Lecture XIX

Laplace Transform of Periodic Functions, Convolution, Applications

1 Laplace transform of periodic function

Theorem 1. *Suppose that $f : [0, \infty) \to \mathbb{R}$ is a periodic function of period $T > 0$, i.e. $f(t + T) = f(t)$ for all $t \geq 0$. If the Laplace transform of f exists, then*

$$F(s) = \frac{\displaystyle\int_0^T f(t)e^{-st}\,dt}{1 - e^{-sT}}. \qquad (1)$$

Proof: We have

$$\begin{aligned}
F(s) &= \int_0^\infty f(t)e^{-st}\,dt \\
&= \sum_{n=0}^\infty \int_{nT}^{(n+1)T} f(t)e^{-st}\,dt \\
&= \sum_{n=0}^\infty \int_0^T f(u + nT)e^{-su-snT}\,du \qquad u = t - nT \\
&= \sum_{n=0}^\infty e^{-snT} \int_0^T f(u)e^{-su}\,du \\
&= \left(\int_0^T f(u)e^{-su}\,du\right) \sum_{n=0}^\infty e^{-snT} \\
&= \frac{\displaystyle\int_0^T f(u)e^{-su}\,du}{1 - e^{-sT}}.
\end{aligned}$$

The last line follows from the fact that

$$\sum_{n=0}^\infty e^{-snT}$$

is a geometric series with common ration $e^{-sT} < 1$ for $s > 0$.

Example 1. *Consider $f(t) = \sin(\omega t)$, which is a periodic function of period $2\pi/\omega$.*

Solution: Using (1), we find

$$F(s) = \frac{1}{1 - e^{-2\pi s/\omega}} \int_0^{2\pi/\omega} e^{-st}\sin(\omega t)\,dt = \frac{\omega}{s^2 + \omega^2}\frac{1 - e^{-2\pi s/\omega}}{1 - e^{-2\pi s/\omega}} = \frac{\omega}{s^2 + \omega^2}$$

Example 2. *Consider a* saw-tooth *function (see Figure 1)*

$$f(t) = \begin{cases} t, & 0 \leq t < 1, \\ f(t-1), & t \geq 1. \end{cases}$$

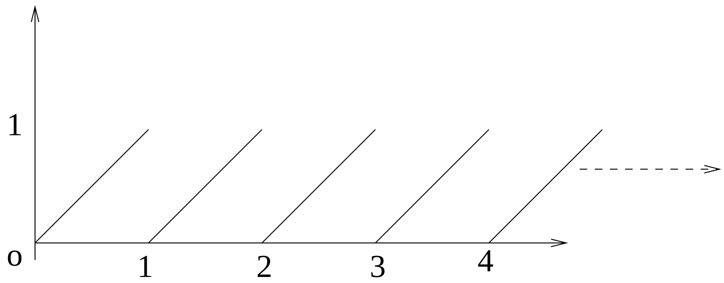

Figure 1: A saw-tooth function.

Solution: Here period $T = 1$. Using (1), we find

$$F(s) = \frac{1}{1-e^{-s}} \int_0^1 te^{-st}\, dt = \frac{1-e^{-s}(1+s)}{s^2(1-e^{-s}} = \frac{1}{s^2} - \frac{e^{-s}}{s(1-e^{-s})}$$

Example 3. *Consider the following function (see Figure 2)*

$$f(t) = \begin{cases} t, & 0 \le t < 1, \\ 2-t, & 1 \le t < 2, \\ f(t-2), & t \ge 2. \end{cases}$$

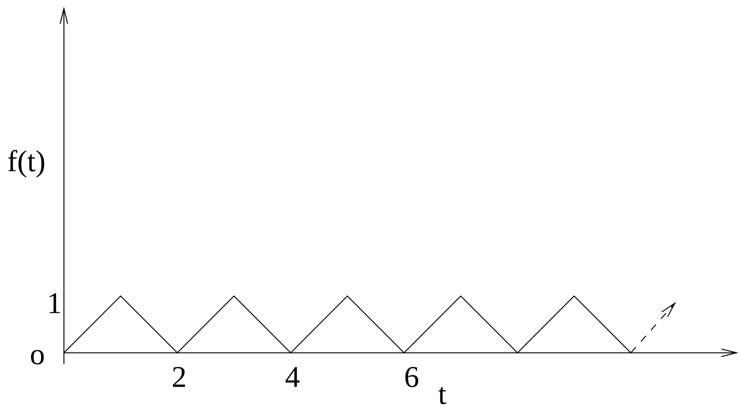

Figure 2: A saw-tooth function.

