Lecture II
Geomterical interpretation

Here we concentrate on a first order ODE of the form

$$y' = f(x, y). \tag{1}$$

From calculas, we know that y' is the slope of the curve $y(x)$ at x. Hence, if (1) has a solution curve passing through the point (x_0, y_0), then the slope of that curve at (x_0, y_0) is $f(x_0, y_0)$. Thus, the value of $f(x, y)$ at each point gives the slope of the solution curve passing through that point.

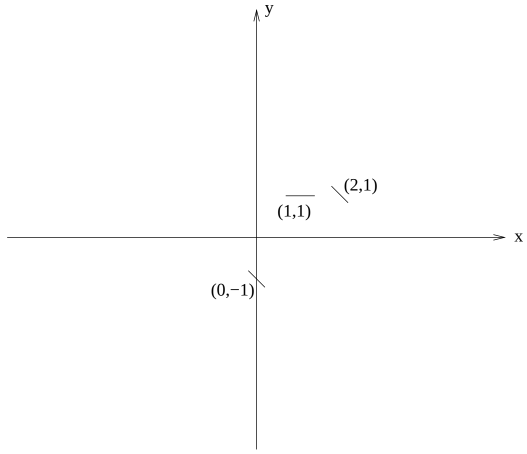

Figure 1: Some lineal elements for $y' = y - x$.

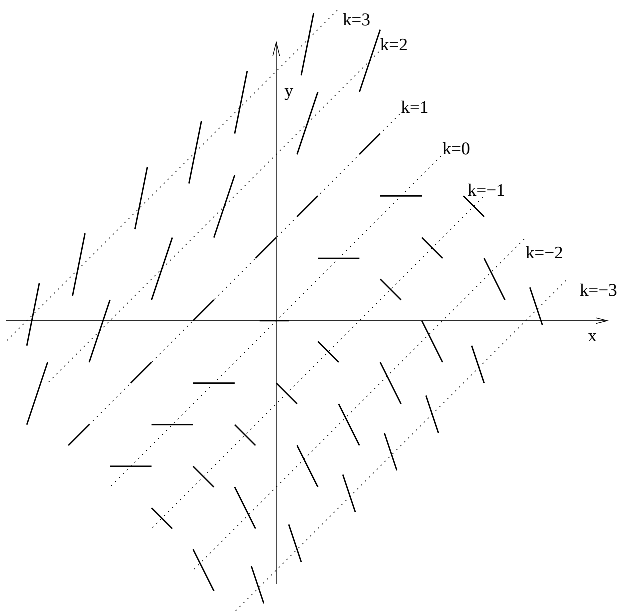

Figure 2: Different isoclines for $y' = y - x$.

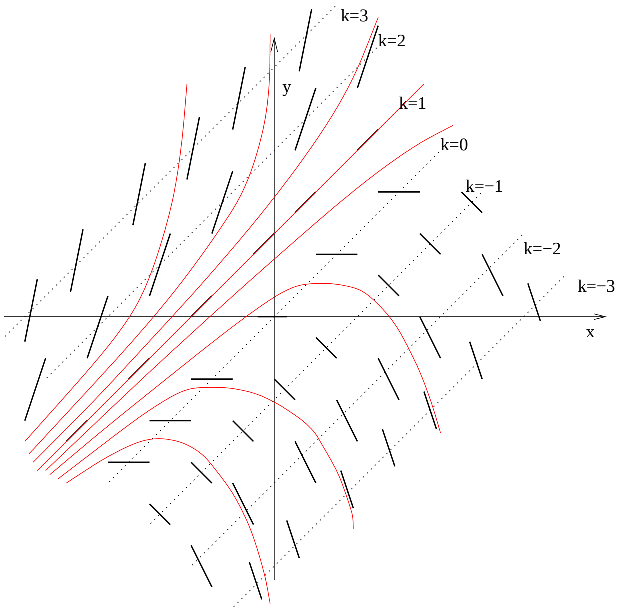

Figure 3: Approximate solution curves (in red) for $y' = y - x$.

Given the differential equation (1), we indicate the slope $f(x, y)$ by a short line segments called lineal elements. For example, lineal elements at three points, for $y' = y - x$, are shown in Figure 1. The lineal elements are also called direction or slope fields.

To draw approximate solution curve through arbitrary points, we need to cover the whole domian with lineal elements. This process is very cumbersome and time consuming. Hence, we use the following method. On the domian we trace the curve $f(x, y) = k$. The curve $f(x, y) = k$, on which the slope y' is constant and equals to k, is called *isoclines*. If $k = 0$, then it has a special name, *nullclines*. Using isoclines, we can draw lineal elements in the domain of $f(x, y)$. These can be seen in Figure 2 for $y' = y - x$.

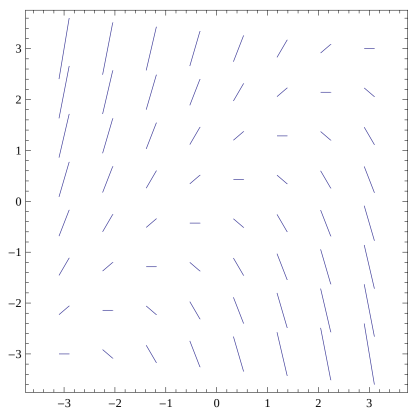

Figure 4: Lineal elements using *Mathematica* for $y' = y - x$.

Since $f(x, y)$ is continuous, we can add more arrows (if needed) between two isoclines. To draw solution curve, we follow the lineal elements since these are the directions of

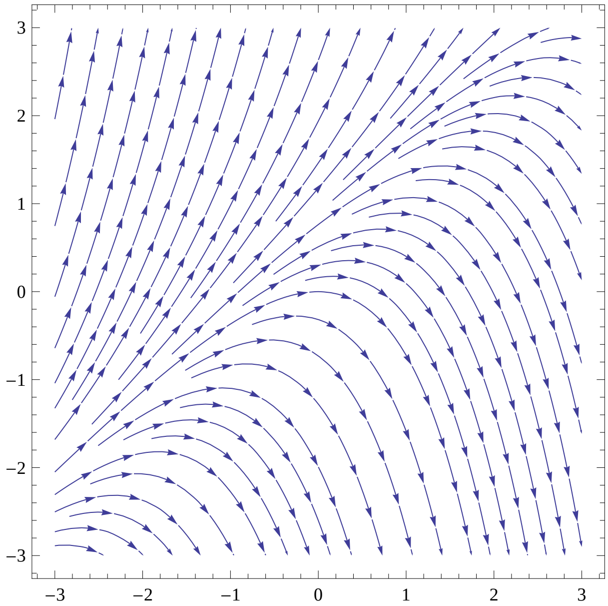

Figure 5: Solution curves using *Mathematica* for $y' = y - x$.

tangents to the curve. It is clear that $y = x + 1$ is a solution curve for $y' = y - x$ (see Figure 3). Other approximate solution curves are show in Figure 3.

Most of the commercial software packages can draw direction field as well as solution curves. For example, the *Mathematica* command

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VectorPlot[{1, y - x}, {x, -3, 3}, {y, -3, 3}, VectorPoints -> 8,
VectorStyle -> Arrowheads[0]]
```

produces Figure 4. Similarly, the *Mathematica* command

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StreamPlot[{1, y - x}, {x, -3, 3}, {y, -3, 3}]
```

produces Figure 5.

Excercise Draw the direction field of the following differential equations:
(i) $y' = 2x$ (ii) $xy' = 2y$