Practice Problems 17: Integration, Riemann's Criterion for integrability (Part II)

1. Let $f : [a, b] \to \mathbb{R}$ be integrable and $[c, d] \subset [a, b]$. Show that f is integrable on $[c, d]$.

2. (a) Let f be bounded on $[a, b]$, $M = \sup\{f(x) : x \in [a, b]\}$, $M' = \sup\{|f(x)| : x \in [a, b]\}$, $m = \inf\{f(x) : x \in [a, b]\}$ and $m' = \inf\{|f(x)| : x \in [a, b]\}$. Show that $M' - m' \leq M - m$.

   (b) Let $f : [a, b] \to \mathbb{R}$ be integrable. Show that $|f|$ and f^2 are integrable.

3. (a) Find $f : [0, 1] \to \mathbb{R}$ such that $|f|$ is integrable but f is not integrable.

   (b) Find $f : [0, 1] \to \mathbb{R}$ such that f^2 is integrable but f is not integrable.

4. Let f and g be two integrable functions on $[a, b]$.

   (a) If $f(x) \leq g(x)$ for all $x \in [a, b]$, show that $\int_a^b f(x)dx \leq \int_a^b g(x)dx$.

   (b) Show that $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$.

   (c) If $m \leq f(x) \leq M$ for all $x \in [a, b]$ show that $m(b - a) \leq \int_a^b f(x)dx \leq M(b - a)$. Use this inequality to show that $\frac{\sqrt{3}}{8} \leq \int_{\pi/4}^{\pi/3} \frac{\sin x}{x} dx \leq \frac{\sqrt{2}}{6}$.

5. Let $f : [a, b] \to \mathbb{R}$ and $f(x) \geq 0$ for all $x \in [a, b]$

   (a) If f is integrable, show that $\int_a^b f(x)dx \geq 0$.

   (b) If f continuous and $\int_a^b f(x)dx = 0$ show that $f(x) = 0$ for all $x \in [a, b]$.

   (c) Give an example of an integrable function f on $[a, b]$ such that $f(x) \geq 0$ for all $x \in [a, b]$ and $\int_a^b f(x)dx = 0$ but $f(x_0) \neq 0$ for some $x_0 \in [a, b]$.

6. Let $f : [0, 1] \to \mathbb{R}$ be a bounded function. Suppose that for any $c \in (0, 1]$, f is integrable on $[c, 1]$.

   (a) Show that f is integrable on $[0, 1]$.

   (b) Show that the function f defined by $f(0) = 0$ and $f(x) = \sin(\frac{1}{x})$ on $(0, 1]$ is integrable,

7. Let $f : [a, b] \to \mathbb{R}$ be a continuous function. Suppose that whenever the product fg is integrable on $[a, b]$ for some integrable function g, we have $\int_a^b (fg)(x)dx = 0$. Show that $f(x) = 0$ for every $x \in [a, b]$.

8. (a) Let $x, y \geq 0$. Show that $\lim_{n\to\infty} (x^n + y^n)^{\frac{1}{n}} = M$ where $M = \max\{x, y\}$.

   (b) Let $f : [a, b] \to \mathbb{R}$ be continuous and $f(x) \geq 0$ for all $x \in [a, b]$. Show that $\lim_{n\to\infty} \left(\int_a^b f(x)^n\right)^{\frac{1}{n}} = M$ where $M = \sup\{f(x) : x \in [a, b]\}$.

9. (a) **(*)** (Cauchy-Schwarz inequality) Let $x_1, x_2, ...x_n, y_1, y_2, ..., y_n \in \mathbb{R}$. By observing that $\sum_{i=1}^n (tx_i - y_i)^2 \geq 0$ for any $t \in \mathbb{R}$, show that $|\sum_{i=1}^n x_i y_i| \leq \left(\sum_{i=1}^n x_i^2\right)^{\frac{1}{2}} \left(\sum_{i=1}^n y_i^2\right)^{\frac{1}{2}}$.

