Lecture IV

Linear equations, Bernoulli equations, Orthogonal trajectories, Oblique trajectories

1 Linear equations

A first order linear equations is of the form

$$y' + p(x)y = r(x) \tag{1}$$

This can be written as

$$(p(x)y - r(x))dx + dy = 0.$$

Here $M = p(x)y - r(x)$ and $N = 1$. Now

$$\frac{1}{N}\left(\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right) = p(x)$$

Hence,

$$\mu(x) = e^{\int p(x)\,dx}$$

is an integrating factor. Multiplying (1) by $\mu(x)$ we get

$$\frac{d}{dx}\left(e^{\int p(x)\,dx}y\right) = r(x)e^{\int p(x)\,dx}$$

Integrating we get

$$e^{\int p(x)\,dx}y = \int^x r(s)e^{\int p(s)\,ds}\,ds + C$$

which on simplification gives

$$y = e^{-\int p(x)\,dx}\left(C + \int^x r(s)e^{\int p(s)\,ds}\,ds\right)$$

Example 1. *Solve* $y' + 2xy = 2x$

Solution: An integrating factor is e^{x^2}. Hence,

$$ye^{x^2} = \int^x 2te^{t^2}\,dt + C \Rightarrow y = 1 + Ce^{-x^2}$$

Comment: The usual notation dy/dx implies that x is the independent variable and y is the dependent variable. In trying to solve first order ODE, it is sometimes helpful to reverse the role of x and y, and work on the resulting equations. Hence, the resulting equation

$$\frac{dx}{dy} + p(y)x = r(y)$$

is also a linear equation.

Example 2. *Solve* $(4y^3 - 2xy)y' = y^2, \quad y(2) = 1$

Solution: We write this as

$$\frac{dx}{dy} + \frac{2}{y}x = 4y$$

Clearly, y^2 is an integrating factor. Hence,

$$xy^2 = \int^y 4y^3\,dy + C \Rightarrow xy^2 = y^4 + C$$

Using initial condition, we find $xy^2 = y^4 + 1$.

2 Bernoulli's equation

This is of the form
$$y' + p(x)y = r(x)y^\lambda, \tag{2}$$
where λ is a real number. Equation (2) is linear for $\lambda = 0$ or 1. Otherwise, it is nonlinear and can be reduced to a linear form by substituting $z = y^{1-\lambda}$

Example 3. *Solve* $y' - y/x = y^3$

Solution: We write this as
$$y^{-3}y' - y^{-2}/x = 1$$
Substitute $y^{-2} = z \Rightarrow -2y^{-3}y' = z'$. This leads to
$$z' + 2z/x = -2$$
This is a linear equation whose solution is
$$zx^2 = -2x^3/3 + C$$
Replacing z we find
$$3\frac{x^2}{y^2} + 2x^3 = C$$

3 Reducible second order ODE

A general 2nd order ODE is of the form
$$F(x, y, y', y'') = 0$$
In some cases, by making substitution, we can reduce this 2nd order ODE to a 1st order ODE. Few cases are described below

Case I: If the *independent variable is missing*, then we have $F(y, y', y'') = 0$. If we substitute $w = y'$, then $y'' = w\frac{dw}{dy}$. Hence, the ODE becomes $F(y, w, w\frac{dw}{dy}) = 0$, which is a 1st order ODE.

Example 4. *Solve* $2y'' - y'^2 - 4 = 0$

Solution: With $w = y'$, the above equation becomes
$$2w\frac{dw}{dy} - w^2 - 4 = 0 \Rightarrow \ln[(w^2 + 4)/C] = y \Rightarrow w = \pm\sqrt{Ce^y - 4}$$
SInce $w = y'$, we find
$$\frac{dy}{\sqrt{Ce^y - 4}} = \pm x + D$$
The integral on the LHS can be evaluated by substitution.

Case II: If the *dependent variable is missing*, then we have $F(x, y', y'') = 0$. If we substitute $w = y'$, then $y'' = w'$. Hence, the ODE becomes $F(x, w, w') = 0$, which is a 1st order ODE.

Example 5. *Solve* $xy'' + 2y' = 0$

Solution: Substitute $w = y'$, then we find

$$\frac{dw}{dx} + \frac{2}{x}w = 0 \Rightarrow w = Cx^{-2}$$

Since $w = y'$, we further get

$$y' = C/x^2 \Rightarrow y = -C/x + D$$

4 Orthogonal trajectories

Definition 1. *Two families of curves are such that each curve in either family is orthogonal (whenever they intersect) to every curve in the other family. Each family of curves is orthogonal trajectories of the other. In case the two families are identical, they we say that the family is self-orthogonal.*

Comment: Orthogonal trajectories has important applications in the field of physics. For example, the equipotential lines and the streamlines in an irrotational 2D flow are orthogonal.

