# Chapter 6

# Elasto-Plastic Fracture Mechanics

## 6.1 Introduction to Plasticity

In this section we take a brief tour through the theory of plasticity. The treatment will not be exhaustive and only those aspects of the theory that are of use to us will be elaborated. Small strains will be assumed all through this chapter.

Consider a cylindrical rod of initial length $l_0$ and cross sectional area $A_0$ being pulled by an axial force $P$ to a final length $l$. The engineering strain and nominal stress are defined in this uniaxial situation as

$$\begin{aligned} \epsilon &= \frac{l - l_0}{l} \\ \sigma &= \frac{P}{A_0}. \end{aligned}$$

The stress-strain behaviour of some typical metals is shown in the Fig. 6.1. If the specimen is deformed to a stress beyond the yield point $A$, plastic flow sets in. While the deformation is reversible before $A$ is reached, unloading from a point $S$ beyond $A$ leads to a permanent strain $\epsilon_p$ that remains in the specimen. Moreover, while reloading, the curve $O'QA'$ is traced which is for all practical purposes, almost coincident with the unloading path $SO'$. Also the new yield stress $A'$ is almost the same as $S$, the stress from which we unloaded. The increase in yield stress as loading proceeds is known as *work hardening.*

As we keep on increasing the load, in some specimens a maximum load is reached where *necking* starts. This involves the reduction in cross section in the middle of the specimen. The onset of necking is a geometric-material instability and the stress state becomes non-uniform in the region around the neck. Note that till the onset of necking the stress state was simply uniaxial all over the length of the specimen. Actually, if we consider the true stress $P/A$ where $A$ is the current area, the stress will continue to increase even beyond $M$. Finally, fracture occurs at $C$ but data in the region $MC$ is very difficult to obtain.

In this chapter we will confine ourselves to just *isotropic hardening.* In this case if the secimen is loaded to point $S$ so that the yield stress is raised to a value $\sigma_t$, and then the load is reduced and the specimen is compressed, yielding in compression occurs at the same value $\sigma_t$. Generally pure metals, in which dislocations can move with equal ease in tension and compression, display this kind of behaviour. Materials which exhibit different yield stresses in tension and compression are called *kinematically hardening.*

We also confine ourselves to *time independent plasticity.* This excludes materials which display strain rate sensitivity. These effects include viscoplasticity or creep like behaviour, as shown in Fig. 6.2.

### 6.1.1 Characterisation of elasto-plastic behaviour

It is evident from Fig. 6.1 that

$$\epsilon = \epsilon_e + \epsilon_p. \tag{6.1}$$

The above additive decomposition of the total strain is true in case of small deformations. If the deformations are large, it is not so obvious. Also, it is easy to see that

$$\epsilon_e = \frac{\sigma}{E}, \tag{6.2}$$

gives the elastic strain. Two moduli can be defined for the uniaxial problem. The tangent modulus $E_t$ defined as

$$E_t = \frac{d\sigma}{d\epsilon} \tag{6.3}$$

or the secant modulus

$$E_s = \frac{\sigma}{\epsilon}. \tag{6.4}$$

Hence, the stress-strain equations can be written as

$$\dot{\sigma} = E_t \dot{\epsilon} \tag{6.5}$$

or,

$$\epsilon_p = \sigma \left( \frac{1}{E_s} - \frac{1}{E} \right). \tag{6.6}$$

Some simple models are used to characterise the stress-strain response of materials. They are shown in Fig. 6.3. The simplest one is the rigid-perfectly plastic response where $E = \infty, E_t = 0$. The elastic-perfectly plastic response assumes $E_t = 0$. The bilinear stress-strain curve assumes a constant value for $E_t$. A more realistic response is the piecewise power-law hardening one, where,

$$\begin{aligned} \frac{\epsilon}{\epsilon_0} &= \frac{\sigma}{\sigma_0} \quad \sigma \leq \sigma_0 && (6.7) \\ &= \left( \frac{\sigma}{\sigma_0} \right)^n \quad \sigma > \sigma_0 && (6.8) \end{aligned}$$

where $n$ is called the hardening exponent. Another popular formula is the Ramberg-Osgood stress strain response, which is a single curve given by

$$\frac{\epsilon}{\epsilon_0} = \frac{\sigma}{\sigma_0} + \alpha \left( \frac{\sigma}{\sigma_0} \right)^n, \tag{6.9}$$

where, $\sigma_0 = E\epsilon_0$. Note that the Ramberg-Osgood response is nonlinear from the very beginning. By varying $n$ from 1 to $\infty$, it can simulate the whole range of metal behaviour from linear elastic to elastic-perfectly plastic.

