Practice Problems 7: Differentiability, Rolle's theorem

1. Which of the following functions are differentiable at $x = 0$?

   (a) $f(x) = x^{\frac{1}{3}}, x \in \mathbb{R}$.

   (b) $f(x) = x^2$ for any rational x and $f(x) = 0$ for any irrational x.

   (c) $f(x) = x \sin x \cos \frac{1}{x}$ for $x \neq 0$ and $f(0) = 0$.

2. Give an example of a function $f : \mathbb{R} \to \mathbb{R}$ which is differentiable only at $x = 1$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x_0 \in \mathbb{R}$.

   (a) If $f(x_0) \neq 0$, show that $|f|$ is also differentiable at x_0.

   (b) If $f(x_0) = 0$, give examples to show that $|f|$ may or may not be differentiable at x_0.

4. Let $f : I \to \mathbb{R}$ be differentiable where I is an interval. If f is increasing on I then show that $f'(x) \geq 0$ for all $x \in I$. If f is strictly increasing on I, is it necessary that $f'(x) > 0$ for all $x \in I$?

5. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable at $x_0 \in \mathbb{R}$. Define $h(x) = \max\{f(x), g(x)\}$ for all $x \in \mathbb{R}$. Show that h is differentiable at x_0 if $f(x_0) \neq g(x_0)$.

6. If $f : \mathbb{R} \to \mathbb{R}$ is differentiable at $c \in \mathbb{R}$, show that $f'(c) = \lim\limits_{n\to\infty} (n\{f(c + 1/n) - f(c)\})$. Does the existence of the limit of this sequence imply the existence of $f'(c)$?

7. Let $f : \mathbb{R} \to \mathbb{R}$ be differentiable at $x = 1$, $f(1) = 1$ and $k \in \mathbb{N}$. Show that
   $$\lim_{n\to\infty} n \left(f(1 + \tfrac{1}{n}) + f(1 + \tfrac{2}{n}) + ... + f(1 + \tfrac{k}{n}) - k \right) = \tfrac{k(k+1)}{2} f'(1).$$

8. Let $f(0) = 0$ and $f'(0) = 1$. For a positive integer k, show that
   $$\lim_{x\to 0} \frac{1}{x} \left\{ f(x) + f(\frac{x}{2}) + f(\frac{x}{3}) + ... + f(\frac{x}{k}) \right\} = 1 + \frac{1}{2} + ... + \frac{1}{k}.$$

9. Let $f : [0, 1] \to \mathbb{R}$ be differentiable and $f(0) = 0$ and $f(1) = 1$. Show that the equation $f'(x) = 2x$ has a solution on $(0, 1)$.

10. Find the number of distinct real solutions of the following equations.

    (a) $2x - \cos^2 x + \sqrt{7} = 0$

    (b) $x^{17} - e^{-x} + 5x + \cos x = 0$

    (c) $x^2 - \cos x = 0$.

11. Let $f : [a, b] \to \mathbb{R}$ be such that $f'''(x)$ exists for all $x \in [a, b]$. Suppose that $f(a) = f(b) = f'(a) = f'(b) = 0$. Show that the equation $f'''(x) = 0$ has a solution.

12. Let $a_1, a_2, ..., a_n$ be real numbers such that $a_1 + a_2 + ... + a_n = 0$. Show that the polynomial $q(x) = a_1 + 2a_2 x + 3a_3 x^2 + ... + n a_n x^{n-1}$ has at least one real root.

13. Let $f, g : \mathbb{R} \to \mathbb{R}$ be differentiable functions. Suppose that $f'(x)g(x) \neq f(x)g'(x)$ for any $x \in \mathbb{R}$. Show that between any two real solutions of $f(x) = 0$, there is at least one real solution of $g(x) = 0$.

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14. Let f and g be continuous on $[a, b]$ and differentiable on (a, b). Suppose $f(a) = f(b) = 0$. Show that there is a point $c \in (a,\ b)$ such that $g'(c)f(c) + f'(c) = 0$.

15. Let $f : [0, 1] \to \mathbb{R}$ be a differentiable function satisfying $f(0) = 0$ and $f(x) > 0$ for all $x \in (0, 1]$. Show that there exists $c \in (0, 1)$ such that $\frac{f'(1-c)}{f(1-c)} = \frac{2f'(c)}{f(c)}$.

16. **(*)** Let $P(x)$ be a polynomial of degree $n, n > 1$ and $P(x_0) = 0$ for some $x_0 \in \mathbb{R}$.

    (a) Show that $P(x) = (x - x_0)Q(x)$ where $Q(x)$ is a polynomial of degree $n - 1$.

