Lecture III

Solution of first order equations

1 Separable equations

These are equations of the form

$$y' = f(x)g(y)$$

Assuing g is nonzero, we divide by g and integrate to find

$$\int \frac{dy}{g(y)} = \int f(x)dx + C$$

What happens if $g(y)$ becomes zero at a point $y = y_0$?

Example 1. $xy' = y + y^2$

Solution: We write this as

$$\int \frac{dy}{y+y^2} = \int \frac{dx}{x} + C \Rightarrow \int \frac{dy}{y} - \int \frac{dy}{1+y} = \ln x + C \Rightarrow \ln y - \ln(1+y) = \ln x + C$$

Note: Strictly speaking, we should write the above solution as

$$\ln|y| - \ln|1+y| = \ln|x| + C$$

When we wrote the solution without the modulas sign, it was (implicitly) assumed that $x > 0, y > 0$. This is acceptable for problems in which the solution domain is not given explicitly. But for some problems, the modulas sign is necessary. For example, consider the following IVP:

$$xy' = y + y^2, \quad y(-1) = -2.$$

Try to solve this.

2 Reduction to separable form

2.1 Substitution method

Let the ODE be

$$y' = F(ax + by + c)$$

Suppose $b \neq 0$. Substituting $ax + by + c = v$ reduces the equation to a separable form. If $b = 0$, then it is already in separable form.

Example 2. $y' = (x + y)^2$

Solution: Let $v = x + y$. Then we find

$$v' = v^2 + 1 \Rightarrow \tan^{-1} v = x + C \Rightarrow x + y = \tan(x + C)$$

2.2 Homogeneous form

Let the ODE be of the form

$$y' = f(y/x)$$

In this case, substitution of $v = y/x$ reduces the above ODE to a seprable ODE.

Comment 1: Sometimes, substitution reduces an ODE to the homogeneous form. For example, if $ae \neq bd$, then h and k can be chosen so that $x = u + h$ and $y = v + k$ reduces the following ODE

$$y' = F\left(\frac{ax + by + c}{dx + ey + f}\right)$$

to a homeogeneous ODE. What happens if $ae = bd$?

Comment 2: Also, an ODE of the form

$$y' = y/x + g(x)h(y/x)$$

can be reduced to the separable form by substituting $v = y/x$.

Example 3. $xyy' = y^2 + 2x^2, \qquad y(1) = 2$

Solution: Substituting $v = y/x$ we find

$$v + xv' = v + 2/v \Rightarrow y^2 = 2x^2(C + \ln x^2)$$

Using $y(1) = 2$, we find $C = 2$. Hence, $y = 2x^2(1 + \ln x^2)$

3 Exact equation

A first order ODE of the form

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{1}$$

is exact if there exits a function $u(x, y)$ such that

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}.$$

Then the above ODE can be written as $du = 0$ and hence the solution becomes $u = C$.

Theorem 1. *Let M and N be defined and continuously differentiable on a rectangle rectangle $R = \{(x, y) : |x - x_0| < a, |y - y_0| < b\}$. Then (1) is exact if and only if $\partial M/\partial y = \partial N/\partial x$ for all $(x, y) \in R$.*

Proof: We shall only prove the necessary part. Assume that (1) is exact. Then there exits a function $u(x, y)$ such that

$$M = \frac{\partial u}{\partial x} \quad \text{and} \quad N = \frac{\partial u}{\partial y}.$$

Since M and N have continuous first partial derivatives, we have

$$\frac{\partial M}{\partial y} = \frac{\partial^2 u}{\partial y \partial x} \quad \text{and} \quad \frac{\partial N}{\partial x} = \frac{\partial^2 u}{\partial x \partial y}.$$

Now continuity of 2nd partial derivative implies $\partial M/\partial y = \partial N/\partial x$.