### Multiaxial generalisations

Like in the uniaxial case assume that the strain tensor can be decomposed additively as

$$\boldsymbol{\epsilon} = \boldsymbol{\epsilon}_e + \boldsymbol{\epsilon}_p \tag{6.10}$$

In incremental form

$$d\boldsymbol{\epsilon} = d\boldsymbol{\epsilon}_e + d\boldsymbol{\epsilon}_p \tag{6.11}$$

The elastic strain increment is related to the total stress by the Hooke's law and so,

$$d\epsilon_{ij} = S_{ijkl} d\sigma_{kl}, \tag{6.12}$$

where $\boldsymbol{S}$ is the compliance tensor.

The basic problem is to find the plastic strain increment $d\boldsymbol{\epsilon}_p$ gien a stress history $\boldsymbol{\sigma}(t)$ and a stress increment $d\boldsymbol{\sigma}$. To solve this problem we need a few tools, namely,

- yield criterion
- flow rule, and
- hardening rule

### Yield criterion

Let us take up the case of the yield criterion first. In the most general sense, a yield criterion is a function of the form

$$\Phi(\boldsymbol{\sigma}, \boldsymbol{\beta}_n) = 0, \tag{6.13}$$

where the $\boldsymbol{\beta}$ are called 'history variables' which contain information about the past history of the material. The function $\Phi$ encloses a space in the 6-dimensional stress space such that

