Chapter 5

Energy methods in fracture

5.1 Introduction to variational methods

Energy methods in mechanics rely on variational methods which deal with minimisation of *functionals* or function of functions. Let us first see how variational calculus works in one dimensions. Consider a function $f(x)$ of x defined in the interval (a, b). Let us define a quantity Π as

$$\Pi = \int_a^b [f(x)]^2 dx, \tag{5.1}$$

then the value of Π depends on the function $f(x)$ as a whole. The quantity Π is said to be a functional of $f(x)$. Common examples of functionals in mechanics include the kinetic energy or the strain energy of a body.

The basic problem of variational calculus is the following: Given a functional

$$\Pi[f] = \int_a^b F(x, f, f')dx \tag{5.2}$$

where $f(x)$ is a smooth and well behaved function of x satisfying the boundary conditions

$$f(a) = u_0 \ \text{and} f(b) = u_1, \tag{5.3}$$

we try to find the particular function $f(x) = y(x)$ that minimises the functional Π. In other words we seek to find a function $y(x)$ such that

$$\Pi[f] \leq \Pi[y], \tag{5.4}$$

for any function $f(x)$ in the neighbourhood of $y(x)$.

Now suppose that we know $y(x)$ and we construct another function $f(x)$ which is a small perturbation of $y(x)$,

$$f(x) = y(x) + \epsilon\eta(x), \tag{5.5}$$

with

$$\eta(a) = \eta(b) = 0. \tag{5.6}$$

Since ϵ is a small number, this necessarily means that $f(x)$ satisfies the boundary conditions at a and b and $f(x)$ lies in the neighbourhood of $y(x)$. Now introduce

$$\phi(\epsilon) = \Pi[y + \epsilon\eta] = \int_a^b F(x, y(x) + \epsilon\eta(x), y'(x) + \epsilon\eta'(x))dx, \tag{5.7}$$

where, since $y(x)$ is supposed to be known Π reduces to just a function of ϵ. Then according to eq. 5.4, we must have

$$\phi(0) \leq \phi(\epsilon), \tag{5.8}$$

for all small values of ϵ. From elementary calculus we know that the condition for a ϕ attaining a minimum at 0 is

$$\phi'(\epsilon)|_{\epsilon=0} = 0. \tag{5.9}$$

Differentiating eq. under the integral sign, we get

$$\frac{d\phi}{d\epsilon} = \int_a^b \left[\frac{\partial F}{\partial f}\eta + \frac{\partial F}{\partial f'}\eta'\right] dx. \tag{5.10}$$

Integrating the last integral by parts we get,

$$\frac{d\phi}{d\epsilon} = \int_a^b \left[\frac{\partial F}{\partial f}\eta - \frac{d}{dx}\frac{\partial F}{\partial f'}\eta\right] dx + \frac{\partial F}{\partial f'}\eta|_a^b. \tag{5.11}$$

The last term vanishes according to eq. 5.6. Now putting $\epsilon = 0$ in the other term, we find,

$$0 = \int_a^b \left[\frac{\partial F(x, y, y'}{\partial y} - \frac{d}{dx} \frac{\partial F(x, y, y')}{\partial y'} \right] \eta(x) dx, \tag{5.12}$$

which has to be valid for all possible functions $\eta(x)$. This immediately implies that

$$F_y(x, y, y') - \frac{d}{dx} F_{y'}(x, y, y') = 0, \tag{5.13}$$

where the subscripts denote differentiation with respect to y and y'. The above equation is known as the *Euler Lagrange* and is the key equation in variational calculus.

It is customary in mechanics to denote $\epsilon\eta(x)$ by $\delta f(x)$ or simply the 'variation of $f(x)$'. Using this notation, we can define the first variation of Π as

$$\delta\Pi = \epsilon\phi'(\epsilon), \tag{5.14}$$

and evidently this can be written as,

$$\delta\Pi = \int_a^b \left[\frac{\partial F(x, y, y')}{\partial f} \eta - \frac{d}{dx} \frac{\partial F(x, y, y')}{\partial f'} \delta f \right] dx + \frac{\partial F(x, y, y'}{\partial f'} \delta f|_a^b, \tag{5.15}$$

which is analogus to the notation of differential calculus. It has to be kept in mind that δf is the variation of a function and mathematically is not the same thing as dx for example. The condition for attainment of an extremum is $\delta\Pi = 0$ for all variations δf with $\delta f(a) = \delta f(b) = 0$.

