Practice Problems 18: Fundamental Theorems of Calculus, Riemann Sum

1. (a) Show that every continuous function on a closed bounded interval is a derivative.
   (b) Show that an integrable function on a closed bounded interval need not be a derivative.

2. (a) Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = 0$ for $-1 \leq x < 0$ and $f(x) = 1$ for $0 \leq x \leq 1$. Define $F(x) = \int_{-1}^{x} f(t)dt$.
      i. Sketch the graphs of f and F and observe that f is not continuous; however, F is continuous.
      ii. Observe that F is not differentiable at 0.
   (b) Give an example of a function f on $[-1, 1]$ such that f is not continuous at 0 but $F(x)$ defined by $F(x) = \int_{-1}^{x} f(t)dt$ is differentiable at 0.

3. Let $f : [a, b] \to \mathbb{R}$ be integrable. Show that $\int_a^b f(t)dt = \lim_{x\to b} \int_a^x f(t)dt$.

4. Prove the second FTC by assuming the integrand to be continuous.

5. Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(x) = 2x \sin \frac{1}{x^2} - (\frac{2}{x}) \cos \frac{1}{x^2}$ for $x \neq 0$ and $f(0) = 0$. Show that $F' = f$ where $F(x) = x^2 \sin \frac{1}{x^2}$ for $x \neq 0$ and $F(0) = 0$ but $\int_{-1}^{1} F'(t)dt$ does not exist.

6. Let $f : [0, 1] \to \mathbb{R}$ be continuous such that $|f(x)| \leq \int_0^x f(t)dt$ for all $x \in [0, 1]$. Show that $f(x) = 0$ for all $x \in [0, 1]$.

7. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous. Define $g(x) = \int_0^x (x - t)f(t)dt$ for all $x \in \mathbb{R}$. Show that $g'' = f$.

8. Let f be continuous on $\mathbb{R}$ and $\alpha \neq 0$. If $g(x) = \frac{1}{\alpha} \int_0^x f(t) \sin[\alpha(x - t)]dt$, show that $f(x) = g''(x) + \alpha^2 g(x)$.

9. Let f be a differentiable function on $[0, 1]$. Show that there exists $c \in (0, 1)$ such that $\int_0^1 f(x)dx = f(0) + \frac{1}{2}f'(c)$.

10. Let $f : [0, 1] \to \mathbb{R}$ be a continuous function such that $\int_0^1 f(x)dx = 1$. Show that there exists a point $c \in (0, 1)$ such that $f(c) = 3c^2$.

11. Let $f : [0, \frac{\pi}{4}] \to \mathbb{R}$ be continuous. Show that $\exists\, c \in [0, \frac{\pi}{4}]$ such that $2 \cos 2c \int_0^{\pi/4} f(t)dt = f(c)$.

12. Let $f : [0, a] \to \mathbb{R}$ be such that $f''(x) > 0$ for every $x \in [0, a]$. Show that $\int_0^a f(x)dx \geq af(\frac{a}{2})$.

13. Let $f : [a, b] \to \mathbb{R}$ be continuous and $\int_a^x f(t)dt = \int_x^b f(t)dt$ for all $x \in [a, b]$. Show that $f(x) = 0$ for all $x \in [a, b]$.

14. Let $f, g : [a, b] \to \mathbb{R}$ be integrable functions. Suppose that f is increasing and g is non-negative on $[a, b]$. Show that there exists $c \in [a, b]$ such that $\int_a^b f(x)g(x)dx = f(b) \int_a^c g(x)dx + f(a) \int_c^b g(x)dx$.

15. Show that the MVT implies the first MVT for integrals: If $f : [a, b] \to \mathbb{R}$ is continuous then there exists $c \in (a, b)$ such that $\int_a^b f(t)dt = f(c)(b - a)$. Observe that the converse can be obtained for functions whose derivatives are continuous.

16. Show that $\int_n^{n+1} \frac{1}{x}dx < \frac{1}{n}$ for every $n \in \mathbb{N}$.

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17. Let $f, g : [a, b] \to \mathbb{R}$ be continuous and $\int_a^b f(x)dx = \int_a^b g(x)dx$. Show that there exists $c \in [a, b]$ such that $f(c) = g(c)$.

