Expanding the above equation we see

$$
\begin{aligned}
\mathcal{V}' - \mathcal{V} &= \int_0^L EI \frac{d^2 w}{dx_1^2} \frac{d^2 \delta w}{dx_1^2} - \int_0^L bp(x_1) \delta w dx_1 \\
&+ \int_0^L \frac{1}{2} EI \left[\frac{d^2 \delta w}{dx_1^2} \right]^2 dx_1. \qquad (5.76)
\end{aligned}
$$

For $\mathcal{V}$ to be a minimum, $\mathcal{V}' > \mathcal{V}$ for all δw and so,

$$
\int_0^L EI \frac{d^2 w}{dx_1^2} \frac{d^2 \delta w}{dx_1^2} - \int_0^L bp(x_1) \delta w dx_1 = 0. \qquad (5.77)
$$

Integrating the above equation by parts yield,

$$
\left[EI \frac{d^2 w}{dx_1^2} \frac{d \delta w}{dx_1} \right]_0^L - \left[EI \frac{d}{dx_1} \left(\frac{d^2 w}{dx_1^2} \delta w \right) \right]_0^L + \int_0^L \left(EI \frac{d^4 w}{dx_1^4} - bp \right) \delta w dx = 0. \qquad (5.78)
$$

As this is valid for any δw, we conclude that

$$
EI \frac{d^4 w}{dx_1^4} - bp(x_1) = 0. \qquad (5.79)
$$

The above equation is the governing differential equation for beam bending. In addition we have the first two boundary terms in eq. 5.78, which, for arbitraray values of the terms containing δw yield at $x_1 = 0$ and $x_1 = L$,

$$
\begin{aligned}
\frac{d^2 w}{dx_1^2} &= 0 \\
\frac{d}{dx_1} \left(\frac{d^2 w}{dx_1^2} \right) &= 0. \qquad (5.80)
\end{aligned}
$$

We recognise from elementary strength of materials that the above equations say that the bending moment and the shear forces vanish at the two ends of the beam.

5.3 Energetics of a cracked body

Now we are in a position to unerstand the energetics of a cracked body. The basic idea is borrowed from Griffith, in that the change in the total energy dE is composed of two parts: one that goes into changing the potential energy of the body by an amount $d\Pi$ and one that is needed to create new surfaces, i. e.

$$dE = d\Pi + dS. \tag{5.81}$$

There is no change in the total energy E of the body while the crack grows from length a to $a + da$, then

$$d\Pi + dS = 0. \tag{5.82}$$

If we believe Griffith's analysis then γ denotes the surface energy density per unit area and $dS = 2\gamma da$ and so, at incipient or during crack growth,

$$\mathcal{G} = 2\gamma. \tag{5.83}$$

Without invoking the concept of surface energy density, we can write in general,

$$\mathcal{G} = -\frac{d\Pi}{da}. \tag{5.84}$$

The quantity $\mathcal{G}$ is called the *energy release rate* and is defined as the rate of decrease of potential energy with increase in crack length. It has to be borne in mind that if there are many cracks growing in the body a denotes the 'total' crack length, i.e. the sum of the lengths of all the existing cracks. Hence $\mathcal{G}$ is a global measure of fracture resistance.

The right side of eq. 5.83 is the fracture resistance that the crack must overcome in order to produce an unit of crack extension. While $\mathcal{G}$ depends on the loading and geometry of the body, the fracture resistance is a material property.

Consider Fig. 5.4 which shows a body with an edge crack being pulled by a force P. The load point displacement is Δ. The length of the crack is a. The potential energy of the body is

$$\Pi = U - P\Delta = -U^* \tag{5.85}$$

where $U = \int_V W dV$ is the total strain energy and U^* the total complementary strain energy.

The expression for the potential energy in this case can be obtained from elementary considerations and it is easy to see that

$$U = \frac{1}{2} P \Delta. \tag{5.86}$$

Hence, the potential energy of the body becomes

$$\Pi = -\frac{P\Delta}{2}. \tag{5.87}$$

Now suppose that during crack growth from a to $a + da$, the load P is kept constant. Substituting into eq. 5.84 we get

$$\mathcal{G} = \frac{\partial \Pi}{\partial a} = \frac{1}{2} P \left(\frac{\partial \Delta}{\partial a} \right)_P. \tag{5.88}$$

Further, if the compliance of the body is C, then

$$\Delta = CP \tag{5.89}$$

and then combining the above two equations we get the relationship between the crack driving force $\mathcal{G}$ and the change in compliance with crack growth,

$$\mathcal{G} = \frac{P^2}{2} \frac{dC}{da} = \frac{1}{2} \frac{\Delta^2}{C^2} \frac{dC}{da}. \tag{5.90}$$

The above equation allows us to measure the energy release rate experimentally.

