Practice Problems 15: Power Series, Taylor's Series

1. For a given $\sum_{n=0}^{\infty} a_n x^n$, let

$$K = \left\{ |x| : x \in \mathbb{R} \text{ and } \sum_{n=0}^{\infty} a_n x^n \text{ is convergent} \right\}$$

be bounded. If $r = \sup K$, then $\sum_{n=0}^{\infty} a_n x^n$

(a) converges absolutely for all $x \in \mathbb{R}$ with $|x| < r$,

(b) diverges for all $x \in \mathbb{R}$ with $|x| > r$.

2. In each of the following cases, determine the values of x for which the power series converges.

(a) $\sum_{n=0}^{\infty} \frac{2^n x^n}{n^n}$ (b) $\sum_{n=0}^{\infty} \frac{(n!)^2 x^n}{(2n)!}$ (c) $\sum_{n=0}^{\infty} (-1)^n n 2^n x^n$

(d) $\sum_{n=0}^{\infty} \frac{(x-2)^{n+1}}{n3^n}$ (e) $\sum_{n=0}^{\infty} (-1)^n \frac{10^n}{n!}(x-10)^n$ (f) $\sum_{n=2}^{\infty} \frac{x^n}{n(\ln n)^2}$

3. (a) Let (S_n) be the sequence of partial sums of the Maclaurin series of $\ln(1+x)$. Show that if $0 \le x \le 1$, then $S_n \to \ln(1+x)$, i.e, the Maclaurin series of $\ln(1+x)$ converges to $\ln(1+x)$ on $[0,1]$.

(b) For each $x \in [0,1]$, $\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$.

(c) Show that $\ln 2 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$.

4. Let $f : (a,b) \to \mathbb{R}$ be infinitely differentiable and $x_0 \in (a,b)$. Suppose that there exists $M > 0$ such that $|f^n(x)| \le M^n$ for all $n \in \mathbb{N}$ and $x \in (a,b)$. Show that Taylor's series of f at x_0 converges to $f(x)$ for all $x \in (a,b)$.

5. Let $f(x) = e^{-\frac{1}{x^2}}$ when $x \neq 0$ and $f(0) = 0$. Show that

(a) $f'(0) = 0$;

(b) for $x \neq 0$, $n \geq 1$, $f^{(n)}(x) = P_n(\frac{1}{x})e^{-\frac{1}{x^2}}$ where P_n is a polynomial of degree $3n$;

(c) $f^{(n)}(0) = 0$ for $n = 1, 2, ...$;

(d) the Maclaurin series of f converges to $f(x)$ only when $x = 0$.

6. **(*)** Let (S_n) be the sequence of partial sums of the Maclaurin series of $\ln(1+x)$. Show that if $-1 < x \le 0$, then $S_n \to \ln(1+x)$ i.e, the Maclaurin series of $\ln(1+x)$ converges to $\ln(1+x)$ on $(-1, 0]$.

7. **(*)(Binomial series)** Let $k \in \mathbb{R}$ and $f : (-1,1) \to \mathbb{R}$ be defined by $f(x) = (1+x)^k$. Denote $\frac{k(k-1)\cdots(k-n+1)}{n!}$ by $\binom{k}{n}$.

(a) Show that the Maclaurin series of f is $1 + \sum_{n=1}^{\infty} \binom{k}{n} x^n$ which is known as the binomial series.

(b) If $k \in \mathbb{N} \cup \{0\}$, show that $f(x) = 1 + \sum_{n=0}^{k} \binom{k}{n} x^n$.

(c) If $k \notin \mathbb{N} \cup \{0\}$, show that $f(x) = 1 + \sum_{n=1}^{\infty} \binom{k}{n} x^n$ for all $x \in (-1,1)$.

(d) Obtain from (b) that $\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n$ and $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$ for $x \in (-1,1)$.

Please write to psraj@iitk.ac.in if any typos/mistakes are found in this set of practice problems/solutions/hints.

8. **(*)** Let $a_n \geq 0$ for all $n \in \mathbb{N}$ and $(a_n^{\frac{1}{n}})$ be a bounded sequence. For each n, define

$$A_n = \sup\{a_k^{\frac{1}{k}} : k \geq n\}$$

. Since (A_n) is bounded and decreasing, let $A_n \to \ell$ for some $\ell > 0$.

(a) If $\ell < 1$, the series $\sum_{n=1}^{\infty} a_n$ converges and if $\ell > 1$, the series diverges.