Solution: Here $f(t)$ is a periodic function of period $T = 2$. Hence, using (1), we find

$$F(s) = \frac{1}{1-e^{-2s}} \int_0^2 f(t)e^{-st}\, dt = \frac{1}{1-e^{-2s}} \left(\int_0^1 te^{-st}\, dt + \int_1^2 (2-t)e^{-st}\, dt \right)$$

Simplifying the RHS, we find

$$F(s) = \frac{(1-e^{-s})^2}{s^2(1-e^{-2s})} = \frac{1-e^{-s}}{s^2(1+e^{-s})} = \frac{1}{s^2}\tanh(s/2)$$

Aliter: Note that

$$f'(t) = \begin{cases} 1, & 0 < t < 1, \\ -1, & 1 < t < 2, \\ f(t-2), & t > 2. \end{cases}$$

Since f' is piecewise continuous and is of exponential order, its Laplace transform exist. Also, f' is periodic with period $T = 2$. Hence,

$$\mathcal{L}\Big(f'\Big) = \frac{1}{1-e^{-2s}}\int_0^2 f'(t)e^{-st}\,dt = \frac{1}{1-e^{-2s}}\left(\int_0^1 e^{-st}\,dt + \int_1^2 -e^{-st}\,dt\right) = \frac{(1-e^{-s})^2}{s(1-e^{-2s})}.$$

Hence,

$$\mathcal{L}\Big(f'\Big) = \frac{1}{s}\tanh(s/2) \implies sF(s) - f(0) = \frac{1}{s}\tanh(s/2) \implies F(s) = \frac{1}{s^2}\tanh(s/2)$$

Comment: Is it possible to do similar calculations (like in aliter) in Example 2? If not, why not?

2 Convolution

Suppose we know that a Laplace transform $H(s)$ can be written as $H(s) = F(s)G(s)$, where $\mathcal{L}\Big(f(t)\Big) = F(s)$ and $\mathcal{L}\Big(g(t)\Big) = G(s)$. We need to know the relation of $h(t) = \mathcal{L}^{-1}\Big(H(s)\Big)$ to $f(t)$ and $g(t)$.

Definition 1. (**Convolution**) *Let f and g be two functions defined in $[0,\infty)$. Then the convolution of f and g, denoted by $f*g$, is defined by*

$$(f*g)(t) = \int_0^\infty f(\tau)g(t-\tau)\,d\tau \tag{2}$$

Note: It can be shown (easily) that $f*g = g*f$. Hence,

$$(f*g)(t) = \int_0^\infty g(\tau)f(t-\tau)\,d\tau \tag{3}$$

We use either (2) or (3) depending on which is easier to evaluate.

Theorem 2. (**Convolution theorem**) *The convolution $f*g$ has the Laplace transform property*

$$\mathcal{L}\Big((f*g)(t)\Big) = F(s)G(s). \tag{4}$$

OR conversely

$$\mathcal{L}^{-1}\Big(F(s)G(s)\Big) = (f*g)(t)$$

Proof: Using definition, we find

$$\begin{aligned}\mathcal{L}\Big((f*g)(t)\Big) &= \int_0^\infty (f*g)(t)e^{-st}\,dt \\ &= \int_0^\infty \left(\int_0^t f(\tau)g(t-\tau)\,d\tau\right)e^{-st}\,dt\end{aligned}$$

The region of integration is the area in the first quadrant bounded by the t-axis and

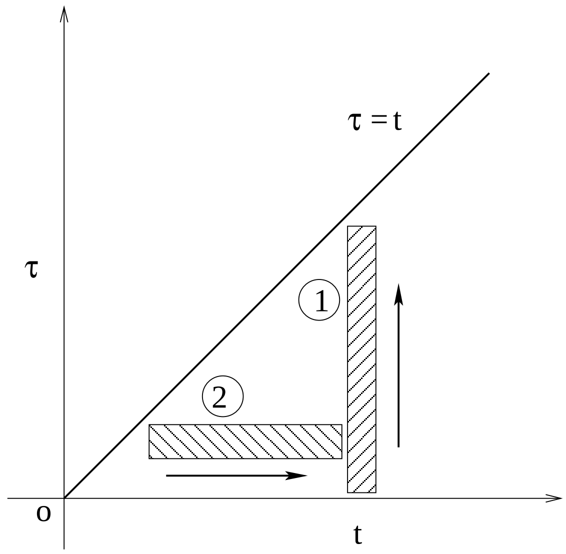

Figure 3: Effects of unit step function on a function $f(t)$. Here $b > a$.

the line $\tau = t$. The variable limit of integration is applied on τ which varies from $\tau = 0$ to $\tau = t$.