A similar expression can be derived for a fixed grip loading where the value of Δ is kept fixed when the crack grows from a to $a + da$. Now,

$$\mathcal{G} da = -d\Pi = -dU, \tag{5.91}$$

and so,

$$\mathcal{G} = -\frac{dU}{da}. \tag{5.92}$$

Remembering that $U = (1/2)P\Delta$,

$$\mathcal{G} = -\left(\frac{\partial U}{\partial a} \right)_\Delta = -\frac{1}{2} \Delta \left(\frac{\partial P}{\partial \Delta} \right)_\Delta. \tag{5.93}$$

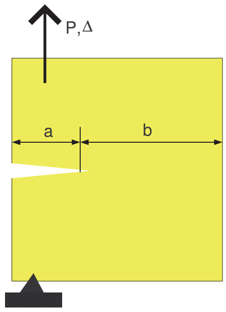

Figure 5.4: An edge cracked specimen for determination of $\mathcal{G}$.

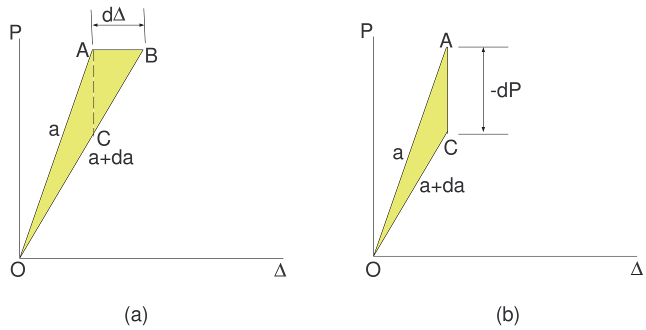

Figure 5.5: Load-displacement plots for (a) dead and (b) fixed grip loadings.

Substituting eq. 5.89 into the above equation, we again arrive at

$$\mathcal{G} = \frac{1}{2}\frac{\Delta^2}{C^2}\frac{dC}{da}, \tag{5.94}$$

showing that the value of $\mathcal{G}$ does not depend on the way the specimen is loaded. This fact can also be seen graphically from Fig. 5.5. Figure 5.5(a) shows the dead loading case while 5.5(b) shows the fixed grip case. The shaded area in both the cases represents the quantity $\mathcal{G}da$. This is easy to see as the shaded area gives the increase in U^* or the decrease in U, both of which from eqs. 5.88 and 5.93, represent $\mathcal{G}da$. The difference in the areas is the triangle ABC, whose area is of higher order compared to OAB or OAC.

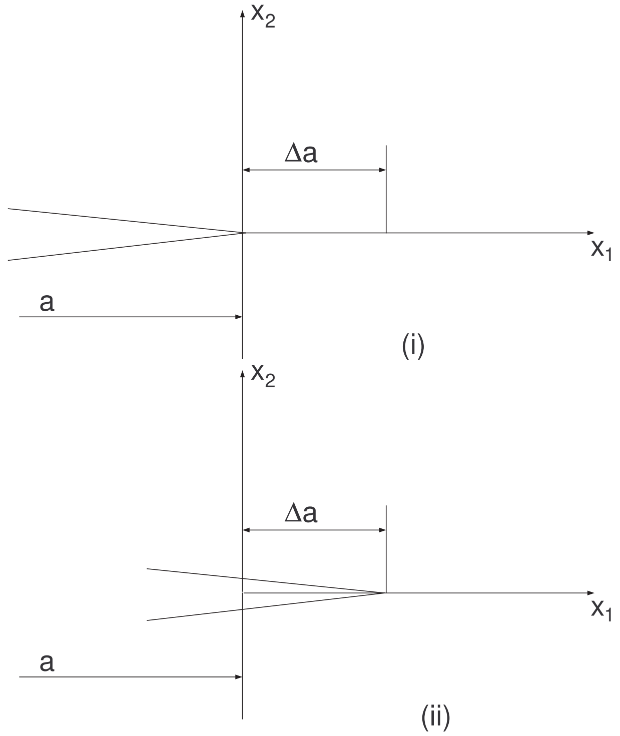

Figure 5.6: Deriving the connection between $\mathcal{G}$ and the stress intensity factor.

