Practice Problems 4: Cauchy criterion, Bolzano-Weierstrass Theorem

1. Show that (x_n) satisfies the Cauchy criterion where (x_n) is defined as

   (a) $x_1 = 2$ and $x_{n+1} = 2 + \frac{1}{x_n}$ for all $n \in \mathbb{N}$;

   (b) $x_1 = 1$ and $x_{n+1} = \frac{1}{2+x_n^2}$ for all $n \in \mathbb{N}$;

   (c) $x_1 = 1$ and $x_{n+1} = \frac{1}{6}(x_n^2 + 8)$ for all $n \in \mathbb{N}$.

2. Let (x_n) satisfy the Cauchy criterion. Show that (x_n) is bounded.

3. Let (x_n) be a sequence of positive real numbers. Prove or disprove the following statements.

   (a) If $x_{n+1} - x_n \to 0$ then (x_n) converges.

   (b) If $|x_{n+2} - x_{n+1}| < |x_{n+1} - x_n|$ for all $n \in \mathbb{N}$ then (x_n) converges.

   (c) If (x_n) satisfies the Cauchy criterion, then there exists an $\alpha \in \mathbb{R}$ such that $0 < \alpha < 1$ and $|x_{n+1} - x_n| \leq \alpha |x_n - x_{n-1}|$ for all $n \in \mathbb{N}$.

4. Let (x_n) be a sequence of integers such that $x_{n+1} \neq x_n$ for all $n \in \mathbb{N}$. Prove or disprove the following statements.

   (a) The sequence (x_n) does not satisfy the Cauchy criterion.

   (b) The sequence (x_n) cannot have a convergent subsequence.

5. Suppose that $0 < \alpha < 1$ and that (x_n) is a sequence satisfying the condition: $|x_{n+1} - x_n| \leq \alpha^n, \quad n = 1, 2, 3, \ldots$. Show that (x_n) satisfies the Cauchy criterion.

6. Let (x_n) be defined by $x_1 = \frac{1}{1!}, x_2 = \frac{1}{1!} - \frac{1}{2!}, \ldots,\ x_n = \frac{1}{1!} - \frac{1}{2!} + \ldots + \frac{(-1)^{n+1}}{n!}$ for $n \in \mathbb{N}$. Show that (x_n) converges.

7. Let $1 \leq x_1 \leq x_2 \leq 2$ and $x_{n+2} = \sqrt{x_{n+1} x_n}$, for $n \in \mathbb{N}$.

   (a) Show that $\frac{x_{n+1}}{x_n} \geq \frac{1}{2}$, $|x_{n+1} - x_n| \leq \frac{2}{3}|x_n - x_{n-1}|$ for all $n \in \mathbb{N}$ and (x_n) converges.

   (b) Observe that $x_{n+2}^2 x_{n+1} = x_{n+1}^2 x_n$ for all $n \in \mathbb{N}$ and find the limit of (x_n).

8. Let $x_1 = 1, x_2 = 2$ and $x_{n+2} = \frac{x_{n+1} + x_n}{2}$ for all $n \in \mathbb{N}$. Using the nested interval theorem, show that (x_n) converges .

9. **(*)** Show that a sequence (x_n) has no convergent subsequence if and only if $|x_n| \to \infty$.

10. **(*)** Show that a sequence (x_n) is bounded if and only if every subsequence of (x_n) has a convergent subsequence.

11. **(*)** Let (x_n) be a sequence in $\mathbb{R}$. We say that a positive integer n is a peak of (x_n) if $x_n > x_m$ whenever $m > n$ (i.e., if x_n is greater than every subsequent term of (x_n)).

    (a) If (x_n) has infinitely many peaks, show that it has a decreasing subsequence.

    (b) If (x_n) has only finitely many peaks, show that it has an increasing subsequence.

    (c) From (a) and (b) conclude that every sequence in $\mathbb{R}$ has a monotone subsequence. Further, conclude that every bounded sequence in $\mathbb{R}$ has a convergent subsequence (*This is an alternate proof of the Bolzano-Weierstrass Theorem*).

Please write to psraj@iitk.ac.in if any typos/mistakes are found in this set of practice problems/solutions/hints.

Practice Problems 4: Hints/Solutions

1. (a) Note that $|x_{n+1} - x_n| = |\frac{1}{x_n} - \frac{1}{x_{n-1}}| = |\frac{x_{n-1}-x_n}{x_n x_{n-1}}| \le \frac{1}{4}|x_{n-1} - x_n|$. Hence (x_n) satisfies the contractive condition and therefore it satisfies the Cauchy criterion.

   (b) Observe that $|x_{n+1} - x_n| = \frac{|x_n^2 - x_{n-1}^2|}{(2+x_n^2)(2+x_{n-1}^2)} \le \frac{|x_n - x_{n-1}||x_n + x_{n-1}|}{4} \le \frac{2}{4}|x_n - x_{n-1}|$.

