Practice Problems 6: Limit, Intermediate Value Theorem

1. Let $\alpha \in \mathbb{R}$ be such that $\lim_{x\to -1} \frac{2x^2-\alpha x-14}{x^2-2x-3}$ exists. Find α and the limit.

2. Let $\lim_{x\to 0} \frac{f(x)}{x^2} = 5$. Show that $\lim_{x\to 0} \frac{f(x)}{x} = 0$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ and $x_0 \in \mathbb{R}$. Suppose $\lim\limits_{x\to x_0} f(x)$ exists. Show that $\lim\limits_{x\to 0} f(x+x_0) = \lim\limits_{x\to x_0} f(x)$.

4. Let $x_0 \in I$ and $f : I\backslash\{x_0\} \to \mathbb{R}$. Suppose $\lim\limits_{x\to x_0} f(x) = L$. If $L > 0$ show that there is $\delta > 0$ such that $f(x) > 0$ for all $x \in [(x_0-\delta, x_0+\delta) \cap I]\backslash\{x_0\}$.

5. Let $x_0 \in I$ and $f, g : I\backslash\{x_0\} \to \mathbb{R}$. Suppose $\lim\limits_{x\to x_0} f(x) = L$ and $\lim\limits_{x\to x_0} g(x) = M$. Then
   (i) $\lim\limits_{x\to x_0} (f+g)(x) = L+M$;
   (ii) $\lim\limits_{x\to x_0} (fg)(x) = LM$;
   (iii) if $L \neq 0$, $\lim\limits_{x\to x_0} (\frac{1}{f})(x) = \frac{1}{L}$.

6. Let $x_0 \in (a,b)$ and $f : (a,b)\backslash\{x_0\} \to \mathbb{R}$. Then $\lim\limits_{x\to x_0} f(x)$ exists if and only if $\lim\limits_{x\to x_0^+} f(x)$ and $\lim\limits_{x\to x_0^-} f(x)$ exist and are equal.

7. Give an example of a function f on $[0,1]$ which is not continuous but it satisfies the intermediate value property (in short, IVP) (We say that f has the property IVP on $[0,1]$ if for every $x, y \in [0,1]$ and α satisfying $f(x) < \alpha < f(y)$ or $f(x) > \alpha > f(y)$ there exists $x_0 \in [x,y]$ such that $f(x_0) = \alpha$).

8. Show that the polynomial $x^4 + 6x^3 - 8$ has at least two real roots.

9. Let $f : [0,1] \to \mathbb{R}$ be continuous. Show that there exists $x_0 \in [0,1]$ such that $f(x_0) = \frac{1}{3}(f(\frac{1}{4}) + f(\frac{1}{2}) + f(\frac{3}{4}))$.

10. Let $f : [0,1] \to R$. Suppose that $f(x)$ is rational for irrational x and that $f(x)$ is irrational for rational x. Show that f cannot be continuous.

11. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function such that $f(x+2\pi) = f(x)$ for all $x \in \mathbb{R}$. Show that there exists $x_0 \in \mathbb{R}$ such that $f(x_0+\pi) = f(x_0)$.

12. Let $f : [0,1] \to \mathbb{R}$ be continuous such that $f(0) = f(1)$. Show that there exists $x_0 \in [0, \frac{1}{2}]$ such that $f(x_0) = f(x_0 + \frac{1}{2})$.

13. Let $f, g : [0,1] \to \mathbb{R}$ be continuous such that $\inf\{f(x) : x \in [0,1]\} = \inf\{g(x) : x \in [0,1]\}$. Show that there exists $x_0 \in [0,1]$ such that $f(x_0) = g(x_0)$.

14. Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Show that f is a constant function if
    (a) $f(x)$ is rational for each $x \in \mathbb{R}$.
    (b) $f(x)$ is an integer for each $x \in \mathbb{Q}$.

15. Show that a polynomial of odd degree with real coefficients has at least one real root.

16. Show that there exists at least one positive real solution to the equation $|x^{31}+x^8+20| = x^{32}$.

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17. Let $f(x) = x^{2n} + a_{2n-1}x^{2n-1} + ... + a_1x + a_0$ where $n \in \mathbb{N}$ and $a_i \in \mathbb{R}$ for $0 \leq i \leq 2n$. Show that f attains its infimum on $\mathbb{R}$.

18. Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + ... + a_1x + a_0$ where $n \in \mathbb{N}$ and $a_i \in \mathbb{R}$ for $0 \leq i \leq n$. If n is even, $a_n = 1$ and $a_0 = -1$, show that $f(x)$ has at least two real roots.

19. A runner runs continuously a eight kilometer race in 40 minutes without taking rest. Show that, somewhere along the race, the runner must have covered a distance of one kilometer in exactly 5 minutes.