$$\Phi < 0 \tag{6.14}$$

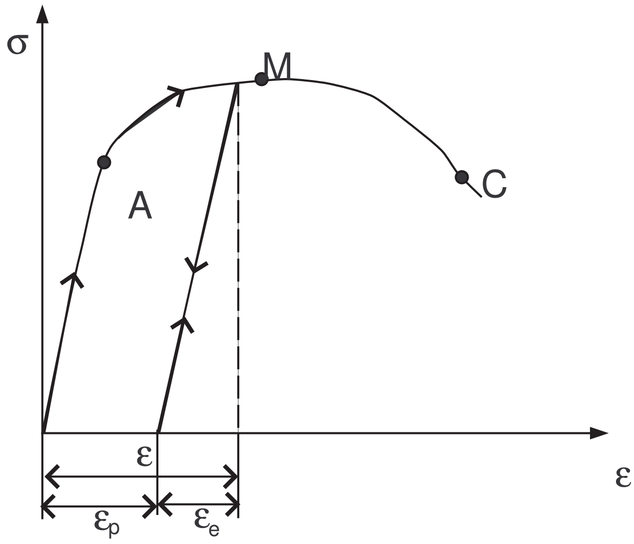


Figure 6.1: Typical uniaxial stress strain curve of a metal

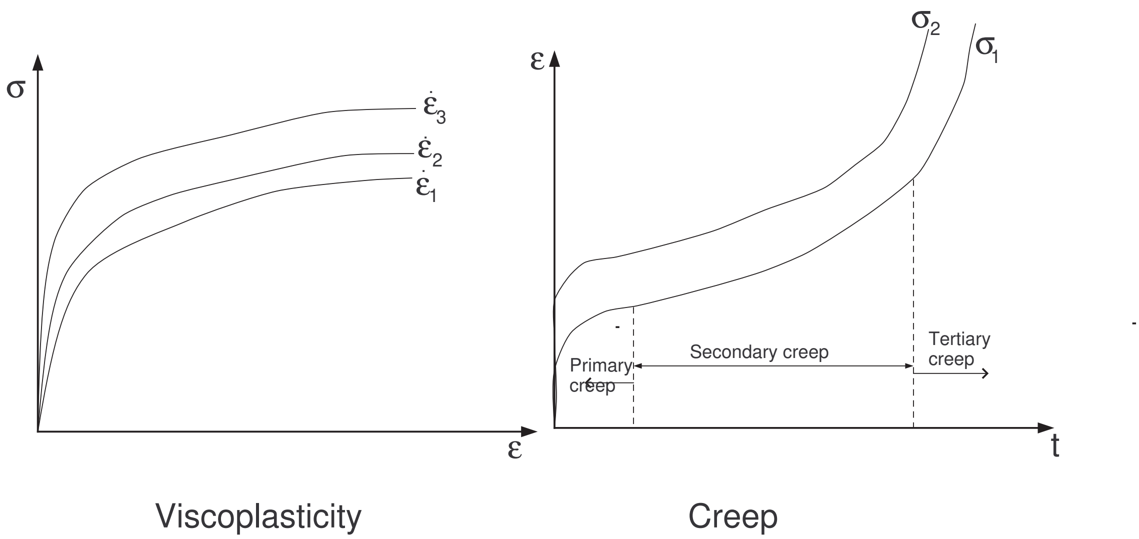


Figure 6.2: Examples of time dependent plasticity

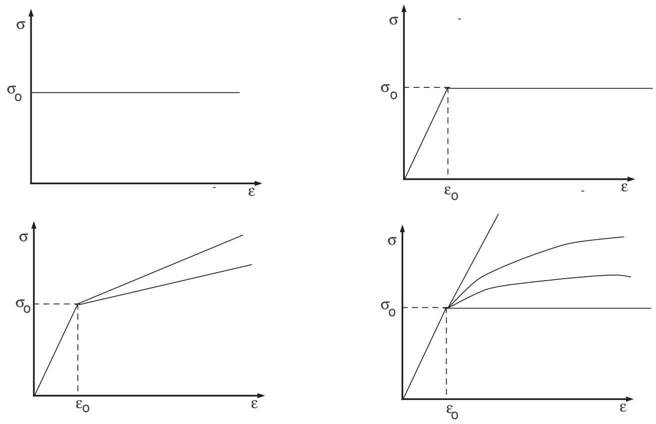


Figure 6.3: Idealisations of uniaxial elasto-plastic behaviour of metals.

implies

$$d\epsilon_{ij}^{p} = 0 \tag{6.15}$$

$$d\epsilon_{ij}^{e} = S_{ijkl} d\sigma_{kl}. \tag{6.16}$$

Let the unit normal to the yield surface in the stress-space be defined as

$$n_{ij} = \frac{\partial \Phi / \partial \sigma_{ij}}{[\partial \Phi / \partial \sigma_{kl} \partial \Phi / \partial \sigma_{kl}]^{1/2}}. \tag{6.17}$$

Let us now consider a stress state $\sigma_{ij}^{A}$ on $\Phi$. For a small increment $d\sigma_{ij}$ the following can be defined,

- Loading, $\boldsymbol{n} \cdot d\boldsymbol{\sigma} > 0$ (when $d\boldsymbol{\sigma}$ is above the tangent plane)
- Unloading, $\boldsymbol{n} \cdot d\boldsymbol{\sigma} < 0$ (when $d\boldsymbol{\sigma}$ is below the tangent plane) and
- Neutral loading $\boldsymbol{n} \cdot d\boldsymbol{\sigma} = 0$

If the material is non-hardening, the yield surface does not depend on the history variables and $\Phi = \Phi(\boldsymbol{\sigma}) = 0$. For hardening materials the yield surface must deform elastically so that elastic unloading can be accomodated. As shown in Fig. 6.4, the yield surface must then expand in the way shown. At the current stress state however, the stress states outside the yield surface are not accessible.

### Flow rule

Let us then concentrate on the flow rule. Given a stress state on the yield surface, and a small stress increment, this rule gives us the direction of the corresponding plastic strain increment. Note that there will be a plastic strain increment only in case of a loading case. In case of a loading,

$$d\boldsymbol{\epsilon}_p = d\lambda \boldsymbol{m}(\boldsymbol{\sigma}, \boldsymbol{\beta}_n), \tag{6.18}$$

where, $\boldsymbol{m}$ is the flow direction and can be considered to be normalised. The scalar $d\lambda$ is called the 'plastic parameter'. Here we should note that the platic strain increment depends on the total stress and not on the stress increment. This postulate is supported by dislocation theories of plasticity.

### Stability postulates

Drucker's stability postulates help in crystalising many of the abstract concepts. Consider Fig. 6.5(a-c). Clearly, in cases (b) and (c), the work done by the stresses is negative. These are examples of unstable materials where it is possible to extract work without any external source of energy.