    (b) Show that $P'(x_0) = 0$ if and only if $P(x) = (x - x_0)^2 R(x)$ where $R(x)$ is a polynomial of degree $n - 2$.

    (c) Show that if all roots of $P(x)$ are real then all roots of $P'(x)$ are also real.

17. **(*)** Let $f : (0, \infty) \to \mathbb{R}$ satisfy $f(xy) = f(x) + f(y)$ for all $x, y \in (0, \infty)$. Suppose that f is differentiable at $x = 1$. Show that f is differentiable at every $x \in (0, \infty)$ and $f'(x) = \frac{1}{x} f'(1)$ for every $x \in (0, \infty)$.

18. **(*)(The IVP of the derivative)** Let $f : I \to \mathbb{R}$ be differentiable. Let $a, b \in I$, $a < b$ and $f'(a) < \lambda < f'(b)$.

    (a) Show that the function $g : [a, b] \to \mathbb{R}$ defined by $g(x) = f(x) - \lambda x$ has a point of minimum in $[a, b]$.

    (b) If c is a point of minimum of g, show that $c \neq a$, $c \neq b$ and $f'(c) = \lambda$.

    (c) Conclude that f' has the IVP (see Problem 7 of PP6 for the definition of IVP).

19. Show that the function $f : [-1, 1] \to \mathbb{R}$ defined by $f(x) = [x]$ is not the derivative of some function.

Practice Problems 7: Hints/Solutions

1. (a) Note that $\lim_{x\to 0} \frac{x^{\frac{1}{3}}-0}{x-0}$ does not exist.

   (b) Observe that $\left|\frac{f(x)-0}{x-0}\right| \le |x| \to 0$ as $x \to 0$. Therefore f is differentiable at $x = 0$.

   (c) Since $\left|\frac{x\sin x \ \cos\frac{1}{x}}{x}\right| \le |\sin x| \to 0$ as $x \to 0$, f is differentiable at $x = 0$.

2. Define $f(x) = (x-1)^2$ for any rational x and $f(x) = 0$ for any irrational x.

3. (a) If $f(x_0) > 0$, then $|f(x)| = f(x)$ in a neighborhood of x_0.

   (b) Take $x_0 = 0$ and consider the functions: (i) $f(x) = x$ (ii) $g(x) = x|x|$.

4. Let x_0 be any element in I. Choose (x_n) in I such that $x_n > x_0$ or $x_n < x_0$ for all n and $x_n \to x_0$. Since f is differentiable at x_0, $f'(x_0) = \lim\limits_{n\to\infty} \frac{f(x_n)-f(x_0)}{x_n-x_0} \ge 0$.

   The function $f(x) = x^3, x \in \mathbb{R}$, is strictly increasing but $f'(0) = 0$.

5. If $f(x_0) > g(x_0)$, i.e., $(f-g)(x_0) > 0$, then in a neighborhood of x_0, $h(x) = f(x)$.

6. Since f is differentiable at c, $f'(c) = \lim\limits_{n\to\infty} \frac{f(c+1/n)-f(c)}{1/n}$.

   Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) = x^2$ for any rational x and $f(x) = 1$ for any irrational x. Then the function is not even continuous at 0 and hence it is not differentiable at 0. However, the limit $\lim\limits_{n\to\infty}(n\{f(1/n) - f(0)\}) = \lim\limits_{n\to\infty} n\frac{1}{n^2}$ exists and is equal to 0.

7. The given limit is $\lim\limits_{n\to\infty}\left(\frac{f(1+\frac{1}{n})-f(1)}{\frac{1}{n}} + 2\frac{f(1+\frac{2}{n})-f(1)}{\frac{2}{n}} + ... + k\frac{f(1+\frac{k}{n})-f(1)}{\frac{k}{n}}\right)$.

8. The given limit is $\lim\limits_{x\to 0}\left(\frac{f(x)-f(0)}{x} + \frac{1}{2}\frac{f(\frac{x}{2})-f(0)}{\frac{x}{2}} + \cdots + \frac{1}{k}\frac{f(\frac{x}{k})-f(0)}{\frac{x}{k}}\right) = 1 + \frac{1}{2} + \cdots + \frac{1}{k}$.

9. Apply Rolle's Theorem for $g(x) = f(x) - x^2$ on $[0, 1]$.

10. (a) Let $f(x) = 2x - \cos^2 x + \sqrt{7}$. Since $f'(x)$ has no real root, by Rolle's theorem $f(x)$ has at most one real root. Now $f(0) > 0$ and $f(-2) < 0$. So by the IVT there exists a real solution for $f(x) = 0$. Therefore $f(x) = 0$ has exactly one real solution.