   (b) **(*)** (Cauchy-Schwarz inequality) Let f and g be any two integrable functions on $[a, b]$. Show that $\left(\int_a^b f(x)g(x)dx\right)^2 \leq \left(\int_a^b |f(x)|^2 dx\right) \left(\int_a^b |g(x)|^2 dx\right)$.

10. **(*)** Let $f : [a, b] \to \mathbb{R}$ be integrable. Suppose that the values of f are changed at a finite number of points. Show that the modified function is integrable.

11. **(*)** Let $f : [a, b]$ be a bounded function and $E \subset [a, b]$. Suppose that E can be covered by a finite number of closed intervals whose total length can be made as small as desired. If f is continuous at every point outside E, show that f is integrable.

Please write to psraj@iitk.ac.in if any typos/mistakes are found in this set of practice problems/solutions/hints.

Practice Problems 17: Hints/Solutions

1. Let $\epsilon > 0$. Since f is integrable on $[a, b]$, there exists a partition $P = \{x_0, x_1, x_2, ..., x_n\}$ (of [a,b]) such that $U(P, f) - L(P, f) < \epsilon$. Let $P_1 = P \cup \{c, d\}$ and $P' = P_1 \cap [c, d]$ which is a partition of $[c, d]$. Then, since $M_i - m_i > 0$, it follows that $U(P', f) - L(P', f) \leq U(P_1, f) - L(P_1, f) \leq U(P, f) - L(P, f) < \epsilon$. Apply the Riemann Criterion.

2. (a) Let $x, y \in [c, d]$. Then $|f(x)| - |f(y)| \leq |f(x) - f(y)| \leq M - m$. Fix y and take supremum for x, we get $M' - |f(y)| \leq M - m$. Take infimum for y.

   (b) To show that $|f|$ is integrable, use the Riemann Criterion and (a).
   For showing f^2 is integrable, use the inequality $(f(x))^2 - (f(y))^2 \leq 2K|f(x) - f(y)|$ where $K = \sup\{|f(x)| : x \in [a, b]\}$ and proceed as in (a).

3. Let $f : [0, 1] \to \mathbb{R}$ be defined by $f(x) = -1$ for x rational and $f(x) = 1$ for x irrational. Then $|f| = f^2$. Note that f is not integrable but $|f|$ is a constant function.

4. (a) Use $\int_a^b g(x)dx - \int_a^b f(x)dx = \int_a^b (g - f)(x)dx$ and Problem 3 of Practice Problems 16

   (b) Since $-|f(x)| \leq f(x) \leq |f(x)|$, $x \in [a, b]$, (b) follows from part (a).

   (c) Use part (a) or $L(P, f) \leq \int_a^b f(x)dx \leq U(P, f)$. On $[\frac{\pi}{4}, \frac{\pi}{3}]$, $\frac{\sin x}{x}$ decreases.

5. (a) This follows from the definition of integrability of f or from Problem 4(a).

   (b) Let $x_0 \in (a, b)$ be such that $f(x_0) > \alpha$ for some $\alpha > 0$. Then by the continuity of f there exists a $\delta > 0$ such that $(x_0 - \delta, x_0 + \delta) \subseteq (a, b)$ and $f(x) > \alpha$ on $(x_0 - \delta, x_0 + \delta)$. Then we can find a partition P of $[a, b]$ such that $\int_a^b f(x)dx \geq L(P, f) > \alpha \times \delta > 0$.

   (c) Let $f(a) = 1$ and $f(x) = 0$ for all $x \in (a, b]$. Then $\int_a^b f(x)dx = 0$ but $f(a) \neq 0$.

