Practice Problems 3: Monotone sequences, subsequences

1. Let $x_n = \frac{1}{n+1} + \frac{1}{n+2} + \cdots \frac{1}{n+n}$ for all $n \in \mathbb{N}$. Show that (x_n) is increasing and bounded.

2. Let (x_n) be a sequence in $\mathbb{R}$. Prove or disprove the following statements.

   (a) If $x_n \to 0$ and (y_n) is a bounded sequence then $x_n y_n \to 0$.

   (b) If $x_n \to \infty$ and (y_n) is a bounded sequence then $x_n y_n \to \infty$.

   (c) If (x_n) is increasing and not bounded then $x_n \to \infty$.

3. Show that the sequence (x_n) is bounded and monotone, and find its limit where (x_n) is defined as

   (a) $x_1 = 2$ and $x_{n+1} = 2 - \frac{1}{x_n}$ for $n \in \mathbb{N}$;

   (b) $x_1 = \sqrt{2}$ and $x_{n+1} = \sqrt{2x_n}$ for $n \in \mathbb{N}$;

   (c) $x_1 = 1$ and $x_{n+1} = \frac{4+3x_n}{3+2x_n}$, for $n \in \mathbb{N}$.

4. Let $0 < b_1 < a_1$ and define $a_{n+1} = \frac{a_n+b_n}{2}$ and $b_{n+1} = \sqrt{a_n b_n}$ for all $n \in \mathbb{N}$. Show that both (a_n) and (b_n) converge.

5. Let $a > 0$ and $x_1 > 0$. Define $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ for all $n \in \mathbb{N}$. Show that the sequence (x_n) converges to $\sqrt{a}$ (*The iterative process given in this problem can be used to find approximate values of $\sqrt{a}$ in case it is irrational. How this iterative process is generated will be discussed in Lecture 11).*

6. Let (x_n) be a sequence in $(0, 1)$. Suppose $4x_n(1 - x_{n+1}) > 1$ for all $n \in \mathbb{N}$. Show that the sequence is monotone and find its limit.

7. Let $x_n = \frac{1-2+3-4+\cdots+(-1)^{n-1}n}{n}$ for all $n \in \mathbb{N}$. Test the convergence of (x_n).

8. Let (x_n) be a sequence and $x_0 \in \mathbb{R}$. Suppose that (x_n) does not converge to x_0. Show that there exist $\epsilon_0 > 0$ and a subsequence (x_{n_k}) such that $|x_{n_k} - x_0| \geq \epsilon_0$ for every k.

9. Let (x_n) be given. Suppose $\lim_{n\to\infty} x_{2n-1} = x_0$ and $\lim_{n\to\infty} x_{2n} = x_0$ for some $x_0 \in \mathbb{R}$. Show that $x_n \to x_0$.

10. Let $x_n = 2 + (-1)^n$ for all $n \in \mathbb{N}$. Show that $\lim_{n\to\infty} (x_1 x_2 \cdots x_n)^{1/n} = \sqrt{3}$.

11. Let (x_n) be a sequence in $\mathbb{R}$ and $x_0 \in \mathbb{R}$. Suppose that every subsequence of (x_n) has at least one subsequence which converges to x_0. Show that $x_n \to x_0$.

12. **(*)** Prove the nested interval theorem directly from the completeness property (i.e., without using Theorem 3.1).

13. **(*)** Let $x_n = (1 + \frac{1}{n})^n$ and $y_n = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + ... + \frac{1}{n!}$ for $n \in \mathbb{N}$.

    (a) Using the binomial theorem, show that (x_n) is increasing.

    (b) Show that $x_n \leq y_n$ for all $n \in \mathbb{N}$. Further, show that (x_n) and (y_n) are bounded.

    (c) For $n > m$, show that $x_n > 1+1+\frac{1}{2!}(1-\frac{1}{n})+\frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n})+\cdots+\frac{1}{m!}(1-\frac{1}{n})\cdots(1-\frac{m-1}{n})$.

    (d) Show that $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.

Please write to psraj@iitk.ac.in if any typos/mistakes are found in this set of practice problems/solutions/hints.

Practice Problems 3: Hints/Solutions

1. Note that $x_{n+1}-x_n = \frac{1}{2n+1}+\frac{1}{2n+2}-\frac{1}{n+1} \geq \frac{2}{2n+2}-\frac{1}{n+1} = 0$ and $0 < x_n \leq \frac{1}{n}+\frac{1}{n}+\cdots+\frac{1}{n} = 1$.

2. (a) True. Find $M \in \mathbb{N}$ such that $0 \leq |x_n y_n| < M|x_n|$. Allow $n \to \infty$.

   (b) False. Take $x_n = n$ and $y_n = \frac{1}{n}$.

   (c) True. Let $M > 0$. Since (x_n) is not bounded (and increasing), there exists $N \in \mathbb{N}$ such that $x_N > M$. As (x_n) is increasing, $x_n \geq x_N$ for all $n \geq N$. Therefore $x_n > M$ for all $n \geq N$.

3. (a) Observe that $x_2 < x_1$. If $x_n < x_{n-1}$, then $x_{n+1} = 2 - \frac{1}{x_n} < 2 - \frac{1}{x_{n-1}} = x_n$. By induction the sequence is decreasing. Note that $x_n > 0$. The sequence converges and the limit is 1.

   (b) Observe that $x_2 > x_1$. Since $x_{n+1}^2 - x_n^2 = 2(x_n - x_{n-1})$, by induction (x_n) is increasing. It can be observed again by induction that $x_n \leq 2$. The limit is 2.