(b) The radius of convergence of the power series $\sum_{n=1}^{\infty} a_n x^n$ is $\frac{1}{\ell}$

(c) Find the radius of convergence of the power series

$$\frac{1}{2}x + \frac{1}{3}x^2 + \frac{1}{2^2}x^3 + \frac{1}{3^2}x^4 + \frac{1}{2^3}x^5 + \frac{1}{3^3}x^6 + \frac{1}{2^4}x^7 + \frac{1}{3^4}x^8 + ...$$

Practice Problems 15: Hints/Solutions

1. (a) If $|x| < r$, then by the definition of supremum there exists $|x_0| \in K$ such that $|x| < |x_0|$. Since $\sum_{n=0}^{\infty} a_n x_0^n$ converges, by Theorem 15.1, $\sum_{n=0}^{\infty} a_n x^n$ converges absolutely.

   (b) Suppose $|x| > r$. By the definition of K, $\sum_{n=0}^{\infty} a_n x^n$ diverges.

2. (a) Since $|\frac{2^n x^n}{n^n}|^{\frac{1}{n}} \to 0$, by the root test the series converges for all $x \in \mathbb{R}$.

   (b) In this case $|\frac{a_{n+1}x^{n+1}}{a_n x^n}| \to |\frac{x}{4}|$ and $\frac{a_{n+1}4^{n+1}}{a_n 4^n} = \frac{(n+1)}{(n+\frac{1}{2})} > 1$. The series converges only for $|x| < 4$ as $(a_n 4^n)$ increases and $a_n 4^n \nrightarrow 0$.

   (c) Use Ratio test . The series converges only for $|x| < \frac{1}{2}$.

   (d) Use Ratio test. The series converges for $|x - 2| < 3$, and hence for $-1 < x < 5$. At $x = 5$ the series diverges and $x = -1$ the series converges.

   (e) Since $|\frac{a_{n+1}}{a_n}(x - 10)| \to 0$, the series converges for all $x \in \mathbb{R}$.

   (f) Apply the Ratio test. The series converges only if $x \in [-1, 1]$.

3. (a) Observe that if $f(x) = \ln(1+x)$, $x \in [0, 1]$, then $f^{(k)}(x) = (-1)^{k-1}\frac{(k-1)!}{(1+x)^k}$ for all $k \in \mathbb{N}$. By Taylor's theorem, $\ln(1 + x) = S_n + \frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}}$ for some $c \in (0, x)$. This implies that $|\ln(1 + x) - S_n| = |\frac{(-1)^n}{n+1}\frac{x^{n+1}}{(1+c)^{n+1}}| \leq |\frac{x^{n+1}}{n+1}| \to 0$.

   (b) Follows from (a).

   (c) Take $x = 1$ in (b).

4. Note that for $x \in (a, b)$, $|E_n(x)| = |\frac{f^{n+1}(c)}{(n+1)!}||x - x_0|^{n+1}$ for some c between x and x_0. This implies that $|E_n(x)| \leq \frac{A^{n+1}}{(n+1)!}$ where $A = M|x - x_0|$. It follows from the ratio test for sequences that $\frac{A^{n+1}}{(n+1)!} \to 0$. This shows that Taylor's series of f converges to $f(x)$.

5. (a) Note that $\lim_{x\to 0^+} \frac{f(x)-f(0)}{x} = \lim_{x\to 0^+} \frac{e^{-\frac{1}{x^2}}}{x} = \lim_{x\to 0^+} \frac{\frac{1}{x}}{e^{\frac{1}{x^2}}} = \lim_{y\to\infty} \frac{y}{e^{y^2}} = 0$, by L'Hospital Rule.

   (b) If $f^{(n)}(x) = P_n(\frac{1}{x})e^{-\frac{1}{x^2}}$, then

$$f^{(n+1)}(x) = \left\{P_n'(\frac{1}{x})(-\frac{1}{x^2}) + P_n(\frac{1}{x})(\frac{2}{x^3})\right\} e^{-\frac{1}{x^2}} = P_{n+1}(\frac{1}{x})e^{-\frac{1}{x^2}}$$

   where $P_{n+1}(t) = -t^2 P_n'(t) + 2t^3 P_n(t)$ which is of degree $3n + 3$ if P_n is of degree $3n$. Use the induction argument.

(c) If $f^{n-1}(0) = 0$ then, as done in (a), $\lim_{x\to 0^+} \frac{f^{(n-1)}(x)-f^{(n-1)}(0)}{x} = \lim_{y\to\infty} \frac{yP_{n-1}(y)}{e^{y^2}} = 0$, i.e., $f^n(0) = 0$.

