Now consider two contours S_1 and S, both starting on one crack face and ending on the other. We can construct a combined contour including the crack faces C_1 and C_2 on which $n_1 = 0$. Thus

$$J_1 - J = \int_{S_1+S+C_1+C_2} \left(W n_1 - T_i \frac{\partial u_i}{\partial x_1} dS \right) \tag{5.119}$$

The contour has to be traversed in such a way that the area enclosed A is always on the left. Then applying the divergence theorem again,

$$\begin{aligned} J_1 - J &= \int_A \left[\frac{\partial W}{\partial x_1} - \frac{\partial}{\partial x_j} \left(\sigma_{ij} \frac{\partial u_i}{\partial x_1} \right) \right] dA \\ &= \int_A \left[\frac{\partial W}{\partial \epsilon_{ij}} \frac{\partial \epsilon_{ij}}{\partial x_1} - \sigma_{ij} \frac{\partial}{\partial x_1} u_{i,j} \right] dA \\ &= \int_A \left[\sigma_{ij} \frac{\partial \epsilon_{ij}}{\partial x_1} - \sigma_{ij} \frac{\partial}{\partial x_1} u_{i,j} \right] dA = 0. \end{aligned} \tag{5.120}$$

This shows that $J_1 = J$ and hence, the J integral is path-independent. Moreover, since the J integral is defined as

$$J = -\frac{d\Pi}{da}, \tag{5.121}$$

in the linear elastic case, J and $\mathcal{G}$ are the same measures and so,

$$J = \frac{\beta}{E} K_I^2 + \frac{\beta}{E} K_{II}^2 + \frac{K_{III}^2}{2\mu}, \tag{5.122}$$

in case of linear elasticity.

The J integral is however, more versatile compared to the $\mathcal{G}$ as will be shown later in connection with elastic-plastic fracture mechanics. For the time being, let us turn to a few examples which illustrate the applications of the J integral.

5.5.1 Applications of the J integral

A bar between rails

Consider Fig. **??** which shows a bar of thickness $2h$ fixed between two rails such that the surfaces $x_2 = \pm h$ cannot move. A semi-infinite crack runs through the center of the bar and we intend to find the stress intensity factor for this crack under the condition that the rails are moved perpendicularly through u_{20}. The contour for the J integral calculation is taken as shown.

On BC and DE, $n_1 = \partial u_i/\partial x_1 = 0$ as the faces $x_2 = \pm h$ are displaqced symmetrically. Edges AB and EF are located far back at $-\infty$ whereby $\sigma_{ij} = 0$. The only contrbution to the line integral therefore comes from the part CD at $x_1 = \infty$ where $e_{22} = u_{20}/h$ and $\partial u_i/\partial x_1 = 0$. Hence J becomes

$$J = \int_{-h}^{h} W|_{x_1=\infty} dx_2. \tag{5.123}$$

For linear elasticity, W is given by eq. 5.52 and working that out for this case gives

$$W|_{x_1=\infty} = \frac{(1-\nu)Ee_{22}^2}{2(1+\nu)(1-2\nu)}, \tag{5.124}$$

which gives

$$J = \frac{(1-\nu)Eu_{20}^2}{(1+\nu)(1-2\nu)h} \tag{5.125}$$

$$K_I = \frac{Eu_{20}}{[(1+\nu)^2(1-2\nu)h]^{1/2}}. \tag{5.126}$$

Coating

Consider a thin coating of thickness $2h$ on an infinite and rigid substrate. The coating is subjected, as shown in Fig. 5.9, to a moment M and an axial force P. Using the contour shown, and considering that the rigid substrate does not contribute anything to the strain energy, the J integral becomes:

$$J = \int_{h}^{0} W|_{x_1=-\infty} ds. \tag{5.127}$$

The contributions to the energy density comes from the force as well as the moment and thus is

$$W = \frac{1}{2}\sigma_{11}e_{11} = \frac{P^2}{Eh^2} + \frac{M^2x_2^2}{2I^2E^2} \tag{5.128}$$

and so,

$$J = \left(P^2 + \frac{12M^2}{h^2}\right)/2Eh. \tag{5.129}$$

5.6 Fracture Toughness

Elementary estimates of the plastic zone size ahead of the crack tip can be found from the LEFm solutions. In Mode I,

$$\sigma_{22} = \frac{K_I}{\sqrt{2\pi x_1}}, \tag{5.130}$$

and so, if the yield criterion is $\sigma_{22} = \sigma_0$, the size of the plastic zone is easily found to be

$$r_y = \frac{1}{2\pi}\left(\frac{K_I}{\sigma_0}\right)^2. \tag{5.131}$$

This identifies the point on the elastic σ_{22} curve at which the stress equals the yield stress. However, local yielding will lead to a redistribution of the stresses and then the plastic zone size can be better estimated using

