Assignment 1

1. Prove the following tensor identities:

   (a) $\delta_{ij}\delta_{ij} = 3$, (b) $\delta_{ik}\delta_{kj} = \delta_{ij}$, (c) $\sigma_{kk} = \sigma_{ij}\delta_{ij}$.

2. Show that $\boldsymbol{z} = \boldsymbol{x} \times \boldsymbol{y}$, where $\boldsymbol{x}, \boldsymbol{y}$ are vectors, can be expressed as $z_i = e_{ijk}x_j y_k$.

3. The Airy stress function $\phi(r, \theta)$ in polar coordinates is given by

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}\right)\left(\frac{\partial^2 \phi}{\partial r^2} + \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta^2}\right). \tag{4.50}$$

   In addition, the stresses are given by:

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \phi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \phi}{\partial \theta} \tag{4.51}$$

$$\sigma_{\theta\theta} = \frac{\partial^2 \phi}{\partial r^2} \tag{4.52}$$

$$\sigma_{r\theta} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \phi}{\partial \theta}\right). \tag{4.53}$$

   Now consider the 2-D wedge shown in Fig. 1. The wedge is made up of a perfectly elastic material and is loaded by $p = Pr^m$ (P and m are constants) on the edge $\theta = 0$. The other edge $\theta = \alpha$ is free of tractions. Find the stresses in the wedge using the Airy stress function,

$$\phi = r^{m+2}\left(a\cos(m+2)\theta + b\sin(m+2)\theta + c\cos m\theta + d\sin m\theta\right). \tag{4.54}$$

   Find the constants a, b, c and$d$. Discuss the *boundedness* (zero, finite or infinite) of the stresses. Find the derivatives $\partial v/\partial r$ in the neighbourhood of the tip. The symbol v denotes the component of displacement in the direction of incresing θ.

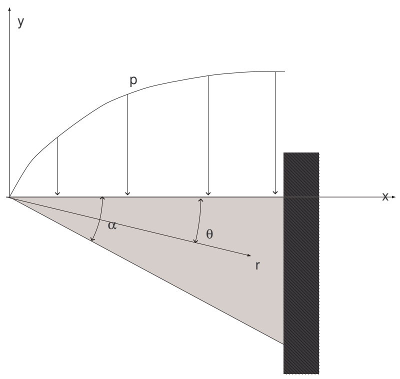

Figure 4.4: Figure for Assignment Problem 3

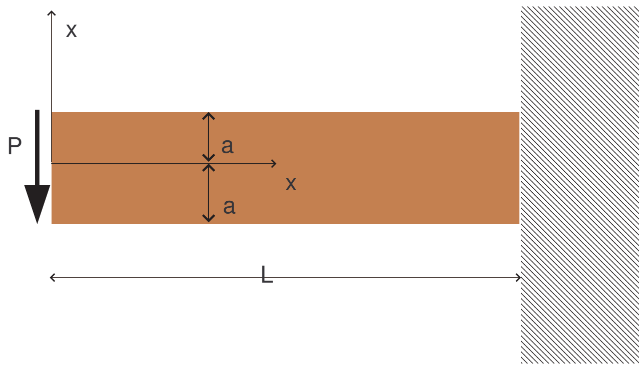

Figure 4.5: Figure for Assignment Problem 4

4. Consider the cantilever beam of depth $2a$ and out of plane thickness b. The co-ordinate axes are taken as shown. (a) List the boundary conditions on stress and displacement for this problem. (b) Assume an Airy stress function of the form

$$\phi = A_1x_1^4+A_2x_1^3x_2+A_3x_1^2x_2^2+A_4x_1x_2^3+A_5x_2^4+B_1x_1^3+B_2x_1^2x_2+B_3x_1x_2^2+B_4x_2^3+C_1x_1^2+C_2x_1x_2+C_3x_2^2. \tag{4.55}$$

and find the stresses and end displacements.

5. If ϕ is the Airy stress function satisfying $\nabla^4\phi = 0$, and $P = \nabla^2\phi$, then we can construct an analytic function $f(z) = P(x, y) + iQ(x, y)$, where P and Q are related through the Cauchy-Riemann conditions. Now, define $\psi(z) = \frac{1}{4}\int(P + iQ)dz =$

$R(x, y) + iI(x, y)$ and prove that,

$$\nabla^2 [\phi - (xR + yI)] = 0. \tag{4.56}$$

6. In the class, we started the derivation of the Kolosoff-Mushkelishvilli equations from the *Goursat theorem*, namely that

$$\phi(x, y) = Re [\bar{z}\psi(z) + \chi(z)], \tag{4.57}$$

where, ϕ is the Airy stress function and ψ and χ are two analytic (holomorphic) functions of the complex variable $z = x + iy$. Using the Goursat theorem, prove the first two K-M equations:

$$\begin{aligned} \sigma_{xx} + \sigma_{yy} &= 2\left[\psi'(z) + \bar{\psi}'(\bar{z})\right] && (4.58) \\ \sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} &= 2[\bar{z}\psi''(z) + \chi''(z)] && (4.59) \\ & && (4.60) \end{aligned}$$

7. Show that if we put $\psi'(z) = \frac{1}{2}Z_1(z)$ and $\chi''(z) = -\frac{1}{2}zZ_1'(z)$ in the K-M equations, then on the line $y = 0$, $\sigma_{xy} = 0$ and $\sigma_{xx} = \sigma_{yy}$. Find the form of the K-M equations in terms of the new function Z_1. Incidentally, these functions are known as *Westergaard* functions and form an useful tool for solving symmetric problems in elasticity.

8. Given $\psi(z) = Az$ and $\chi(z) = Bln(z)$, where A, B are real constants, find the stresses in the thick cylinder of inner radius a and outer radius b and pressures p_2 and p_1 on the outer and inner surfaces respectively.

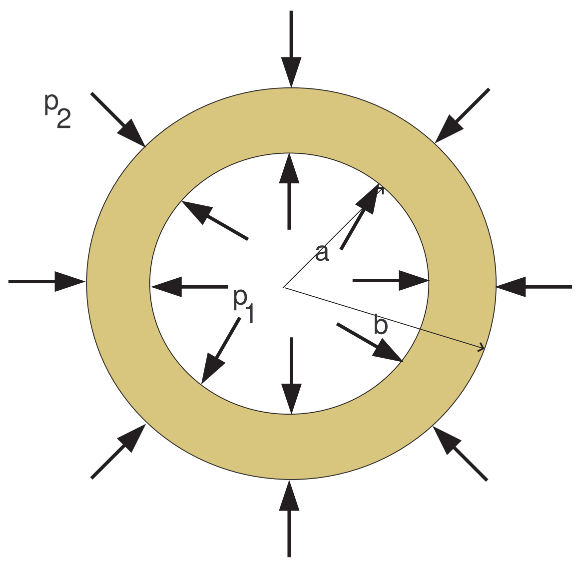

Figure 4.6: Figure for Assignment Problem 8