Lecture I

Introduction, concept of solutions, application

Definition 1. *A differential equation (DE) is a relation that contains a finite set of functions and their derivatives with respect to one or more independent variables.*

Definition 2. *An ordinary differential equation (ODE) is a relation that contains derivatives of functions of only one variable.*

If the number of functions is one, then it is called simple or scalar ODE. Otherwise, we have a system of ODEs.

Note: ODEs are distinguished from partial differential equations (PDEs), which contain partial derivatives of functions of more than one variable.

From now on, we shall mostly deal with simple or scalar ODE.

Definition 3. *The highest derivative that appear in a ODE is the order of that ODE.*

Notations: A general simple ODE of order n can be written as

$$F\left(x, y(x), y^{(1)}(x), y^{(2)}(x), \cdots, y^{(n)}(x)\right) = 0. \tag{1}$$

If the order of the ODE is small, then we shall use $'$ for the derivative. For example, a second order ODE is written as

$$F(x, y, y', y'') = 0.$$

Definition 4. *If the ODE (1) can be written as*

$$a_0(x)y^{(n)}(x) + a_1(x)y^{(n-1)}(x) + \cdots + a_{n-1}(x)y^{(1)}(x) + a_n(x)y(x) = F(x), \tag{2}$$

then the given ODE is linear. If such representation is not possible, then we say that the given ODE is nonlinear.

If the function $F(x)$ is zero, then (2) is called linear homogeneous ODE. If the functions $a_0(x), a_1(x), \cdots, a_n(x)$ are constants, then (2) is called linear ODE with constant coefficients. Similarly, if the functions $a_0(x), a_1(x), \cdots, a_n(x)$ are constants and $F(x)$ is zero, then (2) is called linear homogeneous ODE with constant coefficients.

Example 1. *Both $y''+y^2 = 2$ and $y''+\cos y = 0$ are nonlinear, whereas both $y''+e^x y = x^2$ and $e^x = y'/(\cos x + y'')$ are linear.*

Example 2. *A pendulam is released from rest when the supporting string is at angle α to the vertical. If the air resistence is negligible, find the ODE describing the motion of the pendulam.*

Solution: Let s be the arc length measured from the lowest point as shown in Figure 1. If l is the length of the string, then $s = l\theta$. Now the velocity v and accleration a is given by

$$v = \frac{ds}{dt} = l\frac{d\theta}{dt}, \qquad a = \frac{d^2s}{dt^2} = l\frac{d^2\theta}{dt^2}.$$

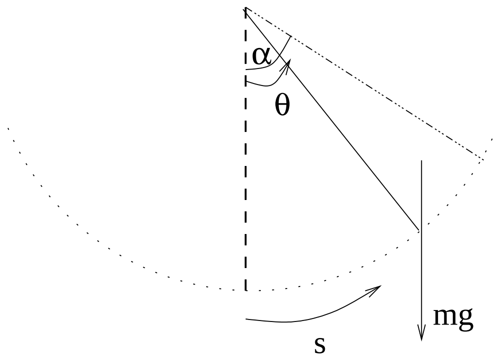

Figure 1: Motion of a pendulam.

Resolving the gravity force, we find

$$\frac{d^2\theta}{dt^2} + \kappa \sin\theta = 0, \tag{3}$$

where $\kappa = g/l$. Note that (3) is of second order and nonlinear. Usually, the nonlinear ODEs are very difficult to solve analytically.

Example 3. *Consider a pond in which only two types of fishes, A and B, live. Fish A feeds on phytoplankton whereas fish B survives by eating A. We are interested to model the populations growths of both the fishes. A model for this kind of Predator-Prey interaction is popularly known as the Lotka–Volterra system.*