(d) Trivial.

6. We use Taylor's theorem with Cauchy remainder (see Problem 15 of PP9). If $f(x) = \ln(1+x)$, $x \in (-1, 0]$, then by Taylor's theorem with Cauchy reminder, there exists $c \in (x, 0)$ such that

$$|f(x) - S_n| = \left|\frac{(x-c)^n x}{n!} f^{(n+1)}(c)\right| = \left|\frac{(x-c)^n x}{(1+c)^{n+1}}\right| = \left|\frac{x-c}{1+c}\right|^n \frac{|x|}{1+c} \leq \left|\frac{x-c}{1+c}\right|^n \frac{|x|}{1+x}.$$

Let $c = \theta x$ for some $0 < \theta < 1$. Note that θ depends on n. Now

$$|f(x) - S_n| \leq \left(\frac{1-\theta}{1+\theta x}\right)^n \left|\frac{x^{n+1}}{1+x}\right|.$$

Since $\frac{1-\theta}{1+\theta x} \in (0, 1)$, $|f(x) - S_n| \leq \left|\frac{x^{n+1}}{1+x}\right|$. Hence $|f(x) - S_n| \to 0$.

7. (a) Since $f^{(n)}(x) = k(k-1)\cdots(k-n+1)(1+x)^{k-n}$, the Maclaurin series of f is $1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots$.

(b) This follows from the fact that $f^{(n)}(0) = 0$ for all $n > k+1$.

(c) The solution is similar to that of Problem 6. We use Taylor's theorem with Cauchy remainder (see Problem 15 of PP9). Let $x \in (-1, 1)$. Observe that

$$\left|\frac{(x-c)^n x}{n!} f^{n+1}(c)\right| = \left|\frac{(x-c)^n x}{n!} k(k-1)\cdots(k-n)(1+c)^{k-n-1}\right|.$$

Let $c = \theta x$ for some $0 < \theta < 1$. Note that θ depends on n. Now,

$$\left|\frac{(x-c)^n x}{n!} f^{n+1}(c)\right| = \left|\left(\frac{1-\theta}{1+\theta x}\right)^n x^{n+1}(1+\theta x)^{k-1} \frac{k(k-1)\cdots(k-n)}{n!}\right|.$$

Observe that $\frac{1-\theta}{1+\theta x} \in (0, 1)$. Further, $(1+\theta x)^{k-1} \leq (1+|x|)^{k-1}$ if $k > 1$ and $(1+\theta x)^{k-1} \leq (1-|x|)^{k-1}$ if $k < 1$. Therefore,

$$\left|\frac{(x-c)^n x}{n!} f^{n+1}(c)\right| \leq (1 \pm |x|)^{k-1} \left|x^{n+1} \frac{k(k-1)\cdots(k-n)}{n!}\right|$$

By the ratio test for sequences, $\left|x^{n+1} \frac{k(k-1)\ldots(k-n)}{n!}\right| \to 0$.

(d) Take $k = -1$ for the first part. Replace x by $-x$ for the second part.

8. (a) If $\ell < 1$, then find $\epsilon > 0$ such that $\ell < \ell + \epsilon < 1$. Since $A_n \to \ell$, there exists $N \in \mathbb{N}$ such that $A_n < \ell + \epsilon$ for all $n \geq N$. That is $a_n^{\frac{1}{n}} < \ell + \epsilon < 1$ for all $n \geq N$. Therefore by Theorem 14.3, the series $\sum_{n=1}^{\infty} a_n$ converges.

If $\ell > 1$, choose $\epsilon > 0$ such that $\ell - \epsilon > 1$. Since $A_n \to \ell$, there exists a subsequence $(a_{n_k}^{\frac{1}{n_k}})$ of $(a_n^{\frac{1}{n}})$ such that $a_{n_k}^{\frac{1}{n_k}} \geq \ell - \epsilon > 1$. Hence $a_n^{\frac{1}{n}} \nrightarrow 0$ and therefore $\sum_{n=1}^{\infty} a_n$ diverges.

(b) Follows from the proof of (a) (Repeat the proof of (a) by replacing a_n by $a_n x^n$).

(c) See Problem 8 of PP14. In this case $\ell = \frac{1}{\sqrt{2}}$ and hence $\frac{1}{\ell} = \sqrt{2}$ is the radius of convergence.