$$\int_0^{r_y} \sigma_{22} dx_1 = \sigma_0 r_p. \tag{5.132}$$

This gives,

$$r_p = \frac{1}{\pi}\left(\frac{K_I}{\sigma_0}\right)^2, \tag{5.133}$$

for a state of plane stress. In plane strain the plastic zone size is given by

$$r_p = \frac{1}{3\pi}\left(\frac{K_I}{\sigma_0}\right)^2. \tag{5.134}$$

Another estimate of the plastic zone can be derived from Dugdale's idea that in front of the crack tip we have an thin, inelastic cohesive zone in which the traction across the surfaces is related to the separation $\delta = u_2^+ - u_2^-$ of the cohesive surfaces (see, Fig. 5.10). Using a contour shrunk onto the cohesive zone, $n_1 = T_1 = 0$ and J becomes

$$J = -\int_0^{\delta_t} \sigma\left[\frac{\partial}{\partial x_1}(u_2^+ - u_2^-)\right] dx_1 = -\int_0^{\delta_t} \sigma \frac{\partial \delta}{\partial x_1} dx_1 = \int_0^{\delta_t} \sigma(\delta) d\delta. \tag{5.135}$$

Here δ_t is the crack tip opening displacement. If the cohesive stress is taken to be σ_0, then

$$J = \sigma_0 \delta_t. \tag{5.136}$$

Now, since J is path independent,

$$J = \mathcal{G} = \frac{K_I^2}{E'} = \sigma_0 \delta_t, \tag{5.137}$$

which goes on to show that the stress intensity factor, the J integral, the energy release rate and the crack tip opening displacement are all valid fracture parameters within the restrictions of small scale yielding.

Now, let us try to get an estimate of the length of the Dugdale cohesive zone. This can be derived based on the principle that the effective Griffith crack length is now $2(a+d)$ and the stress intensity factor in front of the tip is now zero. This is due to the fact that the cohesive zone is formed to relieve the high stresses at the crack tip and once it is formed, the singularity is removed. Considering the portion $a < |x_1| < a+d$, the stresses on the crack faces is $\sigma_\infty - \sigma_0$ while the stresses on the portion $|x_1| < a$ is just σ_∞. Note that we have used the same superposition principle that we used in case of the original fully elastic Griffith crack.

Using the idea that $K_I = (1/\sqrt{\pi a}) \int_{-a}^{a} [(a+t)/(a-t)]^{1/2} p(t) dt$, we get, if $p(t)$ is an even function, the simpler expression

$$K_I = -2\sqrt{\frac{a}{\pi}} \left[\int_0^{\pi/2} p(a \cos\phi) d\phi \right]. \tag{5.138}$$

Using this above equation and setting $K_I = 0$ we get,

$$\sigma_\infty \alpha + (\sigma_\infty - \sigma_0) \left(\frac{\pi}{2} - \alpha \right) = 0, \tag{5.139}$$