Let us change the order of integration, thus apply variable limit on t. Then t would vary from $t = \tau$ to $t = \infty$ and τ would vary from $\tau = 0$ to $\tau = \infty$. Hence, we have

$$\begin{aligned}
\mathcal{L}\Big((f * g)(t)\Big) &= \int_0^\infty \left(\int_\tau^\infty e^{-st} g(t-\tau)\, dt \right) f(\tau)\, d\tau \\
&= \int_0^\infty \left(\int_0^\infty e^{-su} g(u)\, du \right) f(\tau) e^{-s\tau}\, d\tau, \qquad t - \tau = u \\
&= \left(\int_0^\infty e^{-su} g(u)\, du \right) \left(\int_0^\infty e^{-s\tau} f(\tau)\, d\tau \right) \\
&= F(s)G(s)
\end{aligned}$$

Example 4. *Consider the same problem as given in Example 4 of Lecture Note 18, i.e. find inverse Laplace transform of* $1/s(s+1)^2$.

Solution: We write $H(s) = F(s)G(s)$, where $F(s) = 1/s$ and $G(s) = 1/(s+1)^2$. Thus $f(t) = 1$ and $g(t) = te^{-t}$. Hence, using convolution theorem, we find

$$h(t) = \int_0^t f(t-\tau) g(\tau)\, d\tau = \int_0^t \tau e^{-\tau}\, d\tau = 1 - (t+1)e^{-t}.$$

Note: We have used $f(t-\tau)g(\tau)$ in the convolution formula since $f(t) = 1$. This helps a little bit in the evaluation of the integration.

Example 5. *Find inverse Laplace transform of* $1/(s^2 + \omega^2)^2$.

Solution: Let $H(s) = F(s)G(s)$, where $F(s) = 1/(s^2 + \omega^2)$ and $G(s) = 1/(s^2 + \omega^2)$.

Thus, $f(t) = \sin(\omega t)/\omega = g(t)$. Hence,

$$\begin{aligned} h(t) &= \frac{1}{\omega^2}\int_0^t \sin(\omega\tau)\sin(\omega(t-\tau))d\tau \\ &= \frac{1}{2\omega^3}\Big(\sin(\omega t) - \omega t\cos(\omega t)\Big). \end{aligned}$$

3 Applications

Example 6. *(Differential equation) Solve the IVP*

$$y'' + y = t, \quad y(0) = 0, y'(0) = 2$$

Solution: Take Laplace transform on both sides. This gives

$$s^2Y - 2 + Y = \frac{1}{s^2} \implies Y = \frac{1}{s^2(s^2+1)} + \frac{2}{s^2+1}$$

Using partial fraction, we find

$$Y = \frac{1}{s^2} + \frac{1}{s^2+1} \implies y(t) = t + \sin t$$

Aliter: In the method above, we evaluated Laplace transform of the nonhomogeneous term in the right hand side. Now here we don't evaluate it. Let $g(t)$ be nonhomogeneous term (in this case $g(t) = t$). Let $G(s)$ be the Laplace transform of g. Now Take Laplace transform on both sides. This gives

$$s^2Y - 2 + Y = G(s) \implies Y = \frac{G(s)}{s^2+1} + \frac{2}{s^2+1}$$

Taking inverse transform and convolution, we find

$$y(t) = \int_0^t g(t-\tau)\sin(\tau)\,dt + 2\sin t \implies y(t) = \int_0^t (t-\tau)\sin(\tau)\,dt + 2\sin t$$

OR (using integration by parts)

$$y(t) = t + \sin t$$

Example 7. *(Differential equation) Solve the IVP*

$$y'' + 9y = \begin{cases} 8\sin t, & 0 < t < \pi, \\ 0, & t > \pi, \end{cases} \qquad y(0) = 0, y'(0) = 4.$$