5.4 Connection between energy release rate and SIF

We have learnt that $\mathcal{G}$ represents the rate of change in potential energy with crack growth. In absence of body forces

$$\mathcal{G} = -\frac{\partial \Pi}{\partial a} = \frac{d}{da}\int_S T_i u_i dS - \frac{d}{da}U. \tag{5.95}$$

Now the above equation is valid only in the elastic case. In the elasto-plastic case another term representing plastic energy dissipation should be added to the right hand side. Physically however, $\mathcal{G}da$ represents the separation work involved in opening an additional crack length of Δa.

Crack growth is shown in Fig. 5.6. The crack length is a in stage (i) and $a + \Delta a$ in stage (ii). The additional crack opening created in stage (ii) had a traction σ_{2i} acting on it in stage (i). This traction goes to zero through a complex history as the crack opens out. The work done during this transition is

$$-\int_{S^+}\left(\int_{(i)}^{(ii)}\sigma_{2i}du_i\right)dS. \tag{5.96}$$

The work done per unit area of fracture surface created can thus be defined as,

$$\mathcal{C}=-\lim_{\Delta a\to 0}\frac{1}{\Delta a}\int_{S^+}\left(\int_{i}^{ii}\sigma_{2i}du_i\right)dS \tag{5.97}$$

where, $\mathcal{C}$ is called the separation work. In general, there will be a plastic process zone around the surface S^+, but if we consider linear elasticity to hold, the separation work becomes equivalent to $\mathcal{G}$. And

$$-\mathcal{G}\Delta a=-\lim_{\Delta a\to 0}\int_{S^+}\left(\int_{i}^{ii}\sigma_{2i}du_i\right)dS \tag{5.98}$$

is the work done on the elastic continuum to bring it quasi-statically from state (i), where it has tractions T_i operating on S^+ to state (ii) where the new crack surfaces are traction free.

Considering a 2-dimensional problem, with x_1 coincident with the crack direction, $T_i=-\sigma_{2i}$ on the upper surface and $=\sigma_{2i}$ on the lower surface. The displacements are $u_i=u_i^+$ above and u_i^- below. Then the expression for $\mathcal{G}$ becomes

$$\mathcal{G}=\lim_{\Delta a\to 0}\frac{1}{\Delta a}\int_0^{\Delta a}\left[\int_{(i)}^{(ii)}\sigma_{2i}d(u_i^+-u_i^-)\right]dx_1. \tag{5.99}$$

In case of linear elasticity this turns into,

$$\mathcal{G}=\lim_{\Delta a\to 0}\frac{1}{\Delta a}\int_0^{\Delta a}\left[\int_{(i)}^{(ii)}\sigma_{2i}^{(i)}d(u_i^+-u_i^-)^{(ii)}\right]dx_1 \tag{5.100}$$

as all the loads and displacements are proportional. Now considering that S^+ is within the zone of dominance of the crack tip fields, and that the crack tip fields are translated with the tip, we can replace $-x_1$ by $\Delta a-x_1$, which gives us in case of a Mode I crack:

$$\int_0^{\Delta a}\sigma_{2i}^{a}(u_i^+-u_i^-)^b dx_1=\frac{2}{\pi\mu}\frac{\kappa+1}{4}K_I^2\int_0^{\Delta a}\sqrt{\frac{\Delta a-x_1}{x_1}}dx_1. \tag{5.101}$$

The last integral can be evaluated as $\pi\Delta a/2$. Finally we get

$$\mathcal{G} = \frac{\beta}{E} K_I^2, \tag{5.102}$$

with $\beta = 1$ for plane stress and $= 1 - \nu^2$ for plane strain. The analysis can be repeated for Modes II and III giving

$$\mathcal{G} = \frac{\beta}{E} K_I^2 + \frac{\beta}{E} K_{II}^2 + \frac{K_{III}^2}{2\mu}, \tag{5.103}$$

As an application of this method, consider a double cantilever beam specimen shown in Fig. 5.7. Assuming slender beams,

$$\Delta = 2\frac{Pa^3}{3EI}, \tag{5.104}$$

where $I = Bh^3/12$ is the moment of inertia of the rectangular cross section. This implies that the compliance is

$$C = \frac{2a^3}{3EI} \tag{5.105}$$

and therefore, using eq. 5.90,

$$\mathcal{G} = \frac{P^2}{2B}\frac{dC}{da} = \frac{P^2}{2B}\frac{2a^2}{EI}. \tag{5.106}$$

Proceeding further and using the relation between $\mathcal{G}$ and the stress intensity factor we find that for this specimen,

$$K_I = \sqrt{\frac{12}{h^3}}\frac{Pa}{B}. \tag{5.107}$$

Recall that once the stress intensity factor is known, everything about the crack tip stress fields is known. Hence in the elastic case calculation of $\mathcal{G}$ gives us a method for finding the stress intensity factor.