20. (*) Let $f : I \to \mathbb{R}$ be a continuous one-one map. Show that f is either strictly increasing (i.e, $f(x) > f(y)$ whenever $x > y$) or strictly decreasing.

21. (*) Let $f : \mathbb{R} \to [0, \infty)$ be a bijective map. Show that f is not continuous on $\mathbb{R}$.

22. (*) Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function.

    (a) Suppose f attains each of its values exactly two times. Let $f(x_1) = f(x_2) = \alpha$ for some $\alpha \in \mathbb{R}$ and $f(x) > \alpha$ for some $x \in [x_1, x_2]$. Show that f attains its maximum in $[x_1, x_2]$ exactly at one point.

    (b) Using (a) show that f cannot attain each of its values exactly two times.

Practice Problems 6: Hints/Solutions

1. $\alpha = 12$ and the limit is 4.

2. Note that $\frac{f(x)}{x} = \frac{f(x)}{x^2}x$ for $x \neq 0$.

3. Let $\lim_{x\to x_0} f(x) = M$ for some $M \in \mathbb{R}$. Let $x_n \to 0, x_n \neq 0 \;\forall\; n$. Then $x_n + x_0 \to x_0$. Since $\lim_{x\to x_0} f(x) = M$, $f(x_n + x_0) \to M$. This implies that $\lim_{x\to 0} f(x + x_0) = M$.

4. Suppose for every n, there exists $x_n \in (x_0 - \frac{1}{n}, x_0 + \frac{1}{n}) \cap I$ such that $f(x_n) \leq 0$. Then $x_n \to x_0$. Since $f(x_n) \to L, L \leq 0$ which is a contradiction.

5. Apply the definition of limit and Theorem 2.1 to get (i) and (ii). Theorem 2.1 and Problem 4 imply (iii).

6. It follows from the definitions that if $\lim\limits_{x\to x_0} f(x)$ exists then $\lim\limits_{x\to x_0^+} f(x)$ and $\lim\limits_{x\to x_0^-} f(x)$ exist and are equal. To show the converse, let $\lim\limits_{x\to x_0^+} f(x) = \lim\limits_{x\to x_0^-} f(x) = L$ for some $L \in \mathbb{R}$. Let $\epsilon > 0$. Then by Theorem 6.3, there exists $\delta_1 > 0$ and $\delta_2 > 0$ such that

$$|f(x) - L| < \epsilon \text{ whenever } x \in I,\; x > x_0 \text{ and } 0 < |x - x_0| < \delta$$

and
$$|f(x) - L| < \epsilon \text{ whenever } x \in I,\; x > x_0 \text{ and } 0 < |x - x_0| < \delta.$$

Choose $\delta = \min\{\delta_1, \delta_2\}$. Then $|f(x) - L| < \epsilon$ whenever $x \in I,$ and $0 < |x - x_0| < \delta$. This shows that $\lim\limits_{x\to x_0} f(x) = L$.

7. Consider $f(0) = 0$ and $f(x) = \sin\frac{1}{x}$ for $x \neq 0$.

8. Note that $f(0) < 0, f(2) > 0$ and $f(-8) > 0$. Use the IVT for f on $[-8, 0]$ and f on $[0, 2]$.

9. Let $x_1, x_2 \in [0, 1]$ be such that $f(x_1) = inf\{f(x) : x \in [0, 1]\}$ and $f(x_2) = sup\{f(x) : x \in [0, 1]\}$. Note that $f(x_1) \leq \frac{1}{3}(f(\frac{1}{4}) + f(\frac{1}{2}) + f(\frac{3}{4})) \leq f(x_2)$. Apply the IVT.

10. Let g be defined by $g(x) = f(x) - x \;\forall\; x \in [0, 1]$. Then $g(x)$ irrational for all $x \in [0, 1]$. Because of the IVT, g cannot be continuous and hence f cannot be continuous.

11. Consider the function $g(x) = f(x + \pi) - f(x)$ and the values $g(0)$ and $g(\pi)$. Apply the IVT.

12. Consider the function $g(x) = f(x) - f(x + \frac{1}{2})$ and the values $g(0)$ and $g(\frac{1}{2})$. Apply the IVT.

13. Let $x_1, x_2 \in [0, 1]$ be such that $f(x_1) = \inf\{f(x) : x \in [0, 1]\}$ and $g(x_2) = \inf\{g(x) : x \in [0, 1]\}$. Note that $f(x_1) \leq g(x_1)$ and $f(x_2) \geq g(x_2)$. Let $\varphi(x) = f(x) - g(x)$. Apply the IVT for φ.

