Chapter 4

Stress Intensity Factors in Complex cases

Closed form solutions of the stress intensity factor cannot be obtained in most cases. So, in most cases, we have to resort to taken to numerical techniques. However, the K-M equations allow us to find stress intensity factors for quite a few geometries. This chapter will give an overview of how the K-M equations can be used to derive stress intensity factors. Then we will look at some other analytical techniques for calculating the stress intensity factors. Numerical techniques for finding SIFs will be deferred to a later chapter. Finally, a table of some known stress intensity factors available in closed form will be presented.

4.1 Continuation of complex potentials

If $\theta(z)$ is an arbitrary analytic function defined in the domain S^+, then it is possible to use this function to define an associated analytic complex function in an region which is the image of S^+ in its boundary. Take for instance the half plane denoted by $y > 0$ as S^+.

Suppose that $\theta(z)$ is analytic for $z \in S^+$. Then evidently $\theta(\bar{z})$ is defined for all $z \in S^-$, where S^- is the region $y < 0$. We can easily show that the function $\bar{\theta}(\bar{z})$, which is defined for all $z \in S^-$ is analytic in S^-. If,

$$\theta(z) = u(x, y) + iv(x, y), \quad \text{then,} \tag{4.1}$$

$$\bar{\theta}(\bar{z}) = u(x, -y) - iv(x, -y). \tag{4.2}$$

Hence, using the relations

$$\frac{\partial}{\partial z} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right) \quad \text{and} \tag{4.3}$$

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right), \tag{4.4}$$

we easily get

$$\frac{\partial}{\partial \bar{z}} = \frac{1}{2}\left[u_{,x} - v_{,y} - i(u_{,y} + v_{,x})\right] = 0. \tag{4.5}$$

The above equation proves that $\bar{\theta}(\bar{z})$ is analytic in S^-. It is called the associated function of $\theta(z)$.

The above idea is supplemented by the *Continuation Theorem*, which states that if $\theta_1(z)$ and $\theta_2(z)$ are two functions analytic in S^+ and S^- respectively, and if there are a set of common points $z_1, \ddots z_n$, where $\theta_1(z_i) = \theta_2(z_i)$, then the function

$$\theta(x) = \begin{cases} = \theta_1(z) & z \in S^+ \\ = \theta_2(z) & z \in S^-. \end{cases}$$

The idea of analytic continuation is used extensively to solve plane and half plane problems in elasticity. Consider first a half plane problem where the material occupys the upper half plane. We denote the upper half plane by S^+ $(y > 0)$ and the lower half plane by S^-. The stresses and displacements in the body are entirely defined by the values of the complex potential in S^+ and the boundary values on $y = 0+$. Consequently, the potentials ψ, χ can be specified arbitrarily in S^- without affecting the stress and displacement fields.

The method of analytic continuation is then used to replace the definition of one of the potentials, say χ in S^+ with that of ψ in S^-. This leads to significant simplification of the boundary conditions. The most common continuations are phrased in terms of the associated functions $\bar{\psi}(\bar{z})$. This trick will be illustrated in the next section.

4.2 Stress intensity factor using the K-M formulation

Recall the Kolosoff-Mushkelishvilli equations:

$$\sigma_{xx} + \sigma_{yy} = 2\left[\psi'(z) + \bar{\psi}'(\bar{z})\right] \tag{4.6}$$

$$\sigma_{yy} - \sigma_{xx} + 2i\sigma_{xy} = 2\left[\bar{z}\psi''(z) + \chi''(z)\right] \tag{4.7}$$

$$2\mu D = \kappa\psi(z) - z\bar{\psi}'(\bar{z}) - \bar{\chi}'(\bar{z}). \tag{4.8}$$

Using the first two of the three K-M equations, we can easily show that

$$\sigma_{yy} - i\sigma_{xy} = z\bar{\psi}''(\bar{z}) + \bar{\chi}''(\bar{z}) + \bar{\psi}'(\bar{z}) + \psi'(z). \tag{4.9}$$

Now, consider an infinite body with n collinear cracks (see, Fig. 4.1) occupying segments L_r ($a_r \leq x \leq b_r$) on the real axis. This body is loaded by stresses at infinity.