    (b) Let $f(x) = x^{17} - e^{-x} + 5x + \cos x$. Observe that $f'(x) > 0 \ \forall \ x \in \mathbb{R}$, $f(2) > 0$ and $f(-2) < 0$. By the IVT and Rolle's theorem $f(x) = 0$ has exactly one real solution.

    (c) Let $fx) = x^2 - \cos x$. Since $f''(x) > 0$ for all $x \in \mathbb{R}$, by Rolle's theorem, $f'(x)$ has at most one real root and hence $f(x)$ has at most two real roots. Note that $f(\frac{-\pi}{2}) > 0, f(0) < 0$ and $f(\frac{\pi}{2}) > 0$. By the IVT, $f(x)$ has at least two real roots. Therefore $f(x) = 0$ has exactly two real solutions.

11. By Rolle's theorem there exists $d \in (a, b)$ such that $f'(d) = 0$. Again, by applying Rolle's theorem for f'', there exists $c_1 \in (a, d)$ and $c_2 \in (d, b)$ such that $f''(c_1) = 0$ and $f''(c_2) = 0$. Apply Rolle's Theorem for f'' on $[c_1, c_2]$.

12. Let $p(x) = a_1x + a_2x^2 + ... + a_nx^n$. Then $p(0) = 0$ and $p(1) = 0$. By Rolle's theorem, $p'(x) = q(x)$ has a real root.

13. Let $f(a) = f(b) = 0$ and $a < b$. Since $f'(a)g(a) \ne f(a)g'(a)$, $g(a) \ne 0$. Similarly $g(b) \ne 0$. If $g(x) = 0$ has no real solution on (a, b) then $h(x) = \frac{f(x)}{g(x)}$ is well defined on $[a, b]$ and $h(a) = h(b) = 0$. By Rolle's theorem, there exists $c \in (a, b)$ such that $h'(c) = 0$. That is $f'(c)g(c) = f(c)g'(c)$ which is a contradiction.

14. Define $h(x) = f(x)e^{g(x)}$. Since $h(a) = h(b) = 0$, by Rolle's theorem, there exists $c \in (a, b)$ such that $h'(c) = 0$. This implies that $f'(c) + g'(c)f(c) = 0$.

15. Let $g(x) = (f(x))^2 f(1 - x)$. Then $g(0) = g(1) = 0$. Apply Rolle's theorem for g on $[0, 1]$.

16. (a) Use $P(x) = P(x) - P(x_0)$ and $x^k - x_0^k = (x - x_0)(x^{k-1} + x^{k-2}x_0 + ... + xx_0^{k-2} + x_0^{k-1})$.

    (b) Suppose $P(x) = (x - x_0)Q(x)$. Then $P'(x) = Q(x) + (x - x_0)Q'(x)$. If $P'(x_0) = 0$ then $Q(x_0) = 0$. Therefore $Q(x) = (x - x_0)R(x)$ for a polynomial $R(x)$ of degree $n - 2$.

    (c) First observe that if x_0 is a root of $P(x)$ of order k then it is a root of $P'(x)$ of order $k - 1$. Apply the fact that between any two distinct real roots of $P(x)$ there is one real root of $P'(x)$.

17. Observe that $f(1) = 0, f(\frac{1}{x}) = -f(x)$ and $f(\frac{x}{y}) = f(x) - f(y)$. Now
    $f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$
    $= \lim_{h \to 0} \frac{1}{h} f\left(\frac{x+h}{x}\right) = \lim_{k \to 0} \frac{f(1+k)}{kx} = \lim_{k \to 0} \frac{1}{x} \frac{f(1+k) - f(1)}{k} = \frac{1}{x} f'(1)$.

18. (a) By Theorem 5.3, g has a point of minimum in $[a, b]$.

    (b) If $c = a$, then choose $x_n \in (a, b)$ such that $x_n \to a$. Then $g(x_n) - g(a) \geq 0$ and hence $g'(a) = \lim\limits_{n \to \infty} \frac{g(x_n) - g(a)}{x_n - a} \geq 0$ which contradicts the assumption that $f'(a) < \lambda$. Similarly $c \neq b$. By Theorem 7.1, $g'(c) = 0$ and hence $f'(c) = \lambda$.

    (c) It follows from (b) that $f'(c) = \lambda$ for some $c \in (a, b)$. Hence f' has the IVP.

19. Since f does not satisfy the IVP, by Problem 18, f cannot be a derivative.