Appendix H

Assignment 5

H.1 Theoretical Background

This part has been explained in the class. I am adding this here for the sake of completeness.

We will now use the J_2 deformation theory of plasticity to derive the stress and strain fields at the tip of a sharp crack. We assume that the material obeys the Ramberg-Osgood stress strain curve with α and n being two material parameters. In addition, the following simplifying assumption will be made. Inside a region D around the crack tip the stresses are very high and it can be safely assumed that the elastic strains are much smaller than the plastic strains in this region. This allows us to neglect the elastic strains and assume that the total strain is purely plastic, i.e.

$$\epsilon_{ij}^{p} = \frac{3}{2}\alpha\epsilon_0 \left(\frac{\bar{\sigma}}{\sigma_0}\right)^{n-1} \frac{s_{ij}}{\sigma_0} \tag{H.1}$$

The derivation of the Mode I elasto-plastic crack tip fields are due to Rice and Rosengren (1968) and Hutchinson (1968) and are therefore called the HRR fields.

Consider the J integral on a circular domain of radius r around the crack tip. We have,

$$J = r \int_{-\pi}^{\pi} \left(W[\epsilon(r,\theta)] \cos\theta - T_i(r,\theta) \frac{\partial u_i(r,\theta)}{\partial x_1} \right) d\theta. \tag{H.2}$$

Since the deformation theory of plasticity can be viewed as a non-linear elasticity theory, the J integral is path independent. This in turn implies that the term inside the integral sign in eq. H.2 has an inverse r singularity. Moreover, all the terms inside the integral sign are of the form $\sigma_{ij}\epsilon_{ij}$ and so, we can write

$$\sigma_{ij}\epsilon_{ij} \rightarrow \frac{f(\theta)}{r}, \tag{H.3}$$

inside the region D. Now according to the Ramberg-Osgood stress-strain law, $\epsilon_{ij} \sim \sigma_{ij}^n$ and hence, $\sigma_{ij}\epsilon_{ij} \sim \sigma_{ij}^{n+1}$. Evidently then

$$\sigma_{ij} \sim r^{-1/n+1} \quad \text{as } r \rightarrow 0. \tag{H.4}$$

The power law relationship implies therefore that,

$$\begin{aligned} \sigma_{ij}(r,\theta) &= K r^{-1/n+1} \hat{\sigma}_{ij}(\theta) && \text{(H.5)} \\ \epsilon_{ij}(r,\theta) &= \alpha\epsilon_0 K^n r^{-n/n+1} \hat{\epsilon}_{ij}(\theta) && \text{(H.6)} \\ u_i(r,\theta) &= \alpha\epsilon_0 K^n r^{-1/n+1} \hat{u}_i(\theta). && \text{(H.7)} \end{aligned}$$

The quantities with a $\hat{}$ are unknown angular quantities while the r dependence is now known. Compare this with the elastic case where we had to solve an eigenvalue problem to get the order of the singularity.

The problem now boils down to finding the angular functions and the constant K. The constant gives the strength of the singularity. We will start by finding the angular functions first.

Since we know the order of the stress singularity to be of the order $-1/(n+1)$, let us assume that we have a stress function of the form

$$\Psi = K r^s \hat{\Psi}(\theta), \tag{H.8}$$

where, $s = (2n+1)/(n+1)$. The near tip stresses due to this function are given by,

$$\sigma_{\theta\theta} = \frac{\partial^2 \Psi}{\partial r^2} = Ks(s-1)r^{-1/(n+1)}\hat{\Psi}(\theta) \quad \text{(H.9)}$$

$$\sigma_{rr} = -\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial \Psi}{\partial \theta}\right) = K(1-s)r^{-1/(n+1)}\hat{\Psi}'(\theta) \quad \text{(H.10)}$$

$$\sigma_{rr} = \frac{1}{r}\frac{\partial \Psi}{\partial r} + \frac{1}{r^2}\frac{\partial^2 \Psi}{\partial \theta^2} = Kr^{-1/(n+1)}\left[s\hat{\Psi}(\theta) + \hat{\Psi}''(\theta)\right]. \quad \text{(H.11)}$$

We can now calculate the effective stress $\bar{\sigma} = \sqrt{(3/2)s_{ij}s_{ij}}$ taking into account the fact that in view of the incompressibility of plastic flow, $\sigma_{zz} = (1/2)(\sigma_{rr} + \sigma_{\theta\theta})$. Hence, it is easy to show that

$$\bar{\sigma} = Kr^{-1/(n+1)}\left(\frac{3}{4}\left[\hat{\Psi}'' - s^2\hat{\Psi}\right]^2 + 3\left[(1-s)\hat{\Psi}'\right]^2\right)^{1/2}. \quad \text{(H.12)}$$

Once $\bar{\sigma}$ is known, one can use eq. H.1 to calculate the plastic strains. Since elastic strains have been neglected, the total strain equals the plastic strain. These strain components ϵ_{rr}, $\epsilon_{\theta\theta}$ and $\epsilon_{r\theta}$ have to satisfy the compatibility condition

$$\frac{1}{r}\frac{\partial^2}{\partial r^2}(r\epsilon_{\theta\theta}) + \frac{1}{r^2}\frac{\partial^2 \epsilon_{rr}}{\partial \theta^2} - \frac{1}{r}\frac{\partial \epsilon_{rr}}{\partial r} - \frac{2}{r^2}\frac{\partial}{\partial r}\left(r\frac{\partial \epsilon_{r\theta}}{\partial \theta}\right) = 0. \quad \text{(H.13)}$$