This problem is the sum of two problems. The first is the deformation of an uncracked plate under known stresses at infinity and this gives rise to stresses $\sigma_{yy} - i\sigma_{xy} = p(x) + is(x)$ across the segments L_r. The second problem involves the cracked plate, with the crack faces loaded by stresses $\sigma_{yy} - i\sigma_{xy} = -[p(x) + is(x)]$ and with zero stresses at infinity. It is sufficient to determine a solution to the second problem, as the first problem is easy to solve.

The stresses σ_{yy} and σ_{xy} are continuous across $y = 0$ except across the cracks along L_r. The same also applied for the displacements u and v. The continuity of the displacements lead to,

$$\kappa\psi^+(x) - x\bar{\psi}'^+(x) - \bar{\chi}'^+(x) = \kappa\psi^-(x) - x\bar{\psi}'^-(x) - \bar{\chi}'^-(x) \tag{4.10}$$

Using the ideas expressed in the last section, the above equations can be written as

$$\lim_{y\to 0^+} \kappa\psi(z) + z\bar{\psi}'(z)) + \bar{\chi}'(z) = \lim_{y\to 0^-} \kappa\psi(z) + z\bar{\psi}'(z)) + \bar{\chi}'(z). \tag{4.11}$$

Therefore, by analytic continuation, the function

$$\phi(z) = \kappa\psi(z) + z\bar{\psi}'(z) + \bar{\chi}'(z), \tag{4.12}$$

is analytic everywhere on the x-y plane.

Similarly we can repeat the exercise for continuity of stresses across $y = 0$. This yields

$$x\psi''^{+}(x) + \chi''^{+}(x) + \psi'^{+}(x) + \bar{\psi'}{+}(x) = x\psi''^{-}(x) + \chi''^{-}(x) + \psi'^{-}(x) + \bar{\psi}'^{-}(x). \tag{4.13}$$

Proceeding in the same way as in the case of displacements, we can conclude that the function

$$\theta'(z) = \psi'(z) - \bar{\psi}'(\bar{z}) - z\bar{\psi}''(\bar{z}) - \bar{\chi}''(\bar{z}) \tag{4.14}$$

is analytic over the whole plane.

Now eliminating $\chi(z)$ in favour of $\theta(z)$ in the K-M equations, we get,

$$\begin{aligned} \sigma_{yy} - i\sigma_{xy} &= (z-\bar{z})\bar{\psi}''(\bar{z}) + \psi'(z) - \theta'(\bar{z}) + \psi'(\bar{z}) && (4.15)\\ 2\mu D &= \kappa\psi(z) - \psi(\bar{z}) + (\bar{z}-z)\bar{\psi}'(\bar{z}) + \theta(\bar{z}). && (4.16) \end{aligned}$$

Now, we can use the boundary conditions on the crack faces, giving,

$$\begin{aligned} -p(x) - is(x) &= \psi'^{+}(x) + \psi'^{-}(x) - \theta'^{-}(x) && (4.17)\\ -p(x) - is(x) &= \psi'^{-}(x) + \psi'^{+}(x) - \theta'^{+}(x). && (4.18) \end{aligned}$$

Adding the above two equations, we get a Hilbert problem in $2\psi'(z) - \theta'(z)$:

$$-2\left[p(x) + is(x)\right] = (2\psi'^{+}(x) - \theta'^{+}(x)) + (2\psi'^{-}(x) - \theta'^{-}(x)), \tag{4.19}$$

whereas, subtracting the equations yield a simple Hilbert problem in $\theta'(z)$:

$$\theta'^{+}(x) - \theta'^{-}(x) = 0. \tag{4.20}$$

The solution to this problem is given in terms of the Plemelj function with $\gamma = 1/2$, such that:

$$2\psi'(z) - \theta'(z) = \frac{X(z)}{2\pi i}\int_L \frac{p(t) + is(t)}{X^+(t)(t-z)}dt + X(z)P(z). \tag{4.21}$$