5.5 The J integral

The treatment of the energy release rate in the previous section is valid only for linear elastic materials. We now seek a more general treatment which leads to the path in-

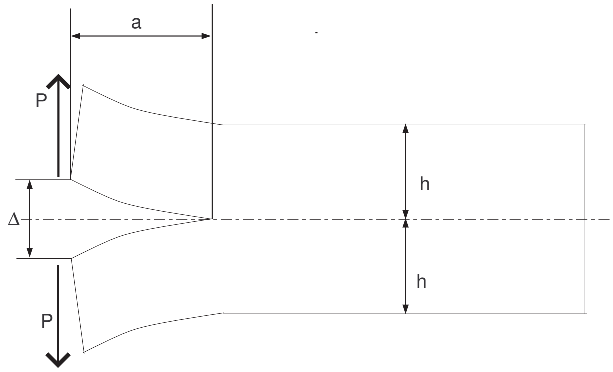

Figure 5.7: Loading in the double cantilever beam (DCB) specimen

dependent J integral. This integral was first used in its present form by J.R. Rice in 1968.

Consider a 2-d body subject to the ususal boundary conditions (see, Fig. **??**). The potential energy in the absence of body forces is

$$\Pi(a) = \int_A W dA - \int_{S_t} T_i u_i dS. \tag{5.108}$$

The tractions are specified quantities which can be taken to be independent of the crack length. So,

$$\frac{d\Pi}{da} = \int_A \frac{dW}{da} dA - \int_{S_0} T_i \frac{du_i}{da} dS. \tag{5.109}$$

The contour of the line integral has been taken to cover the whole boundary of the body since the quantity du_i/da vanishes over S_u. Let us move the origin of the coordinate system to the crack tip by making the simple transformation

$$X_i = x_i - a\delta_{i1}. \tag{5.110}$$

This implies that

$$\frac{d}{da} = \frac{\partial}{\partial a} + \frac{\partial X_1}{\partial a}\frac{\partial}{\partial X_1} = \frac{\partial}{\partial a} - \frac{\partial}{\partial x_1}. \tag{5.111}$$

The eq. 5.109 then becomes

$$\frac{d\Pi}{da} = \int_A \left(\frac{\partial W}{\partial a} - \frac{\partial W}{\partial x_1}\right) dA - \int_{S_0} T_i \left(\frac{\partial u_i}{\partial a} - \frac{\partial u_i}{\partial x_1}\right) dS. \tag{5.112}$$

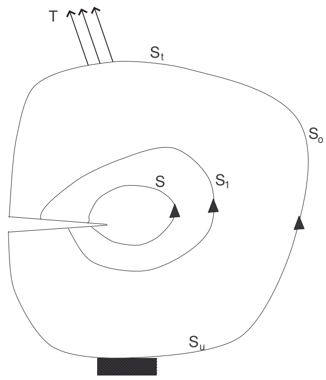

Figure 5.8: Contours for the J integral

Using the constitutive equation of the material it is easy to show that

$$\frac{\partial W}{\partial a} = \sigma_{ij} \left(\frac{\partial u_i}{\partial a} \right)_{,j} . \tag{5.113}$$

Using the divergence theorem, the term

$$\int_A \frac{\partial W}{\partial a} \tag{5.114}$$

can be converted into

$$\int_{S_0} T_i \frac{\partial u_i}{\partial a} dS. \tag{5.115}$$

Equation 5.109 therefore becomes

$$-\frac{d\Pi}{da} = \int_A \frac{\partial W}{\partial x_1} dA - \int_{S_0} T_i \frac{\partial u_i}{\partial x_1} dS, \tag{5.116}$$

which by the application of the divergence theorem once again can be written as

$$-\frac{d\Pi}{da} = \int_{S_0} \left(W n_1 - T_i \frac{\partial u_i}{\partial x_1} \right) . \tag{5.117}$$

Rice defined

$$J = \int_S \left(W n_1 - T_i \frac{\partial u_i}{\partial x_1} dS \right) \tag{5.118}$$

which is defined not on the contour enclosing the body but on an arbitrary contour starting on one crack face and ending on another. The positive sense of the contour is anti-clockwise.