Solution: Consider $g(t) = 8\Big(u_0(t) - u_\pi(t)\Big)\sin t$. Then Laplace transform of the nonhomogeneous term is the same as that of $g(t)$. Now we write $g(t)$ as

$$g(t) = 8u_0(t)\sin t + +8u_\pi(t)\sin(t-\pi).$$

Now taking Laplace transform of the ODE, we get

$$s^2Y - 4 + 9Y = \frac{8}{s^2+1} + 8\frac{e^{-\pi s}}{s^2+1} \implies Y = \frac{4}{s^2+9} + 8\frac{1}{(s^2+1)(s^2+9)} + 8e^{-\pi s}\frac{1}{(s^2+1)(s^2+9)}$$

Using partial fraction, we get

$$Y = \frac{4}{s^2+9} + \left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right) + e^{-\pi s}\left(\frac{1}{s^2+1} - \frac{1}{s^2+9}\right).$$

Now

$$\mathcal{L}\left(\sin t - \frac{1}{3}\sin 3t\right) = \frac{1}{s^2+1} - \frac{1}{s^2+9}.$$

Hence, using shifting theorem and inverse transform, we find

$$y(t) = \frac{4\sin 3t}{3} + \sin t - \frac{1}{3}\sin 3t + u_\pi(t)\left(\sin(t-\pi) - \frac{1}{3}\sin 3(t-\pi)\right)$$

Further, this can be break up as

$$y(t) = \begin{cases} \sin 3t + \sin t, & 0 \le t \le \pi, \\ \dfrac{4}{3}\sin 3t, & t \ge \pi, \end{cases}$$

Example 8. *(Differential equation) (Variable coefficient) Solve the IVP*

$$y'' - 2xy' + 4y = 0, \quad y(0) = 1, y'(0) = 0$$

Solution: Take Laplace Transform on both sides, we find

$$s^2Y - sy(0) - y'(0) + 2\frac{d}{ds}\Big(\mathcal{L}(y')\Big) + 4Y = 0,$$

OR

$$s^2Y - s + 2\frac{d}{ds}\Big(sY - y(0)\Big) + 4Y = 0 \implies 2sY' + (s^2+6)Y = s \implies Y' + \left(\frac{s}{2} + \frac{3}{s}\right)Y = \frac{1}{2}$$

This is linear equation. Hence,

$$Ys^3e^{s^2/4} = \frac{1}{2}\int s^3e^{s^2/4}\,ds + C$$

OR

$$Y = \frac{s^2-4}{s^3} + C\frac{e^{-s^2/4}}{s^3}$$

OR

$$Y = \frac{1}{s} - \frac{4}{s^3} + C\frac{e^{-s^2/4}}{s^3}.$$

Now it can be shown by Bromwich integral method (*not in the syllabus*) that

$$\mathcal{L}\left(\frac{x^2}{2} - \frac{1}{4}\right) = \frac{e^{-s^2/4}}{s^3}$$

Hence, we find

$$y(t) = (1 - 2x^2) + C\left(\frac{x^2}{2} - \frac{1}{4}\right).$$

OR

$$y(t) = (1 - C/4) + (C/2 - 2)x^2$$

Now $y(0) = 1 \implies C = 4$. Hence

$$y(x) = (1 - 2x^2)$$

Comment: If we expand $e^{-s^2/4}/s^3$ then we find

$$\frac{e^{-s^2/4}}{s^3} = \frac{1}{s^3} - \frac{1}{4s} + \text{non-negative power of } s.$$

If we assume $\mathcal{L}^{-1}(s^k) = 0,\ k = 0, 1, 2, \cdots$, then we find

$$\mathcal{L}\left(\frac{x^2}{2} - \frac{1}{4}\right) = \frac{e^{-s^2/4}}{s^3}$$

Example 9. *(Integral equation) Solve*

$$y' + \int_0^t y(t-\tau)e^{-2\tau}d\tau = 1, \quad y(0) = 1.$$

Solution: Take Laplace Transform on both sides, we find

$$sY - y(0) + \frac{Y}{s+2} = \frac{1}{s} \implies Y = \frac{s+2}{s(s+1)} \implies Y = \frac{2}{s} - \frac{1}{s+1}$$

Hence,

$$y(t) = 2 - e^{-t}$$