In the above equation, the Plemelj function $X(z)$ is given by $X(z) = \prod_{r=1}^{n}(z-a_r)^{-1/2}(z-b_r)^{-1/2}$, where the branch is selected such that $z^n X(z) \to 1$ as $|z| \to \infty$. The undetermined function $P(z)$ is a polynomial that is analytic over the whole plane and is of the degree $(n-1)$ with n undetermined coefficients.

The coefficients of $P(z)$ are chosen so that the displacement field is single valued: that is,

$$\int_C \frac{\partial D}{\partial z} dz = 0 \tag{4.22}$$

where C is any closed contour in the body and in particular contours taken around each crack. When eq. is written around each of the n cracks, we get n linear algebraic equations which may then be solved to get the undetermined constants. For the homogeneous Hilbert problem eq. 4.20, with vanishing stresses as $|z| \to \infty$, $\theta'(z) = 0$.

Consider now a special case of a single crack extending from $x = -a$ to $x = a$. The Plemelj function for this case is

$$X(z) = (z^2 - a^2)^{-1/2}. \tag{4.23}$$

The value of $X^+(x)$ can be found by taking $z - a = R_1 \exp(i\theta_1)$ and $z + a = R_2 \exp(i\theta_2)$, such that

$$X^+(x) = i\sqrt{(a^2 - x^2)} \ \text{ for } |x| \leq a. \tag{4.24}$$

The undetermined polynomial function $P(z)$ is a constant (for a single crack) and in fact, the value of the constant is zero. This can be shown (see, England, 1971) by taking a contour along the crack faces and considering that the resultant force on the crack is zero. Therefore, we are left with the solution

$$\psi'(z) = -\frac{\sqrt{z^2 - a^2}}{2\pi}\int_a^{-a} \frac{(a^2 - t^2)^{1/2}[p(t) + is(t)]}{t - z}dt. \tag{4.25}$$

In general, contour integration will be used to evaluate this integral. But in our case, we can use a different route to get the stress intensity factors. From the Mode I and II analyses in the previous chapter, it is easy to show that the stress intensity factors are given by

$$\psi(\zeta) = \frac{K_I - iK_{II}}{\sqrt{2\pi}}\zeta^{1/2}, \tag{4.26}$$

where, we have replaced z by ζ with the understanding that it is measured from the crack tip. It therefore follows immediately from the above relation that,

$$\psi'(\zeta) = \frac{K_I - iK_{II}}{2\sqrt{2\pi}}\zeta^{-1/2}. \tag{4.27}$$

This means that

$$K_I - iK_{II} = \lim_{\zeta \to 0}\left[2(2\pi\zeta)^{1/2}\psi'(\zeta)\right], \tag{4.28}$$

which is the effective definition of the inplane stress intensity factors that we will use for finding them. Using eq. 4.28 and eq. 4.25 we get,

$$K_I - iK_{II} = \frac{1}{(\pi a)^{1/2}}\int_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{1/2}[p(t) + is(t)]dt, \tag{4.29}$$

where we have shifted the origin to the crack tip such that $z = \zeta + a$.

Using a similar analysis for the equivalent Mode III problem, we can show that

$$K_{III} = \frac{1}{(\pi a)^{1/2}}\int_{-a}^{a}\left(\frac{a+t}{a-t}\right)^{1/2} p_3(t)dt, \tag{4.30}$$

where $p_3(t)$ is the out of plane load on the crack.