Example 4. *Solve* $(2x + \sin x \tan y)dx - \cos x \sec^2 y\, dy = 0$

Solution: Here $M = 2x + \sin x \tan y$ and $N = -\cos x \sec^2 y$. Hence, $M_y = N_x$. Hence, the solution is $u = C$, where $u = x^2 - \cos x \tan y$

4 Reduction to exact equation: integrating factor

An integrating factor $\mu(x, y)$ is a function such that

$$M(x, y)\, dx + N(x, y)\, dy = 0 \tag{2}$$

becomes exact on multiplying it by μ. Thus,

$$\mu M\, dx + \mu N\, dy = 0$$

is exact. Hence

$$\frac{\partial(\mu M)}{\partial y} = \frac{\partial(\mu N)}{\partial x}.$$

Comment: If an equation has an integrating factor, then it has infinitely many integrating factors.
Proof: Let μ be an integrating factor. Then

$$\mu M\, dx + \mu N\, dy = du$$

Let $g(u)$ be any continuous function of u. Now multiplying by $\mu g(u)$, we find

$$\mu g(u) M\, dx + \mu g(u) N\, dy = g(u) du \Rightarrow \mu g(u) M\, dx + \mu g(u) N\, dy = d\left(\int^u g(u)\, du\right)$$

Thus,

$$\mu g(u) M\, dx + \mu g(u) N\, dy = dv, \quad \text{whare} \quad v = \int^u g(u)\, du$$

Hence, $\mu g(u)$ is an integrating factor. Since, g is arbitrary, there exists an infinite number of integrating factors.

Example 5. $xdy - ydx = 0$.

Solution: Clearly $1/x^2$ is an integrating factor since

$$\frac{xdy - ydx}{x^2} = 0 \Rightarrow d(y/x) = 0$$

Also, $1/xy$ is an integrating factor since

$$\frac{xdy - ydx}{xy} = 0 \Rightarrow d \ln(y/x) = 0$$

Similarly it can be shown that $1/y^2$, $1/(x^2 + y^2)$ etc. are integrating factors.

4.1 How to find intgrating factor

Theorem 2. *If (2) is such that*

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

is a function of x alone, say $F(x)$, then

$$\mu = e^{\int F\,dx}$$

is a function of x only and is an integrating factor for (2).

Example 6. $(xy-1)dx + (x^2 - xy)dy = 0$

Solution: Here $M = xy - 1$ and $N = x^2 - xy$. Also,

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{1}{x}$$

Hence, $1/x$ is an integrting factor. Multiplying by $1/x$ we find

$$\frac{(xy-1)dx + (x^2 - xy)dy}{x} = 0 \Rightarrow xy - \ln x - y^2/2 = C$$

Theorem 3. *If (2) is such that*

$$\frac{-1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right)$$

is a function of y alone, say $G(y)$, then

$$\mu = e^{\int G\,dy}$$

is a function of y only and is an integrating factor for (2).

Example 7. $y^3 dx + (xy^2 - 1)dy = 0$

Solution: Here $M = y^3$ and $N = xy^2 - 1$. Also,

$$-\frac{1}{M}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = -\frac{2}{y}$$

Hence, $1/y^2$ is an integrting factor. Multiplying by $1/y^2$ we find

$$\frac{y^3 dx + (xy^2 - 1)dy}{y^2} = 0 \Rightarrow xy + \frac{1}{y} = C$$

Comment: Sometimes it may be possible to find integrating factor by inspection. For this, some known differential formulas are useful. Few of these are given below:

$$\begin{aligned}
d\left(\frac{x}{y}\right) &= \frac{ydx - xdy}{y^2} \\
d\left(\frac{y}{x}\right) &= \frac{xdy - ydx}{x^2} \\
d(xy) &= xdy + ydx \\
d\left(\ln\frac{x}{y}\right) &= \frac{ydx - xdy}{xy}
\end{aligned}$$

Example 8. $(2x^2y + y)dx + xdy = 0$

Obviously, we can write this as

$$2x^2ydx + (ydx + xdy) = 0 \Rightarrow 2x^2ydx + d(xy) = 0$$

Now if we divide this by xy, then the last term remains differential and the first term also becomes differential:

$$2xdx + \frac{d(xy)}{xy} = 0 \Rightarrow d\left(x^2 + \ln(xy)\right) = 0 \Rightarrow x^2 + \ln(xy) = C$$