6. (a) Let $M = \sup\{|f(x)| : x \in [0, 1]\}$. If $P_n = \{\frac{1}{n}, x_1, x_2, ..., x_n\}$ is a partition of $[\frac{1}{n}, 1]$ then let $P'_n = \{0, \frac{1}{n}, x_1, x_2, ..., x_n\}$ be a corresponding partition of $[0, 1]$. Then $U(P'_n, f) \leq \frac{M}{n} + U(P_n, f)$ and $L(P'_n, f) \geq -\frac{M}{n} + L(P_n, f)$. Therefore, $U(P'_n, f) - L(P'_n, f) \leq \frac{2M}{n} + U(P_n, f) - L(P_n, f)$. For $\epsilon > 0$, first choose n such that $\frac{2M}{n} < \frac{\epsilon}{2}$ and then choose P_n such that $U(P_n, f) - L(P_n, f) < \frac{\epsilon}{2}$. Apply the Riemann Criterion.

   (b) Since f is continuous on $[c, 1]$ for every c satisfying $0 < c < 1$, f is integrable on $[c, 1]$. Apply part (a).

7. Suppose $f(x_0) > 0$ for some $x_0 \in (a, b)$. Then $f^2(x_0) > 0$. By the argument used in Problem 5(b), $\int_a^b f^2(x)dx > 0$. Choose $g = f$ to conclude.

8. (a) Note that $M \leq (x^n + y^n)^{\frac{1}{n}} \leq (2M^n)^{\frac{1}{n}}$. Use the Sandwich Theorem.

   (b) For $\epsilon > 0$, by the continuity of f, $\exists\ [c, d] \subseteq [a, b]$ such that $f(x) > M - \epsilon\ \ \forall\ x \in [c, d]$. Hence $(M - \epsilon)(d - c)^{\frac{1}{n}} \leq \left(\int_a^b f(x)^n\right)^{\frac{1}{n}} \leq M(b - a)^{\frac{1}{n}}$. Note that $M(b - a)^{\frac{1}{n}} \to M$ and $(M - \epsilon)(d - c)^{\frac{1}{n}} \to M - \epsilon$. Hence there exists $N \in \mathbb{N}$ such that for all $n \geq N$,
   $M - 2\epsilon \leq (M - \epsilon)(d - c)^{\frac{1}{n}} \leq \left(\int_a^b f(x)^n\right)^{\frac{1}{n}} \leq M(b - a)^{\frac{1}{n}} \leq M + \epsilon.$

9. We will see the solution of part (b) and the solution of part (a) is similar. Note that the inequality $\int_a^b (tf(x) - g(x))^2 = t^2 \left(\int_a^b f^2(x)dx\right) - 2t \left(\int_a^b f(x)g(x)dx\right) + \left(\int_a^b g^2(x)dx\right) \geq 0$ holds for all $t \in \mathbb{R}$. Take $t = \frac{\alpha}{\beta}$ where $\alpha = \int_a^b f(x)g(x)dx$ and $\beta = \int_a^b f^2(x)dx$.

10. Suppose the values of f are changed at $c_1, c_2, .., c_p$ and g is the modified function. Let $M = \max\{|g(c_1)|, |g(c_2)|, ..., |g(c_p)|\}$. Let $\epsilon > 0$. Since f is integrable, there exists a partition P of $[a, b]$ such that $U(P, f) - L(P, f) < \frac{\epsilon}{2}$. Cover $c_i's$ by the disjoint intervals $[y_1, y_2], [y_3, y_4], ..., [y_{2p-1}, y_{2p}]$ where $y_i's$ are in $[a, b]$ and $|y_1 - y_2| + |y_3 - y_4| + ... + |y_{2p-1} - y_{2p}| < \frac{\epsilon}{4pM}$. Consider the partition $P_1 = P \cup \{y_1, y_2, ..., y_{2p}\}$. Then $U(P_1, g) - L(P_1, g) \leq U(P_1, f) - L(P_1, f) + \frac{2pM\epsilon}{4pM} < U(P, f) - L(P, f) + \frac{\epsilon}{2} \leq \epsilon$. Apply the Riemann Criterion.

11. Proceed as in Theorem 17.2 and Problem 10.