Practice Problems 2: Convergence of sequences

1. Use Definition 2.1 and show that
   (a) $\frac{n+1}{2n+1} \to \frac{1}{2}$;
   (b) $1 + \frac{1}{5^n} \to 1$.

2. Let $x_n \le y_n$ for all $n \in \mathbb{N}$. Suppose $x_n \to x_0$ and $y_n \to y_0$. Show that $x_0 \le y_0$.

3. Let $x_n = (-1)^n$ for all $n \in \mathbb{N}$. Show that the sequence (x_n) does not converge.

4. Show that $x_n \to 0$ if and only if $|x_n| \to 0$.

5. If $x_n \to x_0$ show that $|x_n| \to |x_0|$.

6. Using the sandwich theorem show that (x_n) converges where (x_n) is defined as
   (a) $x_n = \sqrt{n^2+1} - n$;
   (b) $x_n = (1+n)^{1/n}$;
   (c) $x_n = \frac{1}{1+n^2} + \frac{2}{2+n^2} + ... + \frac{n}{n+n^2}$;
   (d) $x_n = (n!)^{1/n^2}$;
   (e) $x_n = (a^n + b^n)^{1/n}$ where $0 < a < b$;
   (f) $x_n = (\sqrt{2} - 2^{\frac{1}{3}})(\sqrt{2} - 2^{\frac{1}{5}})...(\sqrt{2} - 2^{\frac{1}{2n+1}})$;
   (g) $x_n = n^\alpha - (n+1)^\alpha$ for some $\alpha \in (0,1)$;

7. For any $x \ge 0, y \ge 0$ and $k \in \mathbb{N}$, show that $|x^{\frac{1}{k}} - y^{\frac{1}{k}}| \le |x-y|^{\frac{1}{k}}$. Using this inequality, show that $x_n^{1/k} \to x_0^{1/k}$ whenever $x_n \ge 0$ for all $n \in \mathbb{N}$ and $x_n \to x_0$.

8. Let $x_0 \in \mathbb{Q}$. Show that there exists a sequence (x_n) of irrational numbers such that $x_n \to x_0$.

9. Let A be a non-empty subset of $\mathbb{R}$ and $\beta = \sup A$. Show that there exists a sequence (a_n) such that $a_n \in A$ for all $n \in \mathbb{N}$ and $a_n \to \beta$.

10. Let A be a non-empty subset of $\mathbb{R}$ and $x \in \mathbb{R}$. Define the distance $d(x, A)$ between x and A by $d(x, A) = \inf\{|x-a| : a \in A\}$. If $\beta = \sup A$, show that $d(\beta, A) = 0$.

11. Let (x_n) and $x_0 \in \mathbb{R}$ be given. State whether the following statement is true or false:
    The sequence (x_n) does not converge to x_0 if and only if there exists some $\epsilon_0 > 0$ such that for every $N \in \mathbb{N}$, there exists $n > N$ such that $|x_n - x_0| > \epsilon_0$.

12. (a) Let $M > 0$ be given. Show that $\frac{M^n}{n!} \to 0$.
    (b) Show that $(n!)^{1/n} \to \infty$.

13. Let $x_n > 0$ for all $n \in \mathbb{N}$. Show that $x_n \to 0$ if and only if $\frac{1}{x_n} \to \infty$

14. Let (x_n) be such that $x_n > 0$ for all $n \in \mathbb{N}$. Show that $x_n \to 0$ if and only if $\frac{x_n}{1+x_n} \to 0$.

15. (*) Let $x_n \to x_0$ and $y_n = \frac{x_1 + x_2 + \cdots + x_n}{n}$ for all $n \in \mathbb{N}$. Show that $y_n \to x_0$. Give an example of (x_n) such that (y_n) converges but (x_n) does not.

Please write to psraj@iitk.ac.in if any typos/mistakes are found in this set of practice problems/solutions/hints.

Practice Problems 2: Hints/Solutions

1. (a) Let $\epsilon > 0$ be given. We have to find $N \in \mathbb{N}$ such that $|\frac{n+1}{2n+1} - \frac{1}{2}| = \frac{1}{4n+2} < \epsilon$ for all $n \geq N$. Choose $N \in \mathbb{N}$ such that $N > \frac{1}{4}(\frac{1}{\epsilon} - 2)$.

   (b) Let $\epsilon > 0$ be given. Find $N \in \mathbb{N}$ such that $5^N > \frac{1}{\epsilon}$.

2. Suppose $y_0 < x_0$. Let $\epsilon = \frac{x_0 - y_0}{4}$. Then there exist $N_1, N_2 \in \mathbb{N}$ such that $x_n \in (x_0 - \epsilon, x_0 + \epsilon)$ for all $n \geq N_1$ and $y_n \in (y_0 - \epsilon, y_0 + \epsilon)$ for all $n \geq N_2$. Thus $x_n > y_n$ for every $n \geq N = max\{N_1, N_2\}$ which is a contradiction.

3. Suppose $x_n \to x_0$ for some $x_0 \in \mathbb{R}$. Let $\epsilon = 1/4$. Then there exists $N \in \mathbb{N}$ such that $x_n \in (x_0 - \epsilon, x_0 + \epsilon)$ for all $n \in \mathbb{N}$. Therefore $|x_n - x_m| \leq 2\epsilon = \frac{1}{2}$ for all $n, m \geq N$ which is not possible.

