Lecture XVI

Strum comparison theorem, Orthogonality of Bessel functions

1 Normal form of second order homogeneous linear ODE

Consider a second order linear ODE in the standard form

$$y'' + p(x)y' + q(x)y = 0. \tag{1}$$

By a change of dependent variable, (1) can be written as

$$u'' + Q(x)u = 0, \tag{2}$$

which is called the normal form of (1).
To find the transformation, let use put $y(x) = u(x)v(x)$. When this is substituted in (1), we get

$$vu'' + (2v' + pv)u' + (v'' + pv' + qv)u = 0.$$

Now we set the coefficient of u' to zero. This gives

$$2v' + pv = 0 \Rightarrow v = e^{-\int p/2\,dx}.$$

Now coefficient of u becomes

$$\left(q(x) - \frac{1}{4}p^2 - \frac{1}{2}p'\right) v = Q(x)v.$$

Since v is nonzero, cancelling v we get the required normal form. Also, since v never vanishes, u vanishes if and only if y vanishes. Thus, the above transformation has no effect on the zeros of solution.

Example 1. *Consider the Bessel equation of order $\nu \geq 0$:*

$$x^2y'' + xy' + (x^2 - \nu^2)y = 0, \qquad x > 0.$$

Solution: Here $v = e^{-\int x/2\,dx} = 1/\sqrt{x}$. Now

$$Q(x) = 1 - \frac{\nu^2}{x^2} - \frac{1}{4x^2} + \frac{1}{2x^2} = 1 + \frac{1/4 - \nu^2}{x^2}.$$

Thus, Bessel equation in normal form becomes

$$u'' + \left(1 + \frac{1/4 - \nu^2}{x^2}\right) u = 0. \tag{3}$$

Theorem 1. (Strum comparison theorem) *Let ϕ and ψ be nontrivial solutions of*

$$y'' + p(x)y = 0, \qquad x \in \mathcal{I},$$

and

$$y'' + q(x)y = 0, \qquad x \in \mathcal{I},$$

where p and q are continuous and $p \leq q$ on $\mathcal{I}$. Then between any two consecutive zeros x_1 and x_2 of ϕ, there exists at least one zero of ψ unless $p \equiv q$ on (x_1, x_2).

Proof: Consider x_1 and x_2 with $x_1 < x_2$. WLOG, assume that $\phi > 0$ in (x_1, x_2). Then $\phi'(x_1) > 0$ and $\phi'(x_2) < 0$. Further, suppose on the contrary that ψ has no zero on (x_1, x_2). Assume that $\psi > 0$ in (x_1, x_2). Since ϕ and ψ are solutions of the above equations, we must have

$$\phi'' + p(x)\phi = 0,$$

$$\psi'' + q(x)\psi = 0.$$

Now multiply first of these by ψ and second by ϕ and subtracting we find

$$\frac{dW}{dx} = (q - p)\phi\psi,$$

where $W = \phi\psi' - \psi\phi'$ is the Wronskian of ϕ and ψ. Integrating between x_1 and x_2, we find

$$W(x_2) - W(x_1) = \int_{x_1}^{x_2} (q - p)\phi\psi \, dx.$$

Now $W(x_2) \leq 0$ and $W(x_1) \geq 0$. Hence, the left hand side $W(x_2) - W(x_1) \leq 0$. On the other hand, right hand side is strictly greater than zero unless $p \equiv q$ on (x_1, x_2). This contradiction proves that between any two consecutive zeros x_1 and x_2 of ϕ, there exists at least one zero of ψ unless $p \equiv q$ on (x_1, x_2).

Proposition 1. *Bessel function of first kind J_ν ($\nu \geq 0$) has infinitely number of positive zeros.*

Proof: The number of zeros J_ν is the same as that of nontrivial u that satisfies (3), i.e.

$$u'' + \left(1 + \frac{1/4 - \nu^2}{x^2}\right) u = 0. \tag{4}$$

Now for large enough x, say x_0, we have

$$\left(1 + \frac{1/4 - \nu^2}{x^2}\right) > \frac{1}{4}, \qquad x \in (x_0, \infty). \tag{5}$$

Now compare (4) with

$$v'' + \frac{1}{4}v = 0. \tag{6}$$

Due to (5), between any two zeros of a nontrivial solution of (6) in (x_0, ∞), there exists at least one zero of nontrivial solution of (4). We know that $v = \sin(x/2)$ is a nontrivial solution of (6), which has infinite number of zeros in (x_0, ∞). Hence, any nontrivial solution of (4) has infinite number of zeros in (x_0, ∞). Thus, J_ν has infinite number of zeros in (x_0, ∞), i.e. J_ν has infinitely number of positive zeros. We label the positive zeros of J_ν by λ_n, thus $J_\nu(\lambda_n) = 0$ for $n = 1, 2, 3, \cdots$.

2 Orthogonality of Bessel function J_ν

Proposition 2. (Orthogonality) *The Bessel functions J_ν ($\nu \geq 0$) satisfy*

$$\int_0^1 x J_\nu(\lambda_m x) J_\nu(\lambda_n x) \, dx = \frac{1}{2}\Big(J_{\nu+1}(\lambda_n)\Big)^2 \delta_{mn}, \tag{7}$$

where λ_i are the positive zeros of J_ν, and $\delta_{mn} = 0$ for $m \neq n$ and $\delta_{mn} = 1$ for $m = n$.

