Lecture V

Picard's existence and uniquness theorem, Picard's iteration

1 Existence and uniqueness theorem

Here we concentrate on the solution of the first order IVP

$$y' = f(x, y), \quad y(x_0) = y_0 \tag{1}$$

We are interested in the following questions:
1. Under what conditions, there exists a solution to (1).
2. Under what conditions, there exists a unique solution to (1).

Comment: An ODE may have no solution, unique solution or infinitely many solutions. For, example $y'^2 + y^2 + 1 = 0,\ y(0) = 1$ has no solution. The ODE $y' = 2x,\ y(0) = 1$ has unique solution $y = 1+x^2$, whereas the ODE $xy' = y-1,\ y(0) = 1$ has infinitely many solutions $y = 1 + \alpha x$, α is any real number.

(We only state the theorems. For proof, one may see 'An introduction to ordinary differential equation' by E A Coddington.)

Theorem 1. (Existence theorem)*: Suppose that $f(x, y)$ is continuous function in some region*

$$R = \{(x, y) : |x - x_0| \le a, |y - y_0| \le b\}, \quad (a, b > 0).$$

Since f is continuous in a closed and bounded domain, it is necessarily bounded in R, i.e., there exists $K > 0$ such that $|f(x, y)| \le K \quad \forall (x, y) \in R$. Then the IVP (1) has atleast one solution $y = y(x)$ defined in the interval $|x - x_0| \le \alpha$ where

$$\alpha = \min\left\{a, \frac{b}{K}\right\}$$

(Note that the solution exists possibly in a smaller interval)

Theorem 2. (Uniquness theorem)*: Suppose that f and $\partial f/\partial y$ are continuous function in R* (defined in the existence theorem)*. Hence, both the f and $\partial f/\partial y$ are bounded in R, i.e.,*

$$\text{(a) } |f(x, y)| \le K \quad \textit{and} \quad \text{(b) } \left|\frac{\partial f}{\partial y}\right| \le L \qquad \forall (x, y) \in R$$

Then the IVP (1) has atmost one solution $y = y(x)$ defined in the interval $|x - x_0| \le \alpha$ where

$$\alpha = \min\left\{a, \frac{b}{K}\right\}.$$

Combining with existence thereom, the IVP (1) has unique solution $y = y(x)$ defined in the interval $|x - x_0| \le \alpha$.

Comment: Condition (b) can be replaced by a *weaker* condition which is known as Lipschitz condition. Thus, instead of continuity of $\partial f/\partial y$, we require

$$|f(x, y_1) - f(x, y_2)| \le L|y_1 - y_2| \qquad \forall (x, y_i) \in R.$$

If $\partial f/\partial y$ exists and is bounded, then it necessarily saitisfies Lipschitz condition. On the other hand, a function $f(x,y)$ may be Lipschitz continuous but $\partial f/\partial y$ may not exists. For example $f(x,y) = x^2|y|,\ |x| \le 1, |y| \le 1$ is Lipschitz continous in y but $\partial f/\partial y$ does not exist at $(x,0)$ (prove it!).

***Note 1**: The existence and uniqueness theorems stated above are *local* in nature since the interval, $|x - x_0| \le \alpha$, where solution exists may be smaller than the original interval, $|x - x_0| \le a$, where $f(x,y)$ is defined. However, in some cases, this restrictions can be removed. Consider the linear equation

$$y' + p(x)y = r(x), \tag{2}$$

where $p(x)$ and $r(x)$ are defined and continuous in a the interval $a \le x \le b$. Here $f(x,y) = -p(x)y + r(x)$. If $L = \max_{a \le x \le b} |p(x)|$, then

$$|f(x,y_1) - f(x,y_2)| = |-p(x)(y_1 - y_2)| \le L|y_1 - y_2|$$

Thus, f is Lipschitz continuous in y in the infinite vertical strip $a \le x \le b$ and $-\infty < y < \infty$. In this case, the IVP (2) has a unique solution in the *original* interval $a \le x \le b$.

***Note 2**: Though the theorems are stated in terms of interior point x_0, the point x_0 could be left/right end point.

Comment: The conditions of the existence and uniqueness theorem are sufficeint but not necessary. For example, consider

$$y' = \sqrt{y} + 1, \quad y(0) = 0, \quad x \in [0,1]$$

Clearly f does not satisfy Lipschitz condition near origin. But still it has unique solution. Can you prove this? [Hint: Let y_1 and y_2 be two solutions and consider $z(x) = \left(\sqrt{y_1(x)} - \sqrt{y_2(x)}\right)^2$.]

Comment: The existence and uniqueness theorem are also valid for certain system of first order equations. These theorems are also applicable to a certain higher order ODE since a higher order ODE can be reduced to a system of first order ODE.

Example 1. *Consider the ODE*

$$y' = xy - \sin y, \qquad y(0) = 2.$$

Here f and $\partial f/\partial y$ are continuous in a closed rectangle about $x_0 = 0$ and $y_0 = 2$. Hence, there exists unique solution in the neighbourhood of $(0,2)$.

Example 2. *Consider the ODE*

$$y' = 1 + y^2, \qquad y(0) = 0.$$

Consider the rectangle

$$S = \{(x,y) : |x| \le 100, |y| \le 1\}.$$

Clearly f and $\partial f/\partial y$ are continuous in S. Hence, there exists unique solution in the neighbourhood of $(0,0)$. Now $f = 1 + y^2$ and $|f| \le 2$ in S. Now $\alpha = min\{100, 1/2\} =$

1/2. Hence, the theorems guarantee existence of unique solution in $|x| \leq 1/2$, *which is much smaller than the original interval* $|x| \leq 100$.