where,

$$\alpha = \cos^{-1} \left[\frac{a}{a+d} \right]. \tag{5.140}$$

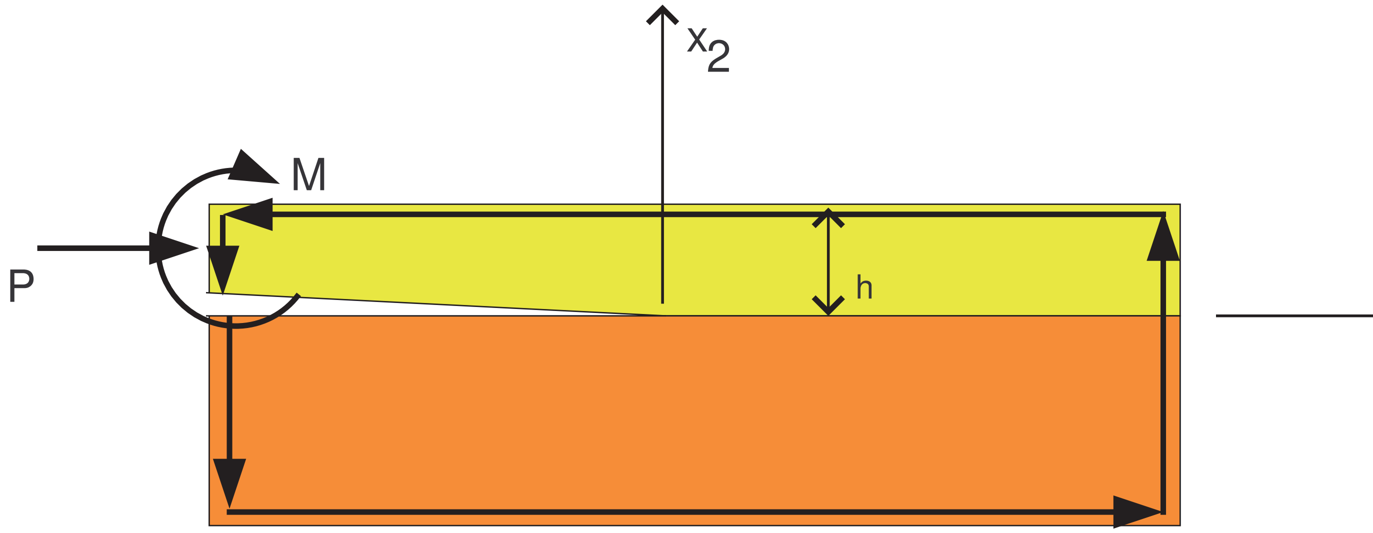

Figure 5.9: Contour for finding the J integral for an elastic layer on a rigid substrate

This leads to the value of the extent of the plastic cohesive zone

$$d = a\left[\sec\left(\frac{\pi\sigma_\infty}{2\sigma_0}\right) - 1\right]. \tag{5.141}$$

Suitable approximations to the above expression yields a plastic zone size

$$d = \frac{\pi}{8}\left(\frac{K_I}{\sigma_0}\right)^2, \tag{5.142}$$

which compares well with the plane stress based estimate of the plastic zone.

Yet another but more exact estimate of the plastic zone size can be determined by using the von Misces' yield criterion

$$(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2 + (\sigma_3 - \sigma_1)^2 = 2\sigma_0^2. \tag{5.143}$$

The principal stresses σ_i are obtained from the equations of the Mode I crack tip stresses

$$\begin{Bmatrix} \sigma_{11} \\ \sigma_{12} \\ \sigma_{12} \end{Bmatrix} = \frac{K_I}{\sqrt{2\pi r}}\cos(\theta/2)\begin{Bmatrix} 1 - \sin(\theta/2)\sin(3\theta/2) \\ \sin(\theta/2)\cos(3\theta/2) \\ 1 + \sin(\theta/2)\sin(3\theta/2) \end{Bmatrix},$$

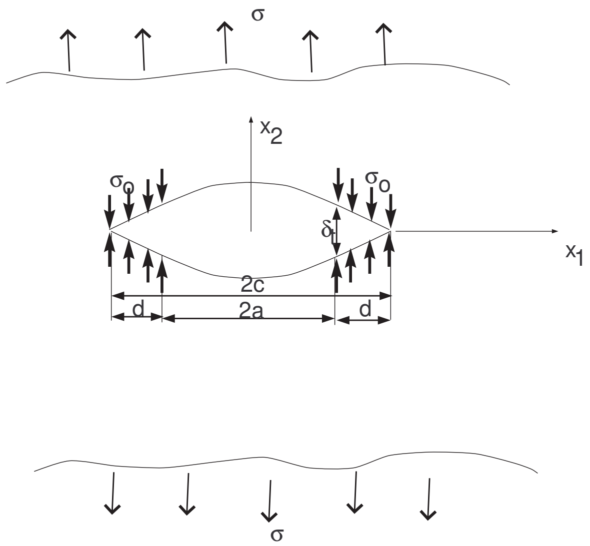

Figure 5.10: Dugdale plastic zone

using

$$\left.\begin{array}{c}\sigma_1\\ \\ \sigma_2\end{array}\right\} = \frac{\sigma_{11}+\sigma_{22}}{2} \pm \left[\left(\frac{\sigma_{11}-\sigma_{22}}{2}\right)^2 + \sigma_{12}^2\right]^{1/2}$$

giving

$$\left\{\begin{array}{c}\sigma_1\\ \\ \sigma_2\end{array}\right\} = \frac{K_I}{\sqrt{2\pi r}}(1 \pm \sin(\theta/2))\cos(\theta/2)$$

and

$$\sigma_3 = \left\{\begin{array}{c} 0 \ \text{ plane stress} \\ 2\nu K_I/\sqrt{2\pi r}\cos(\theta/2) \ \text{ plane strain}\end{array}\right. .$$

Using the above equations the shape of the yielded zone $r_y(\theta)$ comes out to be

$$r_y(\theta) = \frac{K_I^2}{4\pi\sigma_0^2}\left\{\begin{array}{c}\left[(1-2\nu)^2(1+\cos\theta)+\frac{3}{2}\sin^2\theta\right] \ \text{ plane strain} \\ \left[1+\cos\theta+\frac{3}{2}\sin^2\theta\right] \ \text{ plane stress.}\end{array}\right.$$

Comparison of the two yielded zones in case of a fixed Poisson's ratio shows that the plane strain r_y is smaller than the plane stress one. If the thickness of the body B is close to the size of the plastic zone obtained from one of the many estimation techniques described above, plane strain conditions cannot be assumed to prevail. The state of stress then induces through the thickness straining, which might lead to dimpling at the crack tip. The thumb rule is that if $r_p < B/25$ then sufficient material exists to prevent through the thickness straining. There is evidently an intermediate range where inside of the body will have plane strain while close to the boundaries plane stress will prevail.

