Lecture XI
Euler-Cauchy Equation

1 Homogeneous Euler-Cauchy equation

If the ODE is of the form

$$ax^2y'' + bxy'' + cy = 0, \tag{1}$$

where a, b and c are constants; then (1) is called homogeneous Euler-Cauchy equation. Two linearly independent solutions (i.e. basis) depend on the quadratic equation

$$am^2 + (b-a)m + c = 0. \tag{2}$$

Equation (2) is called *characteristic equation* for (1). The ODE (1) is singular at $x = 0$. Hence, we solve (1) for $x \neq 0$. We consider the case when $x > 0$.

Theorem 1. *(i) If the roots of (2) are real and distinct, say m_1 and m_2, then two linearly independent (LI) solutions of (1) are x^{m_1} and x^{m_2}. Thus, the general solution to (1) is*

$$y = C_1x^{m_1} + C_2x^{m_2}.$$

(ii) If the roots of (2) are real and equal, say $m_1 = m_2 = m$, then two LI solutions of (1) are x^m and $x^m \ln x$. Thus, the general solution to (1) is

$$y = (C_1 + C_2 \ln x)x^m.$$

(iii) If the roots of (2) are complex conjugate, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then two real LI solutions of (1) are $x^\alpha \cos(\beta \ln x)$ and $x^\alpha \sin(\beta \ln x)$. Thus, the general solution to (1) is

$$y = x^\alpha\Big(C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)\Big).$$

Proof: We have seen that the trial solution for a constant coefficient equation is e^{mx}. Now since power of x^m is reduced by 1 by a differentiation, let us take x^m as trial solution for (1).

For convenience, (1) is written in the operator form $L(y) = 0$, where

$$L \equiv ax^2\frac{d^2}{dx^2} + bx\frac{d}{dx} + c.$$

We also sometimes write L as

$$L \equiv ax^2D^2 + bxD + c,$$

where $D = d/dx$. Now

$$L(x^m) = \Big(am(m-1) + bm + c\Big)x^m = p(m)x^m, \tag{3}$$

where $p(m) = am^2 + (b-a)m + c$. Thus, x^m is a solution of (1) if $p(m) = 0$.

(i) If $p(m) = 0$ has two distinct real roots m_1, m_2, then both x^{m_1} and x^{m_2} are solutions of (1). Since, $m_1 \neq m_2$, they are also LI. Thus, the general solution to (1) is

$$y = C_1x^{m_1} + C_2x^{m_2}.$$

Example 1. *Solve* $x^2y'' - xy' - 3y = 0$

Solution: The characteristic equation is $m^2 - 2m - 3 = 0 \Rightarrow m = -1, 3$. The general solution is $y = C_1/x + C_2x^3$

(ii) If $p(m) = 0$ has real equal roots $m_1 = m_2 = m$, then x^m is a solution of (1). To find the other solution, note that if m is repeated root, then $p(m) = p'(m) = 0$. This suggests differentiating (3) w.r.t. m. Since L consists of differentiation w.r.t. x only,

$$\frac{\partial}{\partial m}\Big(L(x^m)\Big) = L\left(\frac{\partial}{\partial m}x^m\right) = L(x^m \ln x).$$

Now

$$L(x^m \ln x) = \Big(p'(m) + p(m)\ln x\Big)x^m,$$

where $'$ represents the derivative. Since, m is a repeated root, the RHS is zero. Thus, $x^m \ln x$ is also a solution to (1) and it is independent of x^m. Hence, the general solution to (1) is

$$y = (C_1 + C_2 \ln x)x^m.$$

$\Big($We can also use method of reduction of oder technique i.e. $y_1 = x^m$ and $y_2 = v(x)y_1 = v(x)x^m$. From the given ODE, we find

$$ax^2v'' + (2am + b)xv' + \Big\{am^2 + (b - a)m + c\Big\}\, v = 0$$

Since $m = m_1 = m_2$ is a double root, we must have $am^2 + (b - a)m + c = 0$ and $m = -(b - a)/2a \Rightarrow 2am + b = a$. Hence,

$$ax^2v'' + axv' = 0 \Rightarrow v'' = -\frac{v'}{x} \Rightarrow v' = \frac{1}{x} \Rightarrow v = \ln x$$

Hence $y_2 = x^m \ln x\Big)$

Example 2. *Solve* $x^2y'' - 3xy' + 4y = 0$

Solution: The characteristic equation is $m^2 - 4m + 4 = 0 \Rightarrow m = 2, 2$. The general solution is $y = (C_1 + C_2 \ln x)x^2$.