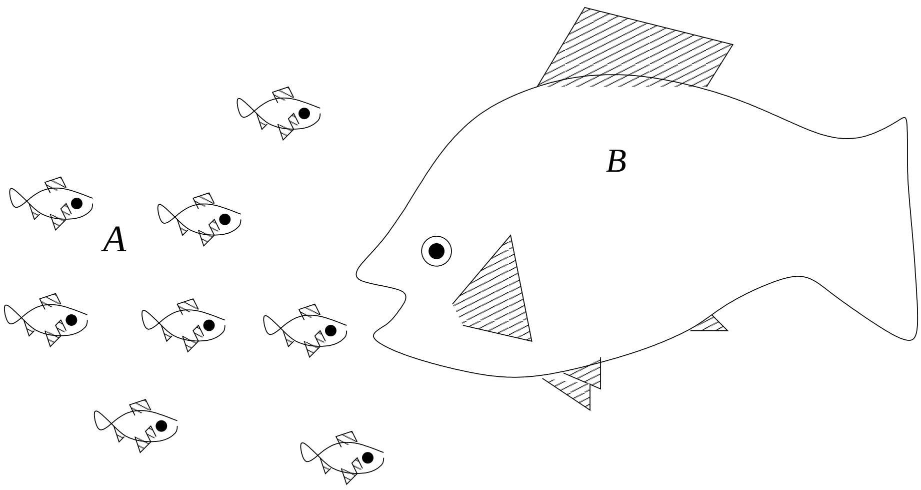

Figure 2: Predator-prey interactions

Solution: In absence of B, fish population A will increase and the rate of increase is proportional to number density of A. Similarly, in absence of A, fish population B will decrease and the rate of decrease is proportional to the number density of B. When both are present, A will decrease and B will increase. This rate of change, as a first approximation, can be taken as proportional to the product of number density of A and B. Thus, we get the following system of ODEs:

$$\begin{aligned}\frac{dA}{dt} &= \alpha A - \beta AB \\ \frac{dB}{dt} &= -\gamma B + \delta AB,\end{aligned}$$

where α, β, γ and δ are positive constants.

Definition 5. *A functon ϕ is said to be the solution of the ODE (1) in an interval $\mathcal{I}$ if ϕ is n times differentiable in $\mathcal{I}$ and*

$$F\left(x, \phi(x), \phi^{(1)}(x), \phi^{(2)}(x), \cdots, \phi^{(n)}(x)\right) = 0, \qquad \forall x \in \mathcal{I}.$$

Example 4. $y = x^2 + c,\ x \in (-\infty, \infty)$ *(c is an arbitray constant) is a solution to* $y' = 2x$. *On the other hand* $y = 1/(1-x),\ x \in (-\infty, 1)$ *or* $x \in (1, \infty)$ *is a solution to* $y' = y^2$. *Note that the last solution is not valid in any interval which contains* $x = 1$.

In general, we are interested to know whether a given ODE, under certain circumstances, has a unique solution or not. Usually, this question can be answered with some extra conditions such as initial conditions. Thus, we consider the so called *initial value problem* (IVP) which takes the following form for a first order ODE:

$$\left.\begin{aligned}&y'(x) = f\big(x, y(x)\big), \quad x \in \mathcal{I}, \\ &y(x_0) = y_0, \quad x_0 \in \mathcal{I},\ y_0 \in \mathcal{J},\end{aligned}\right\} \tag{4}$$

where $\mathcal{I}$ and $\mathcal{J}$ are some intervals.

For higher order ODE, the initial conditions are also applied on the lower derivatives of the function. For example, the IVP for Example 2 can be written as

$$\left.\begin{aligned}&\frac{d^2\theta}{dt^2} + \kappa \sin\theta = 0, \\ &\theta(0) = \alpha, \quad \theta'(0) = 0.\end{aligned}\right\} \tag{5}$$

Note that here, the notation $\theta(0)$ means $\theta(t = 0)$ and similar is the case for $\theta'(0)$.

Definition 6. *A* general solution *of an ODE of order n is a solution which is given in terms of n independent parameters. A* particular solution *of an ODE is a solution which is obtained from the general solution by choosing specific values of independent parameters.*

Example 5. *(i) Consider* $y' = 2x$. *Clearly* $y = C + x^2$ *is the general solution. If we choose* $C = 2$, *then* $y = 2 + x^2$ *is a particular solution. This particular solution can be thought of as the solution of the IVP* $y' = 2x,\ y(0) = 2$ **<u>OR</u>** $y' = 2x,\ y(1) = 3$ **<u>OR</u>** $\cdots$. *Here the particular solution is valid for all* x.