Consider a strain hardening material at a stress state $\boldsymbol{\sigma}^o$, on which an additional stresses are applied and then removed. Drucker's stability postulates state that,

- During loading the additional stresses do positive work.

- During a complete cycle of additional loading and unloading, the additional stresses do positive work if plastic strains are produced and no work if the strains are purely elastic.

Let us now see what these postulates imply. In Fig. 6.6, the initial stress state is $\sigma^o_{ij}$ and we consider the loading path $A-B-C$. Evidently, $A-B$ is purely elastic and in $B-C$ both elastic and plastic strain increments are produced. If we then return along an arbitrary path $C-A$, the strain changes are purely elastic. According to the second postulate we have

$$(\bar{\sigma}_{ij} - \sigma^o_{ij})d\epsilon_{ij} + d\sigma_{ij}d\epsilon^p_{ij} \geq 0, \tag{6.19}$$

since only the plastic deformation does irrecoverable work. Now, it is possible that $\bar{\sigma}_{ij} = \sigma^o_{ij}$, which will require that,

$$d\sigma_{ij}d\epsilon^p_{ij} \geq 0. \tag{6.20}$$

On the other hand we can envisage a situation where the difference $\bar{\sigma}_{ij} - \sigma^o_{ij}$ is very large so that

$$(\bar{\sigma}_{ij} - \sigma^o_{ij})d\epsilon_{ij} \geq 0. \tag{6.21}$$

The above inequality is called the *Maximum plastic work inequality.*

The above eqs. 6.20 and 6.21 have interesting consequences on the nature of the yield surface and the flow rule. For a loading case $\boldsymbol{n} \cdot d\boldsymbol{\sigma} \geq 0$. This implies that in Fig. 6.7 if $d\boldsymbol{\sigma}$ lies in the wedge ACB, then the first stability postulate $d\sigma_{ij}d\epsilon^p_{ij} \geq 0$, is violated. This implies that the wedge must vanish and so A-A and B-B should coincide. This implies that the direction of flow $\boldsymbol{m} = \boldsymbol{n}$. This is known as an *Associated flow rule'* and is the first major consequence of the stability postulates.

The second consequence stems from the second postulate eq. 6.21. If the yield surface is convex as shown in Fig. 6.8(a), then for all $\boldsymbol{\sigma}^o$ lying on or inside the yield surface, the second postulate is satisfied. If however, the yield surface is concave, then, as shown in

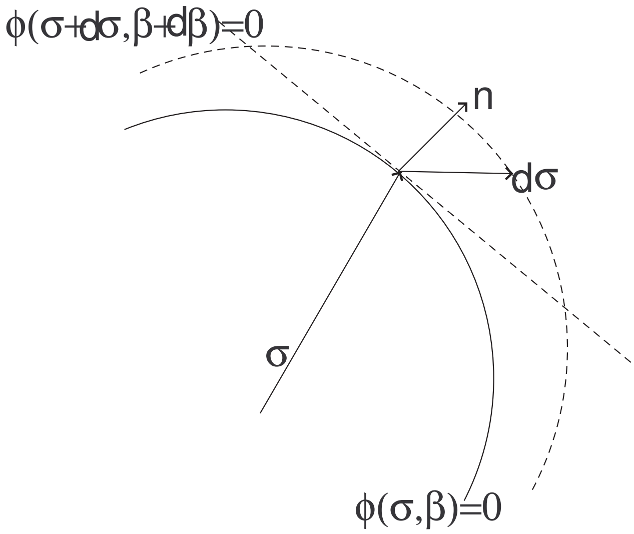


Figure 6.4: Consistency condition

Fig. 6.8 (b), for all $\boldsymbol{\sigma}^o$ lying in the shaded area, the second postulate is violated. So, the yield surface must be convex.