(iii) If $p(m) = 0$ has complex conjugate roots, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then two LI solutions are

$$Y_1 = x^{(\alpha+i\beta)} = x^\alpha e^{i\beta \ln x}, \quad \text{and} \quad Y_2 = x^\alpha e^{-i\beta \ln x}.$$

But these are complex valued. Note that if Y_1, Y_2 are LI, then so are $y_1 = (Y_1 + Y_2)/2$ and $y_2 = (Y_1 - Y_2)/2i$. Hence, two real LI solutions of (1) are $y_1 = x^\alpha \cos(\beta \ln x)$ and $y_2 = x^\alpha \sin(\beta \ln x)$. Thus, the general solution to (1) is

$$y = x^\alpha\Big(C_1 \cos(\beta \ln x) + C_2 \sin(\beta \ln x)\Big).$$

Example 3. *Solve* $x^2y'' - 3xy' + 5y = 0$

Solution: The characteristic equation is $m^2 - 4m + 5 = 0 \Rightarrow m = 2 \pm i$. The general solution is $y = x^2\Big(C_1 \cos(\ln x) + C_2 \sin(\ln x)\Big)$

Comment 1: The solution for $x < 0$ can be obtained from that of $x > 0$ by replacing x by $-x$ everywhere.

Comment 2: Homogeneous Euler-Cauchy equation can be transformed to linear constant coefficient homogeneous equation by changing the independent variable to $t = \ln x$ for $x > 0$.

Comment 3: This can be generalized to equations of the form

$$a(\gamma x + \delta)^2 y'' + b(\gamma x + \delta)y' + cy = 0.$$

In this case we consider $(\gamma x + \delta)^m$ as the trial solution.

2 Nonhomogeneous Euler-Cauchy equation

If the ODE is of the form

$$ax^2 y'' + bxy'' + cy = \tilde{r}(x), \tag{4}$$

where a, b and c are constants; then (4) is called nonhomogeneous Euler-Cauchy equation. We can use the method of variation of parameters as follows. First divide (4) by ax^2 so that the coefficient of y'' becomes unity:

$$y'' + \frac{b}{ax} y'' + \frac{c}{ax^2} y = r(x), \tag{5}$$

where $r(x) = \tilde{r}(x)/ax^2$. Now we already know two LI solutions y_1, y_2 of the homogeneous part. Hence, the particular solution to
(4) is given by

$$y_p(x) = -y_1(x) \int \frac{y_2(x) r(x)}{W(y_1, y_2)} dx + y_2(x) \int \frac{y_1(x) r(x)}{W(y_1, y_2)} dx.$$

Thus, the general solution to (4) is

$$y(x) = C_1 y_1(x) + C_2 y_2(x) + y_p(x).$$

Comment: In few cases, it can be solved also using method of undetermined coefficients. For this, we first convert it to constant coefficient liner ODE by $t = \ln x$. If the the transformed RHS is of special form then the method of undetermined coefficients is applicable.

Example 4. *Consider*

$$x^2 y'' - xy' - 3y = \frac{\ln x}{x}, \qquad x > 0.$$

The characteristic equation is $m^2 - 2m - 3 = 0 \Rightarrow m = -1, 3$. Hence $y_1 = 1/x$ and $y_2 = x^3$. Hence,

$$y_p(x) = y_1(x) u(x) + y_2(x) v(x)$$

where
$$u(x) = -\int \frac{y_2(x)r(x)}{W(y_1,y_2)}dx, \quad v(x) = \int \frac{y_1(x)r(x)}{W(y_1,y_2)}dx.$$
Now $W(y_1, y_2) = 4x$ and $\boxed{r(x) = \ln x/x^3}$! Hence,
$$u(x) = -\int \frac{\ln x}{4x}dx = -\frac{(\ln x)^2}{8}$$
$$v(x) = \int \frac{\ln x}{4x^5}dx = -\frac{\ln x}{16x^4} - \frac{1}{64x^4}$$
Hence,
$$y_p(x) = -\frac{(\ln x)^2}{8x} - \frac{\ln x}{16x} - \frac{1}{64x}$$
Hence the general solution is $y = c_1y_1 + c_2y_2 + y_p$, i.e.
$$y(x) = \frac{A}{x} + Bx^3 - \frac{(\ln x)^2}{8x} - \frac{\ln x}{16x}.$$
Note that last term of y_p is absorbed with y_1.

Aliter: Let us make the transformation $t = \ln x$. Let $y(x) = y(e^t) = u(t)$. Then the given transformed to
$$\ddot{u} - 2\dot{u} - 3u = te^{-t},$$
where $\dot{} = d/dt$. This is the same problem we have solved in lecture 9 using method of undetermined coefficients. The solution is (see lecture 9)
$$u(t) = C_1e^{-t} + C_2e^{3t} - \frac{te^{-t}}{16}(2t+1) = y(e^t),$$
which in terms of original x variable becomes
$$y(x) = \frac{C_1}{x} + C_2x^3 - \frac{\ln x}{16x}(2\ln x + 1),$$