Moreover, in case of plane stress, since the out of plane principal stress is zero, the planes of maximum shear are at 45^o to the free surfaces. Slip along these planes give rise to the so-called shear lips. The variation of the fracture toughness with specimen thickness is shown in Fig. 5.11. At small thickness, the large plastic zones cause the material to behave in a more ductile manner leading to a higher fracture toughness. The critical fracture toughness of the material is defined as the value of K_c in the limit of large thicknesses. This value is called K_{Ic} and is supposedly independent of the thickness of the body. ASTM standard E-399 gives the body dimensions that should be maintained to obtain this quantity.

Another important practical aspect of energy based fracture mechanics is the concept of the R curve. The R curve represents the resistance of the material to crack growth, can be expressed in terms of K, $\mathcal{G}$ or J, and depends mainly on the extent of crack growth Δa rather than the crack length. A typical R curve is shown in Fig. 5.12(a).

Figure 5.12(b) illustrates the use of an R curve. During crack growth, equilibrium between the crack driving force and the fracture resistance is given by

$$\mathcal{G}(a) = R(\Delta a), \tag{5.144}$$

where, $a = a_0 + \Delta a$ is the current crack length. The current configuration is stable if a slight increase of crack length at constant load does not give rise to a driving force that

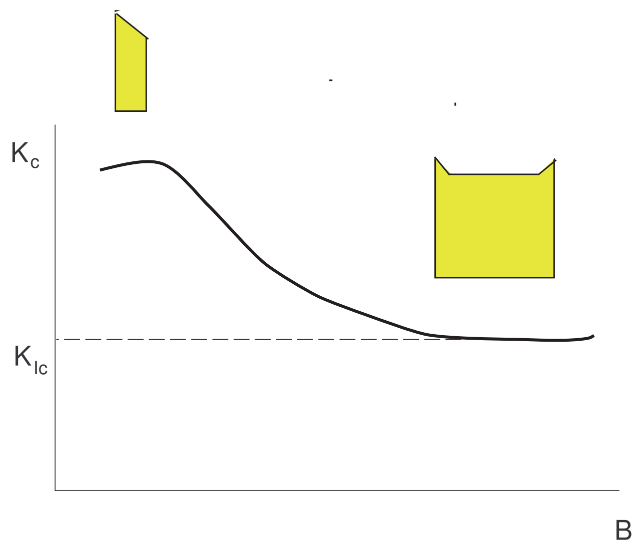

Figure 5.11: Variation of fracture toughness with specimen thickness

exceeds the material's resistance. This means, stable crack growth imples

$$\left(\frac{d\mathcal{G}}{da}\right)_P < \frac{dR}{d\Delta a}. \tag{5.145}$$

That is crack growth will be stable if the rate of increase of the driving force is less than the rate of increase of the material's fracture resistance. If that is not the case, the crack will grow catastrophically. Zones of stable and unstable crack growth are shown in Fig. 5.12(b).

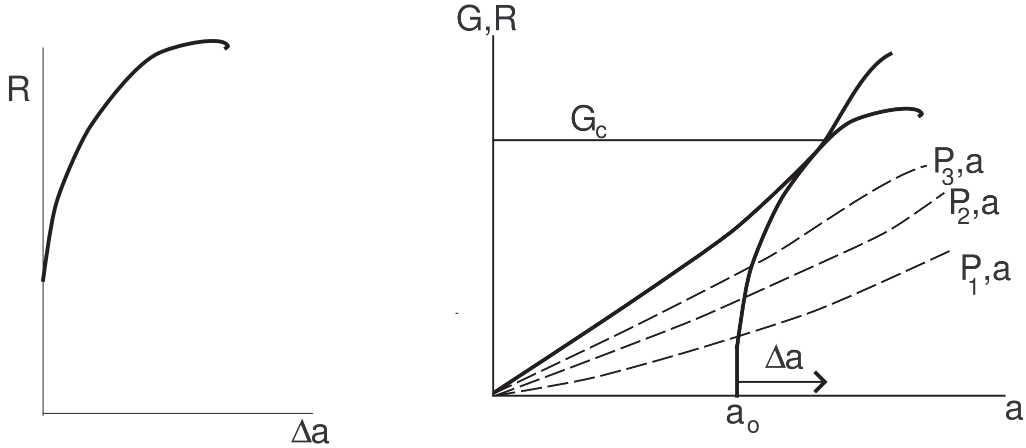

Figure 5.12: Concept of the R curve