### Consistency condition

For loading cases, the *consistency condition* implies that

$$\Phi(\boldsymbol{\sigma} + d\boldsymbol{\sigma}, \beta_n + d\beta_n) = 0, \tag{6.22}$$

implying

$$\Phi(\boldsymbol{\sigma}, \beta_n) + \frac{\partial \Phi}{\partial \sigma_{ij}} d\sigma_{ij} + \sum_n \frac{\partial \Phi}{\partial \beta_n} d\beta_n = 0. \tag{6.23}$$

Now assume that the evolution of the internal variables is given in the form

$$d\beta_n = f(\boldsymbol{\sigma}, \beta_n, d\boldsymbol{\sigma}) = d\lambda h_n(\boldsymbol{\sigma}, \beta_n). \tag{6.24}$$

Why this is an appropriate choice will become clear later. But using this in conjunction with the consistency condition gives us

$$d\lambda = -\frac{(\partial \Phi / \partial \sigma_{ij}) d\sigma_{ij}}{\sum_n (\partial \Phi / \partial \beta_n) h_n}. \tag{6.25}$$

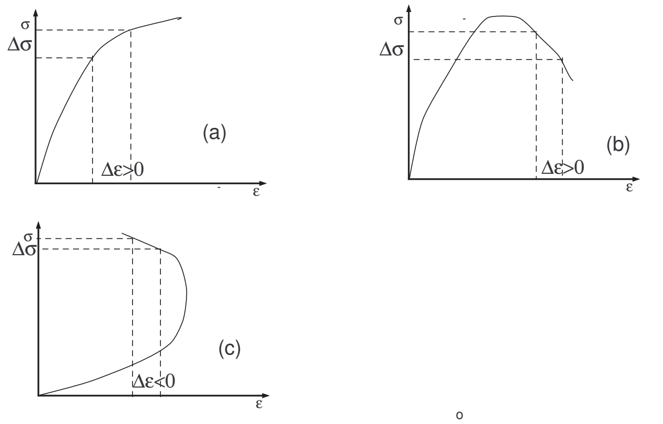


Figure 6.5: Stable and unstable materials

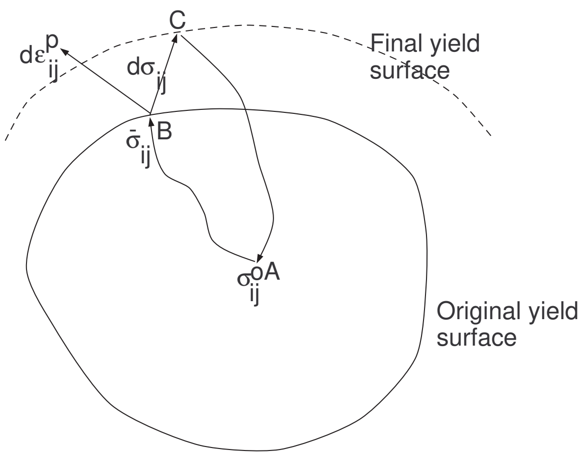


Figure 6.6: Proof of Drucker's postulates

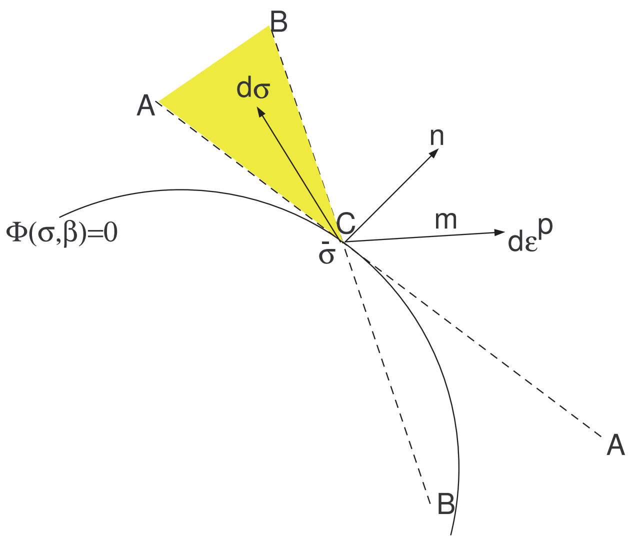


Figure 6.7: Proof of normality of the incremental strain tensor.

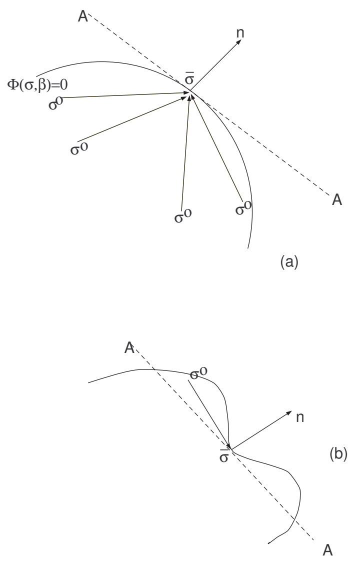


Figure 6.8: Proof of convexity of the yield surface.