Substituting the expressions for ϵ_{ij} in the above equation yields a long and complicated fourth order ordinary differential equation in $\hat{\Psi}(\theta)$. In plane strain, the ode reads

$$\begin{aligned}\hat{\Psi}'''' &+ \frac{N(2+N)}{(1+N)^2}\hat{\Psi}'' + \frac{N}{h^2 + 4N\hat{\Psi}'^2}\left\{\frac{1}{(1+N)^2}\left[4N\hat{\Psi}'' + \frac{1+2N}{1+N}h\right][h^2 + 4\hat{\Psi}'^2]\right. \\ &+ \frac{4(1-N)}{(1+N)^2}\hat{\Psi}'[hh' + 4\hat{\Psi}'\hat{\Psi}''] + \frac{1-N}{N(1+N)}[3hh'^2 + 8h'\hat{\Psi}'\hat{\Psi}'' + 4h(\hat{\Psi}''^2 + \hat{\Psi}'\hat{\Psi}''')] \\ &\left. + \frac{(1-N)(1-3N)}{N^2(1+N)}\frac{h}{h^2 + 4\hat{\Psi}'^2}[hh' + 4\hat{\Psi}'\hat{\Psi}'']^2\right\} = 0. \quad \text{(H.14)}\end{aligned}$$

In the above equation $N = 1/n$ and

$$h = (1+N)f'' + \frac{N(2+N)}{1+N}f. \quad \text{(H.15)}$$

The boundary conditions are derived from the condition that the crack surfaces are traction free, implying that $\sigma_{\theta\theta} = \sigma_{r\theta} = 0$. This means, following eq. H.11 that

$$\hat{\Psi}(\pm\pi) = \hat{\Psi}(\pm\pi) = 0. \quad \text{(H.16)}$$

Since we are looking at the plane strain fields, $\hat{\Psi}(\theta)$ has to be a symmetric function between $-\pi$ and π. Therefore, the condition at $-\pi$ can be replaced by equivalent conditions at $\theta = 0$ as:

$$\hat{\Psi}'(0) = \hat{\Psi}'''(0) = 0. \tag{H.17}$$

This is because of the fact that for a symmetric function, the odd derivatives $()'$, $()'''$, etc. are anti-symmetric and hence zero at $\theta = 0$. In addition we also have

$$\hat{\Psi}(\pi) = \hat{\Psi}(\pi) = 0. \tag{H.18}$$

Thus we have a ordinary differential equation H.14 subjected to four boundary conditions, two each at $\theta = 0$ and $\theta = \pi$. Like in the elastic case, the boundary conditions H.16 can lead to an eigen-value problem. But solving the eigen-value problem will yield nothing but the prder of the singularity, which we have already derived based on the path independence of the J integral. Hence, in this case, we directly attempt to solve the differential eq. H.14 subject to the boundary conditions H.17 and H.18.

The basic numerical scheme for solving the eq. H.14 is now outlined. We use the Runge-Kutta method to solve this problem. The eq. H.14 is solved as an initial value problem with initial values H.17. Note that the initial value for $\hat{\Psi}''(0)$ is not specified. Hence we guess this initial condition. An arbitrary guess will not lead to a solution that satisfies the conditions at $\theta = \pi$ given in eq. H.18. So, we have to iterate over the values of $\hat{\Psi}''(0)$ till the conditions at π are met. In addition, the value of the function is normalised such that $\hat{\Psi}(0) = 1$.

It might seem that four boundary conditions overspecifies a fourth order differential equation. That is however, not the case because the conditions at $\theta = \pi$ imply one another. To see how this is so, consider that

$$T_i = \sigma_{ij} n_j \tag{H.19}$$

and at $\theta = \pi$, only $n_2 = 1$. Hence, for traction free crack faces,

$$T_i(r, \pi) = \sigma_{\theta\theta}(r, \pi) + \sigma_{r\theta}(r, \pi) = 0. \tag{H.20}$$

This means that if $\sigma_{\theta\theta}(r,\pi) = 0$ it automatically implies that $\sigma_{r\theta}(r,\pi) = 0$ and vice versa.

H.2 The task

1. Download the files hrr.m and calcshi.m from the website of the course.
2. Fill in the details missing from calcshi.m.
3. Note down the value of n you have to use from the webpage.
4. Run calcshi.m in Matlab over and over again till you satisfy the boundary conditions at $\theta = \pi$.
5. Make a programme called ass5*yourrollno*.m, which plots $\hat{\Psi}$, $\hat{\Psi}'$ and $s\hat{\Psi} + \hat{\Psi}''$ for $s = (2n+1)/(n+1)$.
6. Send this programme by email alongwith the value of $\hat{\Psi}''(0)$ that you have used.
7. After downloading the programmes, type 'help calcshi' for more details.

A word of caution: I will check your programme only if the curves you plot look reasonable. Also, before emailing the programme please name it in the way I have instructed.