The above eqns. 4.29 and 4.30 can be used to determine the stress intensity factors for a variety of crack problems. The simplest is of course, the Griffith problem, where the remote stresses σ_{2i}^{∞} are specified. So, $p(t) = \sigma_{22}^{\infty}$ and $s(t) = \sigma_{21}^{\infty}$. This leads to

$$K_I = \sigma_{22}^{\infty}\sqrt{\pi a} \;\; K_{II} = \sigma_{22}^{\infty}\sqrt{\pi a}. \tag{4.31}$$

Similarly, for the case where the crack faces are loaded by point forces $p(x) = P\delta(x_1 - b)$ and $s(x) = S\delta(x_1 - b)$, at $x_1 = b$,

$$K_I - iK_{II} = \frac{P - iS}{(\pi a)^{1/2}}\left(\frac{a+b}{a-b}\right)^{1/2}. \tag{4.32}$$

For the problem of collinear cracks Koiter showed that

$$K_I = \sigma_{22}^{\infty}(\pi a)^{1/2}\left[\frac{2b}{\pi a}\tan\left(\frac{\pi a}{2b}\right)\right]^{1/2} \tag{4.33}$$

$$K_{II} = \sigma_{12}^{\infty}(\pi a)^{1/2}\left[\frac{2b}{\pi a}\tan\left(\frac{\pi a}{2b}\right)\right]^{1/2} \tag{4.34}$$

In eq. 4.25, if we had performed the contour integration, we could have, in principle, been able to find closed form expressions for $\psi'(z)$ and $\psi(z)$. These could then be used in the K-M equations to find the displacements D. In case of uniform pressure σ_{22}^{∞} and uniform shear σ_{12}^{∞} on the crack faces, we get,

$$4\mu D = -(\sigma_{22}^{\infty} + i\sigma_{12}^{\infty})\left[(\kappa-1)x \mp i(\kappa+1)(a^2-x^2)^{1/2}\right] \quad \text{on } y = 0\pm,\ |x| \leq a. \tag{4.35}$$

This shows that in case of the Griffith crack, the crack opens out to an ellipse under application of load. Several interesting examples are given at the end of this chapter.

4.3 Other methods of finding stress intensity factors

Several other mathematical techniques exist for finding stress intensity factors. Among them are Westergaard approach and integral transform approach (due to Sneddon, 1969). In the Westergaard approach, we use the following simplifications to the K-M equations:

$$\psi(z) = \frac{1}{2}Z_1(z), \quad \chi''(z) = -\frac{1}{2}zZ_1'(z). \tag{4.36}$$

These equations simplify the K-M equations to

$$\sigma_{xx} + \sigma_{yy} = 2\mathrm{Re}Z_1 \tag{4.37}$$

$$\sigma_{yy} - \sigma_{xx} = y\mathrm{Im}Z_1' \tag{4.38}$$

$$\sigma_{xy} = -y\mathrm{Re}Z_1', \tag{4.39}$$

and obviously, on the line $y = 0$, we have $\sigma_{yy} = \sigma_{xx}$ and $\sigma_{xy} = 0$.

Similarly, if we make the substitution

$$\psi(z) = -i\frac{1}{2}Z_2(z), \quad \chi''(z) = i\frac{1}{2}zZ_2'(z) + iZ_2, \tag{4.40}$$

in the K-M equations, we end up with,

$$\sigma_{xx} + \sigma_{yy} = 2\mathrm{Im}Z_2 \tag{4.41}$$

$$\sigma_{yy} - \sigma_{xx} = -2\mathrm{Im}Z_2 - 2y\mathrm{Re}Z_2' \tag{4.42}$$

$$\sigma_{xy} = \mathrm{Re}Z_2 - y\mathrm{Im}Z_2', \tag{4.43}$$

with the additional property that on the line $y = 0$, $\sigma_{yy} = 0$.

4.4 Application of the principle of superposition

The principle of superposition can be used to great advantage in LEFM to determine the stress intensity factor in a number of cases.