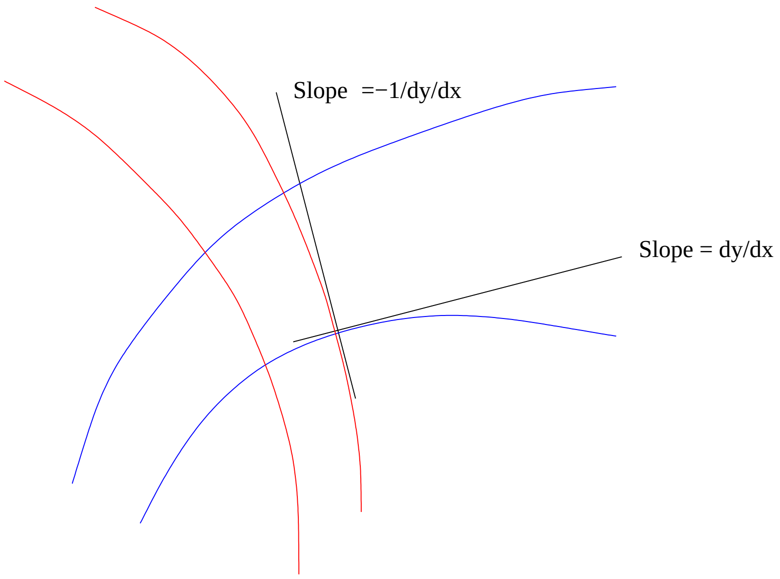

Figure 1: Orthogonal trajectories.

4.1 How to find orthonal trajectories

Suppose the first familiy

$$F(x, y, c) = 0. \tag{3}$$

To find the orthogonal trajectories of this family we proceed as follows. First, differentiate (3) w.r.t. x to find

$$G(x, y, y', c) = 0. \tag{4}$$

Now eliminate c between (3) and (4) to find the differential equation

$$H(x, y, y') = 0 \tag{5}$$

corresponding to the first family. As seen in Figure 1, the differential equation for the other family is obtained by replacing y' by $-1/y'$. Hence, the differetial equation of the orthogonal trajectories is

$$H(x, y, -1/y') = 0 \tag{6}$$

General solution of (6) gives the required orthogonal trajectories.

Example 6. *Find the orthogonal trajectories of familiy of straight lines through the origin.*

Solution: The familiy of straight lines through the origin is given by

$$y = mx$$

The ODE for this familiy is

$$xy' - y = 0$$

The ODE for the orthogonal family is

$$x + yy' = 0$$

Integrating we find

$$x^2 + y^2 = C,$$

which are family of circles with centre at the origin.

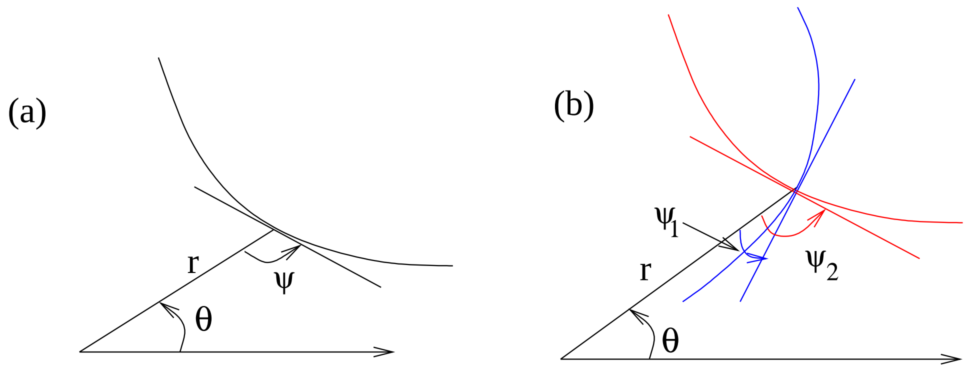

Figure 2: Orthogonal trajectories.