18. Show that $\frac{\pi^2}{9} \leq \int_{\pi/6}^{\pi/2} \frac{x}{\sin x} \leq \frac{2\pi^2}{9}$.

19. Let $f : [0, 1] \to \mathbb{R}$ be an integrable function. Show that $\lim_{n\to\infty} \int_0^1 x^n f(x)dx = 0$.

20. Find $\lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{\sqrt{n^2+kn}}$.

21. Show that $\lim_{n\to\infty} \frac{1}{n^3} \left[\sin \frac{\pi}{n} + 2^2 \sin \frac{2\pi}{n} + ... + n^2 \sin \frac{n\pi}{n}\right] = \int_0^1 x^2 \sin(\pi x)dx$.

22. Show that $\lim_{n\to\infty} \frac{1}{n^{18}} \sum_{k=1}^{n} k^{16} = 0$.

23. (Integration by parts) Let $f, g : [a, b] \to \mathbb{R}$ be such that f' and g' are continuous on $[a, b]$. Show that $\int_a^b f(x)g'(x)dx = f(b)g(b) - f(a)g(a) - \int_a^b f'(x)g(x)dx$.

24. **(*)**(Integration by substitution) Let $\phi : [\alpha, \beta] \to \mathbb{R}$ be differentiable and ϕ' be continuous on $[\alpha, \beta]$. Suppose that $\phi([\alpha, \beta]) = [a, b]$ and $f : [a, b] \to \mathbb{R}$ is continuous. Then $\int_{\phi(\alpha)}^{\phi(\beta)} f(x)dx = \int_\alpha^\beta f(\phi(t))\phi'(t)dt$.

25. (Leibniz Rule) Let f be a continuous function and u and v be differentiable functions on $[a, b]$. If the range of u and v are contained in $[a, b]$, show that $\frac{d}{dx} \int_{u(x)}^{v(x)} f(t)dt = f(v(x))\frac{dv}{dx} - f(u(x))\frac{du}{dx}$.

26. Let $f : [1, \infty) \to \mathbb{R}$ be defined by $f(x) = \int_1^x \frac{\ln t}{1+t}dt$. Solve the equation $f(x) + f(\frac{1}{x}) = 2$.

Practice Problems 18 : Hints/Solutions

1. (a) Follows immediately from the first FTC.

   (b) Consider the function $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = -1$ for $-1 \leq x < 0$, $f(0) = 0$ and $f(x) = 1$ for $0 < x \leq 1$. Then f is integrable on $[1, 1]$. Since f does not have the intermediate value property, it cannot be a derivative (see Problem 18(c) of Practice Problems 7).

2. (a) $F(x) = 0$ for $-1 \leq x \leq 0$ and $F(x) = x$ for $0 < x \leq 1$.

   (b) Let $f : [-1, 1] \to \mathbb{R}$ be defined by $f(\frac{1}{n}) = \frac{1}{n}$ for every $n \in N$ and $f(x) = 0$ otherwise. Then $F(x) = \int_{-1}^x f(t)dt = 0$ for all $x \in [-1, 1]$ and hence F is differentiable at 0 but f is not continuous at 0.

3. Follows from the first FTC.

4. Let $f : [a, b] \to \mathbb{R}$ be continuous and $f = F'$ for some F on $[a, b]$. Define $F_a(x) = \int_a^x f(t)dt$ on $[a, b]$. Then by the first FTC, $F = F_a + C$ for some constant C. Since $F_a(a) = 0$, $C = F(a)$ and hence $F(b) - F(a) = \int_a^b f(t)dt$.

5. Observe that F' is not bounded.

6. Let $M = \sup\{|f(x)| : x \in [0, 1]\}$. Then for $x \in [0, 1]$, $|f(x)| \leq \int_0^x |f(t)|dt \leq Mx$. Now , $|f(x)| \leq \int_0^x |f(t)|dt \leq \int_0^x Mtdt = M\frac{x^2}{2}$. Continue to show that $|f(x)| \leq M\frac{x^n}{n!} \to 0$.

7. Write $g(x) = x \int_0^x f(t)dt - \int_0^x tf(t)dt$ and apply the first FTC.

8. Write $g(x) = \frac{1}{\alpha} \left[\sin(\alpha x) \int_0^x f(t) \cos(\alpha t)dt - \cos(\alpha x) \int_0^x f(t) \sin(\alpha t)dt\right]$ and apply the first FTC.