14. (a) Suppose $f(x) \neq f(y)$ for some $x, y \in \mathbb{R}$. Find an irrational number α between $f(x)$ and $f(y)$. By the IVT, there exists $z \in (x, y)$ such that $f(z) = \alpha$ which is a contradiction.

    (b) Let α be irrational. Find $r_n \in \mathbb{Q}$ such that $r_n \to \alpha$. By the continuity of f, $f(r_n) \to f(\alpha)$. Since each $f(r_n)$ is an integer, $(f(r_n))$ has to be eventually a constant sequence and hence $f(\alpha)$ is an integer. So f takes only integer value for each $x \in \mathbb{R}$. By the IVT, $f(x)$ has to be a constant function.

15. Let $p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ where $a_i \in \mathbb{R}$ for $0 \leq i \leq n$, $a_n \neq 0$ and n is odd. Then $p(x) = x^n(a_n + \frac{a_{n-1}}{x} + \cdots + \frac{a_1}{x^{n-1}} + \frac{a_0}{x^n})$. If $a_n > 0$, then $p(x) \to \infty$ as $x \to \infty$ and $p(x) \to -\infty$ as $x \to -\infty$. Hence there exists some $M > 0$ such that $p(x) > 0$ for all $x > M$ and $p(x) < 0$ for all $x < -M$. Apply the IVT for p on $[-M, M]$.

16. Let $f(x) = \frac{1}{x^{32}}|x^{31} + x^8 + 20| - 1$. Then $f(x) \to \infty$ as $x \to 0$ and $f(x) \to -1$ as $x \to \infty$. By the IVT, there exists $x_0 \in (0, \infty)$ such that $f(x_0) = 0$.

17. Note that $f(x) \to \infty$ as $x \to \infty$ or $x \to -\infty$. Let $\alpha > 0$ be such that $\alpha > f(y)$ for some $y \in \mathbb{R}$. Then there exists $M > 0$ such that $f(x) > \alpha$ for all $|x| > M$. Since f is continuous there exists x_0 such that $f(x_0) = inf\{f(x) : x \in [-M, M]\} = inf\{f(x) : x \in \mathbb{R}\}$

18. Note that $f(0) = -1$ and $f(x) \to \infty$ as $x \to \infty$ or $x \to -\infty$. Apply the IVT.

19. Let x denote the distance, in kilometers, along the course. Let $f : [0, 7] \to \mathbb{R}$, where $f(x) =$ time taken in minutes to cover the distance from x to $x + 1$. Observe that $\sum_{i=0}^{7} f(i) = 40$. Hence $f(i) < 5$ or $f(i) > 5$ is not possible for all $i = 0$ to 7. Therefore, there exists $i, j \in [0, 7]$ such that $f(i) \leq 5 \leq f(j)$. By the IVT there exists $c \in (i, j)$ such that $f(c) = 5$.

20. Case 1: Let $I = [a, b]$. Assume that $f(a) < f(b)$. Let $a < x < b$. Since f is one-one, using the IVT, it is easy to show that $f(a) < f(x) < f(b)$. Let $a < x < y < b$. Then $f(x) < f(y)$. To see this, let $f(y) < f(x)$. Then $f(y) < f(x) < f(b)$. By the IVT, there exists $x_0 \in (y, b)$ such that $f(x_0) = f(x)$ which is a contradiction.

    Case 2: Suppose that I is any interval and f is neither strictly increasing nor strictly decreasing. Then there exist $x_1, x_2, y_1, y_2 \in I$ such that $x_1 < x_2$ but $f(x_1) \leq f(x_2)$ and $y_1 < y_2$ but $f(y_1) \geq f(y_2)$. Find $[a, b]$ such that $x_1, x_2, y_1, y_2 \in [a, b]$ and $[a, b] \subset I$. Case I will lead to a contradiction.

21. If f is continuous, by Problem 21, f is either strictly increasing or strictly decreasing. Suppose f is strictly increasing. Since f is on-to, there exists x_0 such that $f(x_0) = 0$. Then $f(x) < f(x_0)$ for all $x < x_0$ which is a contradiction.

22. (a) Let $\beta = max\{f(x) : x \in [x_1, x_2]\}$. If f attains β on $[x_1, x_2]$ at more than one point, then there exists $\gamma \in (\alpha, \beta)$ such that f attains γ more than twice which is a contradiction.

    (b) Suppose f attains each of its values exactly two times. Let x_1, x_2, α and β be as in (a). Since f attains β exactly once in $[x_1, x_2]$, there exits x_0 lying outside $[x_1, x_2]$ such that $f(x_0) = \beta > \alpha$. Then, by the IVT, every number in (α, β) is attained by f more than twice which is a contradiction.