Consider Fig. 4.2 where evidently, by the principle of superposition, the stress at point H can be written as

$$\sigma_{22} = \sigma_{22}^A + \sigma_{22}^B, \tag{4.44}$$

in terms of the opening stresses in configurations A and B. Provided the point H in all the cases is close to the tip of the crack,

$$\sigma_{22} = \frac{K_I^A}{\sqrt{2\pi r}\cos(\theta/2}(1 + \sin(\theta/2)\sin(3\theta/2)) + \frac{K_I^B}{\sqrt{2\pi r}\cos(\theta/2}(1 + \sin(\theta/2)\sin(3\theta/2)) \tag{4.45}$$

implying that

$$K_I = K_I^A + K_I^B. \tag{4.46}$$

Thus the principle of superposition holds good for the stress intensity factor as well.

Let us focus on the problem of a crack under internal pressure p. This problem can be solved by invoking the principle of superposition. Consider a plate loaded by uniform

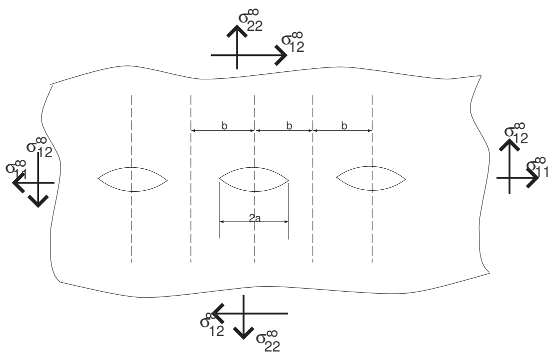

Figure 4.1: Infinite array of collinear cracks spaced $2b$ apart and lengths $2a$.

far-field (see, Fig. 4.3) stresses σ but without a crack (P). This can be looked at as a cracked plate under uniaxial tension but with loads applied on the crack faces to keep them closed. That problem again can be broken down into a problem A involving an unloaded crack under remote tension and the problem B of a crack with the faces forced shut! So,

$$K_I^P = K_I^A + K_I^B. \tag{4.47}$$

But since $K_I^P = 0$,

$$K_I^A = -K_I^B = \sigma\sqrt{\pi a}. \tag{4.48}$$

Reversing the sign of the σ term (for internally pressurised crack) we get that

$$K_I^A = p\sqrt{\pi a}. \tag{4.49}$$

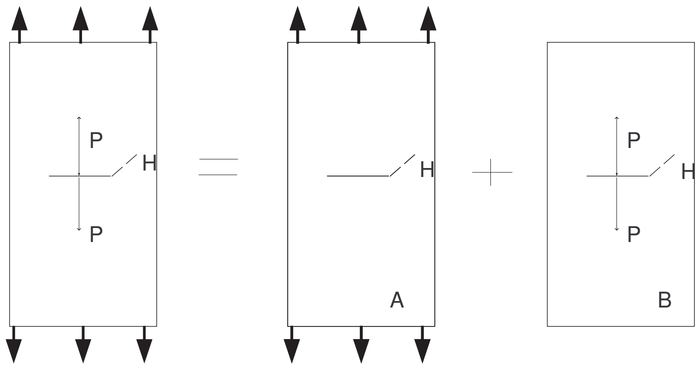

Figure 4.2: Superposition principle for SIFs.

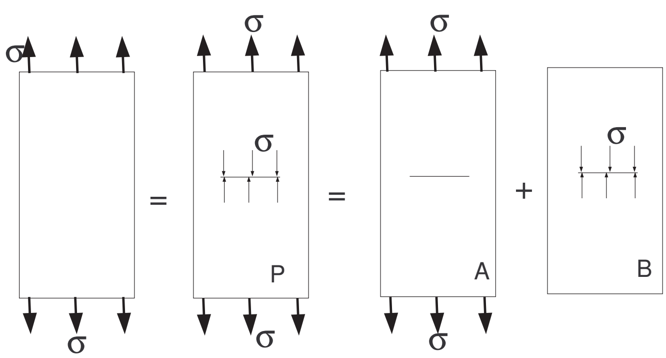

Figure 4.3: Superposition principle applied to a crack loaded with uniform pressure.