$,

$$a_n = \frac{a_n}{a_{n-1}} \frac{a_{n-1}}{a_{n-2}} ... \frac{a_{N+1}}{a_N} a_N \leq \lambda^{n-N} a_N.$$

    Therefore $a_n^{\frac{1}{n}} = (\lambda^{1-\frac{N}{n}}) a_N^{\frac{1}{n}} \leq \lambda + \epsilon$ eventually for any $\epsilon > 0$ as $a_N^{\frac{1}{n}} \to 1$.

(b) Suppose Theorem 14.1 or the Ratio test implies the convergence of a series $\sum_{n=1}^{\infty} a_n$. Then there exists $\lambda$ such that $0 < \lambda < 1$ and $\frac{a_{n+1}}{a_n} \leq \lambda$ eventually. Then, by (a), $a_n^{\frac{1}{n}} \leq \lambda + \frac{(1-\lambda)}{2} < 1$ eventually. Hence by Theorem 14.3, $\sum_{n=1}^{\infty} a_n$ converges. In case, $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \alpha$, then $\alpha \leq \lambda + \frac{(1-\lambda)}{2}$. Hence by the root test, the series converges. For the converse part, see Problem 7.

(c) Follows from (a).