(ii) Consider $y'' - 4y' + 4y = 0$. *For this,* $y(x) = (C_1 + C_2 x)e^{2x}$ *is the general solution, whereas* $y_p(x) = xe^{2x}$ *is a particular solution obtained by choosing* $C_1 = 0, C_2 = 1$. *The conditions on* C_1 *and* C_2 *are equivalent to initial conditions* $y(0) = 0, y'(0) = 1$. *This particular solution is valid for all* x.

(iii) Clearly $y = 1/(C-x)$ is a genreal solution to $y' = y^2$. Now $y = 1/(1-x)$ is a particular solution to the IVP $y' = y^2, y(0) = 1$, which is valid in $(-\infty, 1)$. But $y = -1/(1+x)$ is a particular solution to the IVP $y' = y^2, y(0) = -1$, which is valid in $(-1, \infty)$.
Why the solution $y = -1/(1+x)$ of $y' = y^2, y(0) = -1$ is not valid in $(-\infty, -1)$?

Comment: *An ODE may sometimes have an additional solution that can not be obtained from the general solution. Such a solution is called singular solution.*

Example 3: $y'^2 - xy' + y = 0$ has the general solution $y = cx - c^2$ (verify!). It also has a solution $y_s(x) = x^2/4$ that cannot be obtained from the general solution by choosing specific values of c. Hence, the later solution is a singular solution.

Comment: The solutions $y(x)$ of an ODE often are given *implicitly*. For example, $xy + \ln y = 1$ is an implicit solution of the ODE $(xy+1)y' + y^2 = 0$.

Application: A stone of mass m is dropped from a height H above the ground. Suppose the air resistence is proportional to the speed Derive the ODE that describe the equations of motion.

Solution: Suppose the distance to the ball is measured from the ground. Now the velocity $v = dy/dt$ is positive upward. The air resistence acts in the opposite direction to the motion. Hence,

$$m\frac{d^2y}{dt^2} = -mg - k\frac{dy}{dt}, \qquad y(0) = H, y'(0) = 0. \tag{6}$$

Note that the gravity and the air resistence both oppose the motion if the stone is traveling up. On the other hand, gavity aids and resistence opposes if the stone is traveling down. (What will be the form of the above IVP if the distance y is measured from height H in the downward direction?)

Clearly, (6) is a linear second order equation with constant coefficients. This can be solved using the method to be developed later. Using $v = dy/dt$, (6) becomes a first order ODE:

$$\frac{dv}{dt} + \alpha v = -g, \qquad v(0) = 0, \tag{7}$$

where $\alpha = k/m$. This equation can be solved easily for v. Once, v is found, y is

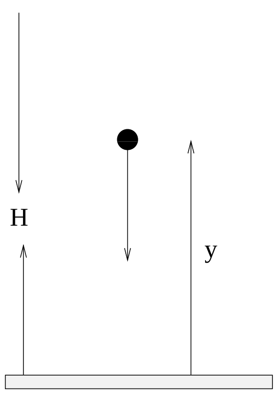

Figure 3: Motion of a falling stone.

obtained from

$$\frac{dy}{dt} = v, \qquad y(0) = H.$$

Even without solving, we can derive some important informations. For example, the stone will attain a terminal velocity $v_t = -g/\alpha$ (why negative sign?) provided H is large enough. (Is it possible for a stone to attain terminal velocity when it is moving up?)

Comment: Care must be taken if the air resistence is proportional to square of the speed. Using the same coordinate system, the governing equation becomes

$$m\frac{d^2y}{dt^2} = -mg + k\left(\frac{dy}{dt}\right)^2, \qquad y(0) = H, y'(0) = 0. \tag{8}$$

The terminal velocity in this case becomes $v_t = -\sqrt{g/\alpha}$.