As an example consider the functional

$$\Pi[u] = \int_0^{\pi/2} [(y')^2 - y^2] dx, \quad y(0) = 0 \text{ and } y(\pi/2) = 1. \tag{5.16}$$

Suppose we want to find the function $u(x)$ that minimises Π. Taking $F(x, y, y') = (y')^2 - y^2$, the Euler-Lagrange equation gives

$$\frac{d^2y}{dx^2} + y = 0, \quad \text{subject to } y(0) = 0 \text{ and } y(\pi/2) = 1. \tag{5.17}$$

This is the differential equation corresponding to the variational statement eq. 5.16.

The Euler-Lagrange equation 5.13 can be generalised to the case where F depends on derivatives higher than y'. The general Euler-Lagrange condition for the case where derivatives upto the nth are involved now reads:

$$\sum_{\nu=0}^{n}(-1)^{\nu}\frac{d^{\nu}}{dx^{\nu}}\left[\frac{\partial}{\partial y^{(\nu)}}F(x,y,y^{(1)},....,y^{(n)})\right]=0. \tag{5.18}$$

The notations δu and $\delta\Pi$ can obviously be used in this case as well.

Another general case arises when F depends on several unknown functions rather than just one, i.e.

$$\Pi=\int_a^b F(x,y_1,y_2,...y_m;y_1',y_2',....y_m')dx. \tag{5.19}$$

In this case we get a set of m Euler Lagrange equations, one for each variable y_i:

$$F_{y_\mu}-\frac{d}{dx}F_{y_\mu'}=0,\quad \mu=1,...,m. \tag{5.20}$$

A final general case arises when F depends on several independent variables rather than just x. Restricting ourselves to just two independent variable x and y (not to be confused with $y(x)$ in the previous expressions for Π),

$$\Pi=\int\int_G F(x,y,u(x,y),u_x,u_y)dxdy, \tag{5.21}$$

where G is a closed domain in the $x-y$ plane with a boundary curve C. The Euler-Lagrange equations now become a bit more complicated. Let us first recall that,

$$\frac{d}{dx}F_{u_x}=\frac{\partial}{\partial x}F_{u_x}+\frac{\partial F_{u_x}}{\partial u}\frac{\partial u}{\partial x}+\frac{\partial F_{u_x}}{\partial u_x}\frac{\partial u_x}{\partial x}+\frac{\partial F_{u_x}}{\partial u_y}\frac{\partial u_y}{\partial x}, \tag{5.22}$$

and in a similar fashion, d/dy also. Then the Euler Lagrange equation reads

$$F_u-\frac{d}{dx}F_{u_x}-\frac{d}{dy}F_{u_y}=0. \tag{5.23}$$

An example would probably make things a little clearer. Consider the following variational statement

$$\Pi[u]=\int\int_D\left[u_x^2+u_y^2\right]dxdy, \tag{5.24}$$

in a domain D. Using eq. 5.23 we get

$$F_u = 0 \quad \frac{d}{dx} F_{u_x} = 2\frac{\partial u_x}{\partial x} \quad \frac{d}{dy} F_{u_y} = 2\frac{\partial u_y}{\partial y}, \tag{5.25}$$

which implies that $u(x, y)$ must satisfy the harmonic equation

$$u_{xx} + u_{yy} = 0. \tag{5.26}$$

Similarly, $F = (u_{xx}^2 + u_{yy}^2 + 2u_{xy})^2$ will lead to the biharmonic equation that we encountered in Chapter 2.

5.2 Theorem of Minimum Potential energy

Consider a body in static equilibrium under the action of specified surface and body forces. The boundary surface S of the body is consists of two parts S_t and S_u on which tractions and displacements are specified (see Fig. **??**). Let us assume that we have a displacement field u_i that satisfies the boundary conditions on S_u and lead to stresses that satisfy the equilibrium conditions. Now consider a perturbation on the actual displacement field u_i of the form $u_i + \delta u_i$, where δu_i is completely arbitrary except that it vanishes on S_u. Consider the *virtual work* done by the surface and body forces

$$\int_V b_i \delta u_i dV + \int_S T_i \delta u_i dS, \tag{5.27}$$

where $T_i = \sigma_{ij} n_j$ according to the Cauchy theorem. Substituting the Cauchy relation into eq. 5.27 and using the Gauss theorem for converting surface integrals into volume integrals, we get,

$$\begin{aligned} \int_S T_i \delta u_i dS &= \int_S \sigma_{ij} n_j \delta u_i dS \\ &= \int_V (\sigma_{ij} \delta u_i)_{,j} dV \\ &= \int_V \sigma_{ij,j} \delta u_i dV + \int_V \sigma_{ij} \delta u_{i,j} dV. \end{aligned} \tag{5.28}$$