Since, the above equation is separable, we can solve it exactly and find $y(x) = \tan(x)$. *This solution is valid only in* $(-\pi/2, \pi/2)$ *which is also much smaller than* $[-100, 100]$ *but nevertheless bigger than that predicted by the existence and uniqueness theorems.*

Example 3. *Consider the IVP*

$$y' = x|y|, \quad y(1) = 0.$$

Since f *is continuous and satisfy Lipschitz condition in the neighbourhood of the* $(1, 0)$, *it has unique solution around* $x = 1$.

Example 4. *Consider the IVP*

$$y' = y^{1/3} + x, \quad y(1) = 0$$

Now

$$|f(x, y_1) - f(x, y_2)| = |y_1^{1/3} - y_2^{1/3}| = \frac{|y_1 - y_2|}{|y_1^{2/3} + y_1^{1/3} y_2^{1/3} + y_2^{1/3}|}$$

Suppose we take $y_2 = 0$. *Then*

$$|f(x, y_1) - f(x, 0)| = \frac{|y_1 - 0|}{|y_1^{2/3}|}$$

Now we can take y_1 *very close to zero. Then* $1/|y_1^{2/3}|$ *becomes unbounded. Hence, the relation*

$$|f(x, y_1) - f(x, 0)| \leq L|y_1 - 0|$$

does not always hold around a region about $(1, 0)$.

Since f *does not satisfy Lipschitz condition, we can not say whether unique solution exits or does not exist (remember the existence and uniqueness conditios are sufficient but not necessary).*

On the other hand

$$y' = y^{1/3} + x, \quad y(1) = 1$$

has unique solution around $(1, 1)$.

Example 5. *Discuss the existence and unique solution for the IVP*

$$y' = \frac{2y}{x}, \quad y(x_0) = y_0$$

Solution: Here $f(x, y) = 2y/x$ and $\partial f/\partial y = 2/x$. Clearly both of these exist and bounded around (x_0, y_0) if $x_0 \neq 0$. Hence, unique solution exists in a interval about x_0 for $x_0 \neq 0$.

For $x_0 = 0$, nothing can be said from the existence and uniqness theorem. Fortunately, we can solve the actual problem and find $y = Ax^2$ to be the general solution. When $x_0 = 0$, there exists no solution when $y_0 \neq 0$. If $y_0 = 0$, then we have infinite number of solutions $y = \alpha x^2$ (α any real number) that satisfy the IVP $y' = 2y/x, \quad y(0) = 0$.

2 Picard iteration for IVP

This method gives approximate solution to the IVP (1). Note that the IVP (1) is equivalent to the integral equation

$$y(x) = y_0 + \int_{x_0}^{x} f(t, y(t))\,dt \tag{3}$$

A rough approximation to the solution $y(x)$ is given by the function $y_0(x) = y_0$, which is simply a horizontal line through (x_0, y_0). (don't confuse function $y_0(x)$ with constant y_0). We insert this to the RHS of (3) in order to obatin a (perhaps) better approximate solution, say $y_1(x)$. Thus,

$$y_1(x) = y_0 + \int_{x_0}^{x} f(t, y_0(t))\,dt = y_0 + \int_{x_0}^{x} f(t, y_0)\,dt$$

The nest step is to use this $y_1(x)$ to genereate another (perhaps even better) approximate solution $y_2(x)$:

$$y_2(x) = y_0 + \int_{x_0}^{x} f(t, y_1(t))\,dt$$

At the n-th stage we find

$$y_n(x) = y_0 + \int_{x_0}^{x} f(t, y_{n-1}(t))\,dt$$

Theorem 3. *If the function $f(x, y)$ satisfy the existence and uniqueness theorem for IVP (1), then the succesive approximation $y_n(x)$ converges to the unique solution $y(x)$ of the IVP (1).*

Example 6. *Apply Picard iteration for the IVP*

$$y' = 2x(1 - y), \quad y(0) = 2.$$

Solution: Here $y_0(x) = 2$. Now

$$y_1(x) = 2 + \int_0^x 2t(1-2)\,dt = 2 - x^2$$

$$y_2(x) = 2 + \int_0^x 2t(t^2 - 1)\,dt = 2 - x^2 + \frac{x^4}{2}$$

$$y_3(x) = 2 + \int_0^x 2t\left(t^2 - \frac{t^4}{2} - 1\right)\,dt = 2 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!}$$

$$y_4(x) = 2 + \int_0^x 2t\left(\frac{t^6}{3!} - \frac{t^4}{2} + t^2 - 1\right)\,dt = 2 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \frac{x^8}{8!}$$

By induction, it can be shown that

$$y_n(x) = 2 - x^2 + \frac{x^4}{2} - \frac{x^6}{3!} + \cdots + (-1)^n \frac{x^{2n}}{n!}$$

Hence, $y_n(x) \to 1 + e^{-x^2}$ as $n \to \infty$. Now $y(x) = 1 + e^{-x^2}$ is the exact solution of the given IVP. Thus, the Picard iterates converge to the unique solution of the given IVP.

Comment: Picard iteration has more theoretical value than practical value. It is used in the proof of existence and uniqueness theorem. On the other hand, finding approximate solution using this method is almost impractical for complicated function $f(x, y)$.