   (c) We have $|x_{n+1} - x_n| \le \frac{|x_n - x_{n-1}||x_n + x_{n-1}|}{6} \le \frac{4}{6}|x_n - x_{n-1}|$.

2. Since (x_n) satisfies the Cauchy criterion, there exists $N \in \mathbb{N}$ such that $|x_n - x_N| < 1$ for all $n \ge N$. Hence $|x_n| \le \max\{|x_1|, |x_2|, \cdots, |x_{N-1}|, 1 + |x_N|\}$ for all $n \in \mathbb{N}$.

3. (a) False. Choose $x_n = \sqrt{n}$ and observe that $x_{n+1} - x_n = \frac{1}{\sqrt{n+1}+\sqrt{n}} \to 0$.

   (b) False. For $x_n = \sqrt{n}$, $|x_{n+2} - x_{n+1}| = |\sqrt{n+2} - \sqrt{n+1}| < \frac{1}{\sqrt{n+1}+\sqrt{n}} = |x_{n+1} - x_n|$.

   (c) False. Take $x_n = \frac{1}{n}$. If for some $\alpha > 0$, $|\frac{1}{n+1} - \frac{1}{n}| \le \alpha|\frac{1}{n} - \frac{1}{n-1}|$ for all $n \in \mathbb{N}$, then $\frac{n-1}{n+1} \le \alpha$. Allow $n \to \infty$ to get $\alpha \ge 1$.

4. (a) True. Because $|x_{n+1} - x_n| \nrightarrow 0$ as $n \to \infty$.

   (b) False. Consider $x_n = (-1)^n$.

5. For $n > m$, we have $|x_n - x_m| \le |x_n - x_{n-1}| + |x_{n-1} - x_{n-2}| + \cdots + |x_{m+1} - x_m|$
   $\le \alpha^{n-1} + \alpha^{n-2} + \cdots + \alpha^m = \alpha^m[1 + \alpha + \cdots + \alpha^{n-1-m}] \le \frac{\alpha^m}{1-\alpha} \to 0$ as $m \to \infty$.
   Thus (x_n) satisfies the Cauchy criterion.

6. Observe that $|x_{n+1} - x_n| \le \frac{1}{(n+1)!} \le (\frac{1}{2})^n$. Apply Problem 5.

7. Since $1 \le x_n \le 2$, $\frac{x_{n+1}}{x_n} \ge \frac{1}{2}$. Observe that $x_{n+1}^2 - x_n^2 = x_n x_{n-1} - x_n^2 = x_n(x_{n-1} - x_n)$. Thus $|x_{n+1} - x_n| = |\frac{x_n}{x_{n+1}+x_n}||x_{n-1} - x_n| \le \frac{2}{3}|x_n - x_{n-1}|$.

8. Define $[a_1, b_1] = [x_1, x_2]$, $[a_2, b_2] = [x_3, x_2]$, $[a_3, b_3] = [x_3, x_4]$, $[a_4, b_4] = [x_5, x_4], \cdots$ and apply the nested interval theorem.

9. Suppose $|x_n| \to \infty$. If (x_{n_k}) is a subsequence of (x_n), then observe that $|x_{n_k}| \to \infty$ as $k \to \infty$. To prove the converse, let $|x_n| \nrightarrow \infty$. Then there exists $M > 0$ such that for every $N \in \mathbb{N}$, we find $n > N$ such that $|x_n| < M$. Hence there exists $n_1 > 1$ such that $|x_{n_1}| < M$. Similarly, there exists $n_2 > n_1$ such that $|x_{n_2}| < M$. This way, we find a bounded subsequence (x_{n_k}) of (x_n). Hence by Bolzano-Weierstrass theorem, (x_{n_k}) has a convergent subsequence and therefore (x_n) has a convergent subsequence.

10. Suppose (x_n) is bounded. Then by the Bolzano-Weirestrass theorem, every subsequence of (x_n) has a convergent subsequence. To prove the converse, suppose (x_n) is not bounded then there exists a subsequence (x_{n_k}) of (x_n) such that $|x_{n_k}| \to \infty$. Observe that (x_{n_k}) cannot have a convergent subsequence.

11. (a) Suppose (x_n) has infinitely many peaks. Let n_1 be the first peak and n_2 be the second and so on. Thus all the peaks can be listed as $n_1 < n_2 < n_3 < ....$ Note that the subsequence (x_{n_k}) is decreasing.

    (b) Suppose there are only finite peaks and let N be the last peak. Since $n_1 = N + 1$ is not a peak, there exists $n_2 > n_1$ such that $x_{n_2} \ge x_{n_1}$. Since $n_2 > N$, n_2 is not a peak and hence there exists $n_3 > n_2$ such that $x_{n_3} \ge x_{n_2}$. This way, we find an increasing subsequence (x_{n_k}).

    (c) This follows immediately from (a) and (b).