Proof: We know that $J_\nu(x)$ satisfies

$$y'' + \frac{1}{x}y' + \left(1 - \frac{\nu^2}{x^2}\right)y = 0.$$

If $u = J_\nu(\lambda x)$ and $v = J_\nu(\mu x)$, then u and v satisfies

$$u'' + \frac{1}{x}u' + \left(\lambda^2 - \frac{\nu^2}{x^2}\right)u = 0, \tag{8}$$

and

$$v'' + \frac{1}{x}v' + \left(\mu^2 - \frac{\nu^2}{x^2}\right)v = 0. \tag{9}$$

Multiplying (8) by v and (9) by u and subtracting, we find

$$\frac{d}{dx}\Big[x(u'v - uv')\Big] = \left(\mu^2 - \lambda^2\right)xuv. \tag{10}$$

Integrating from $x = 0$ to $x = 1$, we find

$$\left(\mu^2 - \lambda^2\right)\int_0^1 xuv\,dx = u'(1)v(1) - u(1)v'(1).$$

Now $u(1) = J_\nu(\lambda)$ and $v(1) = J_\nu(\mu)$. Let us choose $\lambda = \lambda_m$ and $\mu = \lambda_n$, where λ_m and λ_n are positive zeros of J_ν. Then $u(1) = v(1) = 0$ and thus find

$$(\lambda_n^2 - \lambda_m^2)\int_0^1 xJ_\nu(\lambda_m x)J_\nu(\lambda_n x)\,dx = 0.$$

If $n \neq m$, then

$$\int_0^1 xJ_\nu(\lambda_m x)J_\nu(\lambda_n x)\,dx = 0.$$

Now from (10), we find [since $u'(x) = \lambda J_\nu'(\lambda x)$ etc]

$$\frac{d}{dx}\Big[x\Big(\lambda J_\nu'(\lambda x)J_\nu(\mu x) - \mu J_\nu(\lambda x)J_\nu'(\mu x)\Big)\Big] = \left(\mu^2 - \lambda^2\right)xJ_\nu(\lambda x)J_\nu(\mu x).$$

We differentiate this with respect to μ and then put $\mu = \lambda$. This leads to

$$2\lambda xJ_\nu(\lambda x)J_\nu(\lambda x) = \frac{d}{dx}\Big[x\Big(x\lambda J_\nu'(\lambda x)J_\nu'(\lambda x) - J_\nu(\lambda x)J_\nu'(\lambda x) - x\lambda J_\nu(\lambda x)J_\nu''(\lambda x)\Big)\Big]$$

Integrating between $x = 0$ to $x = 1$, we find

$$2\lambda\int_0^1 xJ_\nu^2(\lambda x)\,dx = \lambda\Big(J_\nu'(\lambda)\Big)^2 - J_\nu(\lambda)J_\nu'(\lambda) - \lambda J_\nu(\lambda)J_\nu''(\lambda).$$

OR

$$\int_0^1 xJ_\nu^2(\lambda x)\,dx = \frac{1}{2}J_\nu'(\lambda)^2 - \frac{J_\nu(\lambda)}{2}\left(\frac{J_\nu'(\lambda)}{\lambda} + J_\nu''(\lambda)\right)$$

[This last relation can be written as (NOT needed for the proof!)

$$\int_0^1 xJ_\nu^2(\lambda x)\,dx = \frac{1}{2}J_\nu'(\lambda)^2 + \frac{1}{2}\left(1 - \frac{\nu^2}{\lambda^2}\right)J_\nu^2(\lambda)$$]

Now if we take $\lambda = \lambda_n$, where λ_n is a positive zero of J_ν, then we find

$$\int_0^1 x J_\nu^2(\lambda_n x)\, dx = \frac{1}{2}\Big(J_\nu'(\lambda_n)\Big)^2.$$

Now

$$\Big(x^{-\nu} J_\nu(x)\Big)' = -x^{-\nu} J_{\nu+1}(x) \Rightarrow J_\nu'(x) - \frac{\nu}{x} J_\nu(x) = -J_{\nu+1}(x),$$

we find by substituting $x = \lambda_n$

$$J_\nu'(\lambda_n) = -J_{\nu+1}(\lambda_n).$$

Thus, finally we get

$$\int_0^1 x J_\nu^2(\lambda_n x)\, dx = \frac{1}{2} J_{\nu+1}^2(\lambda_n).$$

Theorem 2. (Fourier-Bessel series) *Suppose a function f is defined in the interval $0 \le x \le 1$ and that it has a Fourier-Bessel series expansion:*

$$f(x) \sim \sum_{n=1}^{\infty} c_n J_\nu(\lambda_{\nu n} x),$$

where $\lambda_{\nu n}$ are the positive zeros of J_ν. Using orthogonality, we find

$$c_n = \frac{2}{J_{\nu+1}^2(\lambda_{\nu n})} \int_0^1 x f(x) J_\nu(\lambda_{\nu n} x)\, dx.$$

Suppose that f and f' are piecewise continuous on the interval $0 \le x \le 1$. Then for $0 < x < 1$,

$$\sum_{n=1}^{\infty} c_n J_\nu(\lambda_{\nu n} x) = \begin{cases} f(x), & \text{where } f \text{ is continuous} \\ \dfrac{f(x^-) + f(x^+)}{2}, & \text{where } f \text{ is discontinuous} \end{cases}$$

At $x = 0$, it converges to zero for $\nu > 0$ and to $f(0+)$ for $\nu = 0$. On the other hand, it converges to zero at $x = 1$.