Using this in the flow rule,

$$d\epsilon_{ij}^{p} = d\lambda m_{ij} = -\frac{(\partial\Phi/\partial\sigma_{ij})d\sigma_{ij}}{\sum_n(\partial\Phi/\partial\beta_n)h_n}m_{ij}, \tag{6.26}$$

which evidently means

$$d\epsilon_{ij}^{p} = \left[\frac{(\partial\Phi/\partial\sigma_{pq})(\partial\Phi/\partial\sigma_{pq})^{1/2}}{\sum_n(\partial\Phi/\partial\beta_n)h_n}\right](n_{kl}d\sigma_{kl})m_{ij}. \tag{6.27}$$

The term inside the square brackets is written as $1/H$ and so,

$$d\boldsymbol{\epsilon}^{p} = \frac{1}{H}(\boldsymbol{n}\cdot d\boldsymbol{\sigma})\boldsymbol{n}, \tag{6.28}$$

since by the stability postulates the strain increment tensor has to be normal to the yield surface.

### Isotropic elasto-plastic solids

The elastic part of the strain is given in terms of only two independent elastic constants $E$ and $\nu$ by

$$\epsilon_{ij}^{e} = \frac{1}{E}\left[(1+\nu)\sigma_{ij} - \nu\sigma_{kk}\delta_{ij}\right]. \tag{6.29}$$

For isotropic solids the yield function can be given in terms of the invariants of the stress tensor

$$I_1 = \sigma_{kk} \tag{6.30}$$

$$I_2 = \frac{1}{2}(\sigma_{ij}\sigma_{ij} - \sigma_{ii}\sigma_{jj}) \tag{6.31}$$

$$I_3 = det[\boldsymbol{\sigma}]. \tag{6.32}$$

The invariants can be expressed in terms of the principal stresses as

$$I_1 = \sigma_1 + \sigma_2 + \sigma_3 \tag{6.33}$$

$$I_2 = -(\sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1) \tag{6.34}$$

$$I_3 = \sigma_1\sigma_2\sigma_3. \tag{6.35}$$

Recall that the eigenvalue problem for the stress tensor is given by:

$$\sigma^3 - I_1\sigma^2 + I_2\sigma - I_3 = det|\sigma_{ij} - \sigma\delta_{ij}| = 0. \tag{6.36}$$

So, instead of representing the yield condition in the 6-dimensional stress space, it can be represented in the 3-dimensional principal stress space. It should be noted that for an isotropic elastic-plastic solid the yield surface must satisfy some symmetry conditions namely

$$\hat{\Phi}(\sigma_1, \sigma_2, \sigma_3, \beta_n) = \hat{\Phi}(\sigma_2, \sigma_1, \sigma_3, \beta_n) = \hat{\Phi}(\sigma_3, \sigma_1, \sigma_2, \beta_n) = .... \tag{6.37}$$

Extensive experimental evidence has shown that yield is not affected by hydrostatic stress in polycrystalline solids and especially in metals. The imediate consquence of this fact is that the yield function depends only on the deviatoric stress

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}. \tag{6.38}$$

The yield function can now be expressed in terms of the invariants of the deviatoric stress tensor $J_2$ and $J_3$ ($J_1 = 0$), where

$$J_2 = \frac{1}{2}s_{ij}s_{ij} \tag{6.39}$$
$$J_3 = \frac{1}{3}s_{mn}s_{nk}s_{km}. \tag{6.40}$$

The yield function $\hat{\Phi}(J_2, J_3, \beta_n)$ yields itself to nice geometric interpretations. Consider a 3-dimensional space characterised by $\sigma_1, \sigma_2, \sigma_3$ (Fig. 6.9). A plane $\sigma_1 + \sigma_2 + \sigma_3 = 0$ in this space is called the '$\pi$ plane'. If we now define unit vectors $\boldsymbol{e}_i$, then the normal to the $\pi$ plane is given by

$$\boldsymbol{N} = \frac{(\boldsymbol{e}_1 + \boldsymbol{e}_2 + \boldsymbol{e}_3)}{\sqrt{3}}, \tag{6.41}$$

so that, for any stress state lying in the $\pi$ plane $\boldsymbol{\sigma} \cdot \boldsymbol{N} = 0$. The normal $\boldsymbol{N}$ is called the hydrostatic line i.e for any vector parallel to $\boldsymbol{N}$, $\sigma_1 = \sigma_2 = \sigma_3$. Since hydrostatic stress does not influence the process of yielding, the component of any stress state $\bar{OA}$ along $\boldsymbol{N}$

will never intersect the yield surface. This implies that the yield surface is a right cylinder (not necessarily of circular cross section) with generators parallel to $\boldsymbol{N}$.