4. Let $\epsilon > 0$ and $n \in \mathbb{N}$. Then $x_n \in (-\epsilon, \epsilon)$ if and only if $|x_n| \in (-\epsilon, \epsilon)$.

5. Observe that $0 \leq ||x_n| - |x_0|| \leq |x_n - x_0|$. Since $|x_n - x_0| \to 0$, $|x_n| \to |x_0|$.

6. (a) Since $0 < x_n = \frac{1}{\sqrt{n^2+1}+n} < \frac{1}{n}$ for all $n \in \mathbb{N}$, by sandwich theorem $x_n \to 0$.

   (b) We have $1 \leq x_n \leq (2n)^{1/n}$ for all $n \in \mathbb{N}$. Therefore, by the sandwich theorem $x_n \to 1$.

   (c) For all $n \in \mathbb{N}$, $(1 + 2 + ... + n)\frac{1}{n+n^2} \leq x_n \leq (1 + 2 + ... + n)\frac{1}{1+n^2}$. Thus $x_n \to \frac{1}{2}$.

   (d) Observe that $1 \leq x_n \leq (n^n)^{1/n^2} = n^{1/n}$ for all $n \in \mathbb{N}$. This implies that $x_n \to 1$.

   (e) For all $n \in \mathbb{N}$, $b = (b^n)^{1/n} \leq x_n \leq (2b^n)^{1/n} = 2^{1/n}b$. By the sandwich theorem $x_n \to b$.

   (f) We have $0 < x_n < (\sqrt{2} - 1)^n$ for all $n \in \mathbb{N}$. Hence, by the sandwich theorem, $x_n \to 0$.

   (g) For all $n \in \mathbb{N}$, $-x_n = n^\alpha[(1 + \frac{1}{n})^\alpha - 1] < n^\alpha[1 + \frac{1}{n} - 1] = \frac{1}{n^{1-\alpha}}$. Hence $x_n \to 0$.

7. Suppose $x \geq y$. Then $x = (x^{1/k})^k = [(x^{1/k} - y^{1/k}) + y^{1/k}]^k \geq (x^{1/k} - y^{1/k})^k + y$.

8. For each $n \in \mathbb{N}$, find an irrational x_n such that $x_0 < x_n < x_0 + \frac{1}{n}$. Use the sandwich theorem.

9. Let $n \in \mathbb{N}$. Since $\beta - \frac{1}{n}$ is not an upper bound, find $a_n \in A$ such that $\beta - \frac{1}{n} < a_n \leq \beta$. By the sandwich theorem $a_n \to \beta$.

10. By Problem 9, there exists a sequence (a_n) in A such that $a_n \to \beta$. Now $0 \leq d(\beta, A) \leq |\beta - a_n|$. By the sandwich theorem $d(\beta, A) = 0$.

11. True.

12. (a) Use ratio test to show that $\frac{M^n}{n!} \to 0$.

    (b) Let $M > 0$. By (a), there exists $N \in \mathbb{N}$ such that $\frac{M^n}{n!} < 1$ for all $n \in \mathbb{N}$. Hence $(n!)^{1/n} > M$ for all $n \geq N$. This shows that $(n!)^{1/n} \to \infty$.

13. Suppose $x_n > 0$ for all $n \in \mathbb{N}$ and $x_n \to 0$. Let $M > 0$ be given. Choose $\epsilon = \frac{1}{M}$. Since $x_n \to 0$, there exists $N \in \mathbb{N}$ such that $x_n = |x_n - 0| < \epsilon$ for all $n \geq \mathbb{N}$. This shows that $\frac{1}{x_n} > M$ for all $n \geq N$ which proves that $\frac{1}{x_n} \to \infty$. The converse is proved similarly.

14. Observe that $\frac{x_n}{1+x_n} = \frac{1}{\frac{1}{x_n}+1}$ and apply Problem 13.

15. (*) Let ϵ be given. Since $x_n \to x_0$, there exists $N \in \mathbb{N}$ such that $|x_n - x_0| < \frac{\epsilon}{2}$ for all $n \geq N$. For $n \geq N$, we have

$$|y_n - x_0| = \frac{1}{n}\left|\sum_{i=1}^{n}(x_i - x_0)\right| \leq \frac{1}{n}\sum_{i=1}^{N}|x_i - x_0| + \frac{1}{n}\sum_{i=N+1}^{n}|x_i - x_0|.$$

Let $M = \sum_{i=1}^{N}|x_i - x_0|$. As N is fixed, M is a constant. Thus, for $n \geq N$,

$$|y_n - x_0| \leq \frac{M}{n} + \left(\frac{n-N}{n}\right)\frac{\epsilon}{2} < \frac{M}{n} + \frac{\epsilon}{2}.$$

Choose $N_1 > N$ such that $\frac{M}{n} < \frac{\epsilon}{2}$ for all $n \geq N_1$. Then $|y_n - x_0| < \epsilon$ for all $n \geq N_1$ which proves that $y_n \to x_0$.

If we take $x_n = (-1)^n$ for all $n \in \mathbb{N}$, then (y_n) converges whereas (x_n) does not.