The first term in the above equation is equal to $-\int_V b_i \delta u_i dV$. On account of the symmetry of σ_{ij}, the second term can be written as

$$\int_V \sigma_{ij} \left[\frac{1}{2}(\delta u_{i,j} + \delta u_{j,i})\right] dV = \int_V \sigma_{ij} \delta e_{ij} dV. \tag{5.29}$$

Therefore, eq. **??** becomes

$$\int_V b_i \delta u_i dV + \int_S T_i \delta u_i dS = \int_V \sigma_{ij} \delta e_{ij} dV. \tag{5.30}$$

The eq. 5.30 is known as the *Principle of virtual work* and is remarkable because δu_i is an arbitrary function with the only condition that it should vanish over S_u.

Now consider a class of materials for which a strain energy function $W(e_{ij})$ exists, so that

$$\sigma_{ij} = \frac{\partial W}{\partial e_{ij}}. \tag{5.31}$$

Introducing this on the right side of eq. 5.30 we get,

$$\int_V \sigma_{ij} \delta e_{ij} dV = \int_V \frac{\partial W}{\partial e_{ij}} \delta e_{ij} dV = \delta \int_V W dV. \tag{5.32}$$

In addition, if the body forces and tractions are conservative so that they can be obtained in terms of potentials G and g respectively, i.e.

$$\begin{aligned} b_i &= \frac{\partial G}{\partial u_i} \\ T_i &= \frac{\partial g}{\partial u_i}, \end{aligned} \tag{5.33}$$

then the virtual work equation can be written as

$$\delta \mathcal{V} = 0, \tag{5.34}$$

where,

$$\mathcal{V} = \int_V (W + G) dV + \int_S g dS. \tag{5.35}$$

The function $\mathcal{V}$ is called the *poential energy* of the system. This equation states that of all the displacement fields satisfying the boundary conditions, those that satisfy the equilibrium equation are distinguished by the fact that they make the potential energy of the system a *extremum*. Another important point to note is that the derivation of eq. 5.35 has not taken the linearity of the material into consideration. If the function W exists then, irrespective of the nature of the material, the eq. 5.35 is valid.

It will now be shown that this extremum is indeed a minimum given the positive definiteness of the function W. Let us compare the potential energy $\mathcal{V}$ of a system with the actual displacements u_i with that of another system $\mathcal{V}'$ which has displacements $u_i + \delta u_i$.

$$\mathcal{V}' - \mathcal{V} = \int_V [W(e_{ij} + \delta e_{ij}) - W(e_{ij})]dV - \int_V b_i \delta u_i dV - \int_S T_i \delta u_i dS. \tag{5.36}$$

We can now expand $W(e_{ij} + \delta e_{ij})$ as

$$W(e_{ij} + \delta e_{ij}) = W(e_{ij}) + \frac{\partial W}{\partial e_{ij}} \delta e_{ij} + \frac{1}{2} \frac{\partial^2 W}{\partial e_{ij} \partial e_{kl}} \delta e_{ij} \delta e_{kl} + ... \tag{5.37}$$

which leads to

$$\mathcal{V}' - \mathcal{V} = \int_V \frac{\partial W}{\partial e_{ij}} dV - \int_V b_i \delta u_i dV - \int_S T_i \delta u_i dS + \int_V \frac{1}{2} \frac{\partial^2 W}{\partial e_{ij} \partial e_{kl}} \delta e_{ij} \delta e_{kl} dV. \tag{5.38}$$

From the eq. 5.30 we see that all terms on the right hand side except the last term vanish.

Now from eq. 5.37 we see that as $e_{ij} \to 0$, the linear term vanishes, (the constant term is immaterial anyway) and so, to second order,

$$\frac{1}{2} \frac{\partial^2 W}{\partial e_{ij} \partial e_{kl}} \delta e_{ij} \delta e_{kl} = W(\delta e_{ij}). \tag{5.39}$$

Therefore, from eq. 5.38 we get,

$$\mathcal{V}' - \mathcal{V} = \int_V W(\delta e_{ij}) dV, \tag{5.40}$$

which is positive if W is positive definite and so,

$$\mathcal{V}' - \mathcal{V} \geq 0. \tag{5.41}$$

This proves that $\mathcal{V}$ is a minimum. Accordingly our principle is called *the principle of minimum potential energy.*