   (c) Note that $x_2 > x_1$. Since $x_{n+1} - x_n = \frac{x_n - x_{n-1}}{(3+2x_n)(3+2x_{n-1})}$, by induction (x_n) is increasing. Note that $x_{n+1} = 1 + \frac{1+x_n}{3+2x_n} \leq 2$. The limit is $\sqrt{2}$.

4. By the AM-GM inequality $b_n \leq a_n$. Therefore $0 \leq a_{n+1} \leq \frac{a_n + a_n}{2} = a_n$. Note that $b_{n+1} \geq \sqrt{b_n b_n} = b_n$ and $b_n \leq a_n \leq a_1$. Both (a_n) and (b_n) are bounded.

5. Note that $x_n > 0$ and $x_{n+1} - x_n = \frac{1}{2}(x_n + \frac{a}{x_n}) - x_n = \frac{1}{2}(\frac{a - x_n^2}{x_n})$. Further, by the A.M -G.M. inequality, $x_{n+1} \geq \sqrt{a}$. Therefore (x_n) is decreasing and bounded below.

6. By the AM-GM inequality $\frac{x_n + (1-x_{n+1})}{2} \geq \sqrt{x_n(1-x_{n+1})} > \frac{1}{2}$. Therefore $x_n > x_{n+1}$. Suppose $x_n \to x_0$ for some x_0. Then $4x_0(1-x_0) \geq 1$ which implies that $(2x_0 - 1)^2 \leq 0$. Therefore $x_0 = \frac{1}{2}$.

7. Here $x_{2n} = -\frac{1}{2}$ and $x_{2n+1} = \frac{n+1}{2n+1} \to \frac{1}{2}$. The sequence does not converge.

8. By Problem 11 of PP2, there exists $\epsilon_0 > 0$ such that for every $N \in \mathbb{N}$, there exists n such that $n > N$ and $|x_n - x_0| \geq \epsilon_0$. First take $N_1 = 1$ and choose $n_1 > N_1$ such that $|x_{n_1} - x_0| \geq \epsilon_0$. Then take some $N_2 > n_1$ and choose $n_2 > N_2$ such that $|x_{n_2} - x_0| \geq \epsilon_0$. Note that $n_2 > n_1$. We have found x_{n_1} and x_{n_2} where $n_2 > n_1$. Proceed.

9. Suppose that (x_n) does not converge to x_0. Use Problem 8 to arrive at a contradiction.

10. Let $y_n = (x_1 x_2 \ldots x_n)^{\frac{1}{n}}$. Then $y_{2n-1} = (3^{n-1})^{\frac{1}{2n-1}}$ for $n \geq 1$ and $y_{2n} = (3^n)^{\frac{1}{2n}}$ for $n \geq 1$. Since $y_{2n} \to \sqrt{3}$ and $y_{2n-1} \to \sqrt{3}$, $y_n \to \sqrt{3}$.

11. Suppose that (x_n) does not converge to x_0. Apply Problem 8 to get a contradiction.

12. Since $[a_n, b_n] \supseteq [a_{n+1}, b_{n+1}]$ for all n, if we let $A = \{a_n : n \in \mathbb{N}\}$, then every b_n is an upper bound for A. Let $x = \sup A$. Then $a_n \leq x \leq b_n$ for all $n \in \mathbb{N}$. For showing that $\bigcap_{n=1}^{\infty} [a_n, b_n]$ is a singleton, see the last part of the proof of Theorem 3.1.

13. (a) By the binomial theorem

$$\begin{aligned}
x_n &= 1 + n \cdot \frac{1}{n} + \frac{n(n-1)}{1 \cdot 2} \cdot \frac{1}{n^2} + \cdots + \frac{n(n-1)\cdots 1}{1 \cdots 2 \cdots n} \cdot \frac{1}{n^n} \\
&= 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n}) + \frac{1}{3!}(1 - \frac{1}{n})(1 - \frac{2}{n}) + \cdots + \frac{1}{n!}(1 - \frac{1}{n})\cdots(1 - \frac{n-1}{n}) \qquad (3.1) \\
&< 1 + 1 + \frac{1}{2!}(1 - \frac{1}{n+1}) + \cdots + \frac{1}{(n+1)!}(1 - \frac{1}{n+1})(1 - \frac{2}{n+1})\cdots(1 - \frac{n}{n+1}) \\
&= x_{n+1}
\end{aligned}$$

(b) Note that $x_n \le 1+1+\frac{1}{2!}+\frac{1}{3!}+\ldots+\frac{1}{n!} = y_n \le 1+1+\frac{1}{2}+\frac{1}{2^2}+\ldots+\frac{1}{2^{n-1}} \le 3$. Therefore, $2 \le x_n \le y_n \le 3$ for all $n \in \mathbb{N}$.

(c) Let $n > m$. It follows from equation (3.1) that

$$x_n > 1+1+\frac{1}{2!}(1-\frac{1}{n})+\frac{1}{3!}(1-\frac{1}{n})(1-\frac{2}{n})+\cdots+\frac{1}{m!}(1-\frac{1}{n})\cdots(1-\frac{m-1}{n}). \qquad (3.2)$$

(d) Fixing m in inequality (3.2) and allowing $n \to \infty$, we get that $\lim_{n\to\infty} x_n \ge y_m$. Allowing $m \to \infty$, we get $\lim_{n\to\infty} x_n \ge \lim_{n\to\infty} y_n$. Since $x_n \le y_n$ for all $n \in \mathbb{N}$, we have $\lim_{n\to\infty} x_n = \lim_{n\to\infty} y_n$.