It further turns out that solids governed by the $J_2, J_3$ based yield criterion are plastically incompressible. This is rather easy to see. Recall that based on Drucker's postulates

$$d\epsilon_{ij}^p = d\lambda n_{ij}. \tag{6.42}$$

Since the volumetric plastic strain is given by

$$d\epsilon_{11}^p + d\epsilon_{22}^p + d\epsilon_{33}^p \tag{6.43}$$

we need to show that $n_{kk} = 0$. Now,

$$\frac{\partial \Phi}{\partial \sigma_{ij}} = \frac{\partial \Phi}{\partial J_2}\frac{\partial J_2}{\partial \sigma_{ij}} + \frac{\partial \Phi}{\partial J_3}\frac{\partial J_3}{\partial \sigma_{ij}}. \tag{6.44}$$

It is easy to see that

$$\frac{\partial J_2}{\partial \sigma_{ij}} = s_{ij}, \quad \text{and} \tag{6.45}$$

$$\frac{\partial J_3}{\partial \sigma_{ij}} = s_{in}s_{jn} - \frac{2}{3}J_2\delta_{ij}. \tag{6.46}$$

This implies that $n_{ii} = 0$ and consequently $\epsilon_{ii}^p = 0$. The preservation of plastic volume has been borne out by various experiments and dislocation theory.

Let us now turn to the incremental theory of plasticity. Though we will

not use it connection with fracture, an understanding will be useful.

Recall the flow rule

$$d\epsilon_{ij}^p = \frac{1}{H}\boldsymbol{n}\cdot d\boldsymbol{\sigma} n_{ij}. \tag{6.47}$$

Using the definition of $n_{ij}$, and the fact that

$$\frac{\partial \Phi}{\partial \sigma_{ij}} = \frac{\partial \Phi}{\partial J_2}\frac{\partial J_2}{\partial \sigma_{ij}} \tag{6.48}$$

we can show that

$$d\epsilon_{ij}^p = d\lambda s_{ij}, \tag{6.49}$$

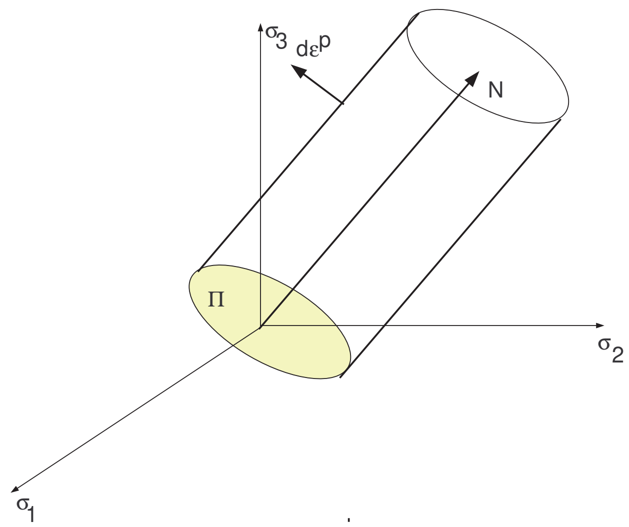


Figure 6.9: Yield surface in the principal stress space.

where

$$d\lambda = \frac{1}{2HJ_2} dJ_2. \tag{6.50}$$

This equation says nothing about the yield condition except that $\Phi(J_2, \beta_n) = 0$. If we now use the von Misces yield condition

$$\Phi(J_2, \beta_n) = J_2 - \tau_y^2(\bar{\epsilon}^p) = 0, \tag{6.51}$$

the only history variable $\beta_n$ is $\bar{\epsilon}^p$, where

$$\bar{\epsilon}^p = \int \sqrt{\frac{2}{3} d\epsilon_{ij}^p d\epsilon_{ij}^p}. \tag{6.52}$$