4.2 *Orthogonal trjactories in polar coordinates

Consider a curve in polar cordinate. The angle ψ between the radial and tangent directions is given by

$$\tan\psi = \frac{r\,d\theta}{dr}$$

Consider the curve with angle ψ_1. The curve that intersects it orthogonally has angle $\psi_2 = \psi_1 + \pi/2$. Now

$$\tan\psi_2 = -\frac{1}{\tan\psi_1}$$

Thus, at the point of orthogonal intersection, the value of

$$\frac{r\,d\theta}{dr} \tag{7}$$

for the second familiy should be negative raciprocal of the value of (7) of the first family. To illustrate, consider the differential equation for the first family:

$$Pdr + Qd\theta = 0.$$

Thus we find $r\,d\theta/dr = -Pr/Q$. Hence, the differential equation of the orthogonal family is given by

$$\frac{r\,d\theta}{dr} = \frac{Q}{Pr}$$

or

$$Q\,dr - r^2\,P\,d\theta = 0$$

General solution of the last equation gives the orthogonal trajectories.

Example 7. *Find the orthogonal trajectories of familiy of straight lines through the origin.*

Solution: The familiy of straight lines through the origin is given by

$$\theta = A$$

The ODE for this familiy is

$$d\theta = 0$$

The ODE for the orthogonal family is

$$dr = 0$$

Integrating we find

$$r = C,$$

which are family of circles with centre at the origin.

4.3 Oblique trajectories

Here the two families of curves intersect at an arbitrary angle $\alpha \neq \pi/2$. Suppose the first familiy

$$F(x, y, c) = 0. \tag{8}$$

To find the oblique trajectories of this family we proceed as follows. First, differentiate (8) w.r.t. x to find

$$G(x, y, y', c) = 0. \tag{9}$$

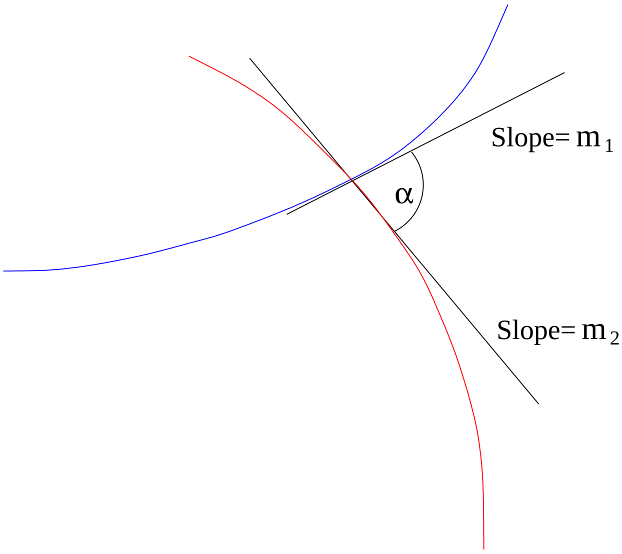

Figure 3: Oblique trajectories.

Now eliminate c between (8) and (9) to find the differential equation

$$H(x, y, y') = 0. \tag{10}$$

Now if m_1 is the slope of this family, then we write (10) as

$$H(x, y, m_1) = 0, \tag{11}$$

Let m_2 be the slope of the second famílily. Then

$$\pm \tan\alpha = \frac{m_1 - m_2}{1 + m_1 m_2}.$$

Thus, we find

$$m_1 = \frac{m_2 \pm \tan\alpha}{1 \mp m_2 \tan\alpha}$$

Hence, from (11), the ODE for the second family satisfies

$$H\left(x, y, \frac{m_2 \pm \tan\alpha}{1 \mp m_2 \tan\alpha}\right) = 0,$$

Replacing m_2 by y', the ODE for the second family is written as

$$H\left(x, y, \frac{y' \pm \tan\alpha}{1 \mp y' \tan\alpha}\right) = 0. \tag{12}$$

General solution of (12) gives the required oblique trajectories.
Note: If we let $\alpha \to \pi/2$, we obtained ODE for the orthogonal trajectories.

Example 8. *Find the oblique trajectories that intersects the familiy $y = x + A$ at an angle of* 60^o

Solution: The ODE for the given family is

$$y' = 1$$

For the oblique trajectories, we replace

$$y' \quad \text{by} \quad \frac{y' \pm \tan(\pi/3)}{1 \mp y' \tan(\pi/3)} = \frac{y' \pm \sqrt{3}}{1 \mp \sqrt{3}y'}$$

Thus, the ODE for the oblique trajectories is given by

$$\frac{y' \pm \sqrt{3}}{1 \mp \sqrt{3}y'} = 1$$

Simplifying we obtain

$$y' = \frac{1-\sqrt{3}}{1+\sqrt{3}} \quad \text{OR} \quad y' = \frac{1+\sqrt{3}}{1-\sqrt{3}}$$

Hence, the oblique trajectories are either

$$y = \frac{1-\sqrt{3}}{1+\sqrt{3}}x + C_1$$

Or

$$y = \frac{1+\sqrt{3}}{1-\sqrt{3}}x + C_2$$