9. Let $F(x) = \int_0^x f(t)dt$. Apply the Extended MVT to F on $[0, 1]$.

10. Consider the function $F(x) = \int_0^x f(t)dt - x^3$ on $[0, 1]$. Apply Rolle's theorem.

11. Let $F(x) = \int_0^x f(t)dt$ and $G(x) = \sin 2x$. Apply the CMVT for F and G on $[0, \pi/4]$.

12. Let $x_0 \in (0, a)$. Then by Taylor's theorem, $f(x) \geq f(x_0) + f'(x_0)(x - x_0)$. Then $\int_0^a f(x)dx \geq af(x_0) - ax_0 f'(x_0) + \frac{a^2}{2} f'(x_0)$. Choose $x_0 = \frac{a}{2}$.

13. Let $F(x) = \int_a^x f(t)dt$. Then $F'(x) = f(x)$. The given condition implies that $F(x) = F(b) - F(x)$. Therefore, $F'(x) = 0$ which implies that $f(x) = 0$.

14. Define $h(x) = f(b) \int_a^x g(x)dx + f(a) \int_x^b g(x)dx$ for all $x \in [a, b]$. Now $h(a) = f(a) \int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx \leq f(b) \int_a^b g(x)dx = h(b)$. Apply the IVP.

15. Let $f : [a, b] \to \mathbb{R}$ be continuous. Define $F(x) = \int_a^x f(t)dt$. Then by the MVT, there $\exists\, c \in (a, b)$ such that $F(b) - F(a) = F'(c)(b - a)$. Apply the First FTC. Conversely, let $f : [a, b] \to \mathbb{R}$ be differentiable and f' be continuous. Then by the MVT for integrals, $\exists\, c \in (a, b)$ such that $\int_a^b f'(x)dx = f'(c)(b - a)$. This implies that $f(b) - f(a) = f'(c)(b - a)$.

16. Use the first MVT for integrals.

17. Use the first MVT for integrals.

18. Use the second MVT for integrals (See Problem 2 of Assignment 6).

19. Note that f is bounded on $[0, 1]$. Apply the second MVT for integrals.

20. $\lim_{n\to\infty} \sum_{k=1}^n \frac{1}{\sqrt{n^2+kn}} = \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^n \frac{1}{\sqrt{1+\frac{k}{n}}} \to \int_0^1 \frac{dx}{\sqrt{1+x}} = 2(\sqrt{2} - 1)$.

21. Note that $\frac{1}{n^3} \left[\sin\frac{\pi}{n} + 2^2 \sin\frac{2\pi}{n} + ... + n^2 \sin\frac{n\pi}{n}\right] = \sum_{k=1}^n \frac{1}{n}(\frac{k}{n})^2 \sin\frac{k\pi}{n}$ which is a Riemann sum.

22. Note that $\frac{1}{n^{18}} \sum_{k=1}^n k^{16} = \frac{1}{n} \left[\frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^{16}\right]$ and $\frac{1}{n} \sum_{k=1}^n \left(\frac{k}{n}\right)^{16} \to \int_0^1 x^{16}dx$.

23. Let $h(x) = f(x)g(x)$. Then $h' = f'g + fg'$. Therefore $\int_a^b h'(x)dx = h(b) - h(a)$.

24. Define $F(x) = \int_{\phi(\alpha)}^x f(u)du$. Therefore $\frac{d}{dt}F(\phi(t))) = F'(\phi(t))\phi'(t) = f(\phi(t))\phi'(t)$. Now $\int_\alpha^\beta f(\phi(t))\phi'(t)dt = [F(\phi(t))]_\alpha^\beta = F(\phi(\beta))$.

25. Note that $\frac{d}{dx} \int_{u(x)}^{v(x)} f(t)dt = \frac{d}{dx} \left[\int_0^{v(x)} f(t)dt - \int_0^{u(x)} f(t)dt\right]$. Apply the first FTC.

26. Observe that $f(\frac{1}{x}) = \int_1^{1/x} \frac{\ln t}{1+t}dt = \int_1^x \frac{\ln y}{y(1+y)}dy$, by taking $t = \frac{1}{y}$. Therefore $f(x) + f(\frac{1}{x}) = \int_1^x \frac{\ln t}{1+t}(1 + \frac{1}{t})dt = \int_1^x \frac{\ln t}{t}dt = \frac{1}{2}(\ln x)^2$. Now $f(x) + f(\frac{1}{x}) = 2$ implies that $\ln x = \pm 2$ which implies that $x = e^2$ as $x > 1$.