The usual definition for $J_2$ holds and in terms of the principal stresses

$$J_2 = \frac{1}{6}\left[(\sigma_1 - \sigma_2)^2 + (\sigma_2 - \sigma_3)^2(\sigma_3 - \sigma_1)^2\right]. \tag{6.53}$$

We are now in a position to interpret the history parameter $\tau_y$ physically. In pure shear when $\sigma_{12} = \sigma_{21} = \tau$ and all other stresses are zero, $s_{12} = s_{21} = \tau$. So, under this uniaxial shear, $\tau_y = \tau$. Hence $\tau_y$ is the yield stress in the pure shear test. It is easy to show that in the pure uniaxial tension case $\sigma_y = \tau_y/\sqrt{3}$, which implies that the yield stress in tension is just $\sqrt{3}$ times the yield stress in shear. Geometrically, the von Misces' criterion can be shown to be a cylinder on the $\pi$ plane with radius $\sqrt{2}\tau_y$.

So, now everything in the incremental constitutive equation is known and

$$d\epsilon_{ij} = \frac{1+\nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk}\delta_{ij} + \frac{1}{2J_2H} dJ_2 s_{ij}. \tag{6.54}$$

In the above equation only $H$ needs now to be determined and we do it through uniaxial tests. To do that we specialise the above equation to an uniaxial state of stress. We easily find that in the uniaxial case,

$$\sigma = \sqrt{3J_2} \tag{6.55}$$

and $dJ_2 = (2/3)\sigma d\sigma$. Putting these results in the eq. 6.54 we find that

$$\frac{1}{H} = \frac{3}{2}\left[\frac{1}{E_t} - \frac{1}{E}\right]. \tag{6.56}$$

Here, $E_t = d\sigma/d\epsilon$, which can be found if the stress-strain plot in uniaxial tesnion is known. For example in case of the Ramberg-Osgood case,

$$\frac{1}{E_t} - \frac{1}{E} = \frac{n\alpha\epsilon_0}{\sigma_0}\left(\frac{\sigma}{\sigma_0}\right)^n. \tag{6.57}$$

Then for the general multiaxial case,

$$d\epsilon_{ij} = \frac{1+\nu}{E} d\sigma_{ij} - \frac{\nu}{E} d\sigma_{kk}\delta_{ij} + \frac{3}{4J_2}\left[\frac{1}{E_t} - \frac{1}{E}\right] dJ_2 s_{ij}. \tag{6.58}$$

A simpler theory is the $J_2$ deformation theory of plasticity where the history of deformation is neglected, the material is supposed to behave like a non-linear elastic solid. Then in analogy to the strain energy density function in the elastic case, we can assume the existence of a function $U$ such that

$$\frac{\partial U}{\partial \sigma_{ij}} = \epsilon^p_{ij}. \tag{6.59}$$

In case of a $J_2$ based theory

$$\epsilon^p_{ij} = \frac{\partial U}{\partial J_2}\frac{\partial J_2}{\partial \sigma_{ij}} = \frac{\partial U}{\partial J_2} s_{ij}. \tag{6.60}$$

We again have to turn to an uniaxial test to determine $\partial U/\partial J_2$. By uniaxial tensile calculations similar to what we did for the incremental theory,

$$\frac{\partial U}{\partial J_2} = \frac{3}{2}\left[\frac{1}{E_s} - \frac{1}{E}\right], \tag{6.61}$$

where, $E_s = \sigma/\epsilon^p$. Again, based on the Ramberg-Osgood stress-strain curve

$$\frac{1}{E_s} - \frac{1}{E} = \frac{\epsilon_0}{\sigma_0}\alpha\left(\frac{\sigma}{\sigma_0}\right)^{n-1}, \tag{6.62}$$

and

$$\epsilon^p_{ij} = \frac{3}{2}\frac{\epsilon_0\alpha}{\sigma_0}\left(\frac{\bar{\sigma}}{\sigma_0}\right)^{n-1} s_{ij} \tag{6.63}$$

Generalisation to the multiaxial case has been done simply by replacing $\epsilon^p$ by $\bar{\epsilon}^p$ and $\sigma$ by $\bar{\sigma}$. Under proportional loading, where $s_{ij}(t) = \mu s^0_{ij}$, the incremental and deformation theories coincide. We will use the deformation theory of plasticity to find the state of stress at a crack tip under large scale plastic conditions.

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