Lecture VIII

Homogeneous linear ODE with constant coefficients

1 Homogeneous 2nd order linear equation with constant coefficients

If the ODE is of the form

$$ay'' + by' + cy = 0, \qquad x \in \mathcal{I}, \tag{1}$$

where a, b, c are constants, then two independent solutions (i.e. basis) depend on the quadratic equation

$$am^2 + bm + c = 0. \tag{2}$$

Equation (2) is called *characteristic equation* for (1).

Theorem 1. *(i) If the roots of (2) are real and distinct, say m_1 and m_2, then two linearly independent (LI) solutions of (1) are e^{m_1x} and e^{m_2x}. Thus, the general solution to (1) is*

$$y = C_1e^{m_1x} + C_2e^{m_2x}.$$

(ii) If the roots of (2) are real and equal, say $m_1 = m_2 = m$, then two LI solutions of (1) are e^{mx} and xe^{mx}. Thus, the general solution to (1) is

$$y = (C_1 + C_2x)e^{mx}.$$

(iii) If the roots of (2) are complex conjugate, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then two real LI solutions of (1) are $e^{\alpha x}\cos(\beta x)$ and $e^{\alpha x}\sin(\beta x)$. Thus, the general solution to (1) is

$$y = e^{\alpha x}(C_1\cos(\beta x) + C_2\sin(\beta x)).$$

Proof: For convenience (specially for higher order ODE) (1) is written in the operator form $L(y) = 0$, where

$$L \equiv a\frac{d^2}{dx^2} + b\frac{d}{dx} + c.$$

We also sometimes write L as

$$L \equiv aD^2 + bD + c,$$

where $D = d/dx$. Now

$$L(e^{mx}) = (am^2 + bm + c)e^{mx} = p(m)e^{mx}, \tag{3}$$

where $p(m) = am^2 + bm + c$. Thus, e^{mx} is a solution of (1) if $p(m) = 0$.

(i) If $p(m) = 0$ has two distinct real roots m_1, m_2, then both e^{m_1x} and e^{m_2x} are solutions of (1). Since, $m_1 \neq m_2$, they are also LI. Thus, the general solution to (1) is

$$y = C_1e^{m_1x} + C_2e^{m_2x}.$$

Example 1. *Solve* $y'' - y' = 0$

Solution: The characteristic equation is $m^2 - m = 0 \Rightarrow m = 0, 1$. The general solution is $y = C_1 + C_2 e^x$

(ii) If $p(m) = 0$ has real equal roots $m_1 = m_2 = m$, then e^{mx} is a solution of (1). To find the other solution, note that if m is repeated root, then $p(m) = p'(m) = 0$. This suggests differentiating (3) w.r.t. m. Since L consists of differentiation w.r.t. x only,

$$\frac{\partial}{\partial m}\Big(L(e^{mx})\Big) = L\left(\frac{\partial}{\partial m} e^{mx}\right) = L(xe^{mx}).$$

$$L(xe^{mx}) = p(m)xe^{mx} + p'(m)e^{mx},$$

where $'$ represents the derivative. Since, m is a repeated root, the RHS is zero. Thus, xe^{mx} is also a solution to (1) and it is independent of e^{mx}. Hence, the general solution to (1) is

$$y = (C_1 + C_2 x)e^{mx}.$$

$\Big($We can also solve by reduction of oder technique i.e. $y_1 = e^{mx}$ and $y_2 = v(x)y_1 = v(x)e^{mx}$. From the given ODE, we find

$$av'' + (2am + b)v' + (am^2 + bm + c)v = 0$$

Since $m = m_1 = m_2$ is a double root, we must have $am^2 + bm + c = 0$ and $m = -b/2a \Rightarrow 2am + b = 0$. Hence, $v" = 0 \Rightarrow v' = 1 \Rightarrow v = x$ and hence $y_2 = xe^{mx}\Big)$

Example 2. *Solve* $y'' - 2y' + y = 0$

Solution: The characteristic equation is $m^2 - 2m + 1 = 0 \Rightarrow m = 1, 1$. The general solution is $y = (C_1 + C_2 x)e^x$

(iii) If the roots of (2) are complex conjugate, say $m_1 = \alpha + i\beta$ and $m_2 = \alpha - i\beta$, then two LI solutions are $Y_1 = e^{(\alpha+i\beta)x}$ and $Y_2 = e^{(\alpha-i\beta)x}$. But these are complex valued. Note that if Y_1, Y_2 are LI, then so does $y_1 = (Y_1 + Y_2)/2$ and $y_2 = (Y_1 - Y_2)/2i$. Hence, two real LI solutions of (1) are $y_1 = e^{\alpha x}\cos(\beta x)$ and $y_2 = e^{\alpha x}\sin(\beta x)$. Thus, the general solution to (1) is

$$y = e^{\alpha x}(C_1 \cos(\beta x) + C_2 \sin(\beta x)).$$

Example 3. *Solve* $y'' - 2y' + 5y = 0$

Solution: The characteristic equation is $m^2 - 2m + 5 = 0 \Rightarrow m = 1 \pm 2i$. The general solution is $y = e^x(C_1 \cos 2x + C_2 \sin 2x)$

2 Homogeneous n-th order linear equation with constant coefficients

Now the ODE is of the form

$$a_0 y^{(n)}(x) + a_1 y^{(n-1)}(x) + a_2 y^{(n-2)}(x) + \cdots + a_{n-1}y^{(1)}(x) + a_n y(x) = 0, \qquad x \in \mathcal{I}, \quad (4)$$

where the superscript (i) denotes the i-th derivative and all a_i's are constants. As in the case of 2nd order linear equation, the LI solutions of (4) depends on the characteristic equations

$$a_0 m^n + a_1 m^{n-1} + \cdots + a_{n-1} m + a_n = 0. \tag{5}$$

Obviously, this equation has n roots. As in the case of 2nd order equation, the following can be proved.

Theorem 2. *The fundamental set of solutions $\mathcal{B}$ for (4) is obtained using the following two rules:*

Rule 1: *If a root m of (5) is real and repeated k times, then this root gives k number of LI solutions $e^{mx}, xe^{mx}, x^2e^{mx}, \cdots, x^{k-1}e^{mx}$ to $\mathcal{B}$.*

Rule 2: *If the roots $m = \alpha \pm i\beta$ of (5) is complex conjugate ($\beta \neq 0$) and are repeated k times each, then they contribute $2k$ number of LI solutions $e^{\alpha x}\cos(\beta x), e^{\alpha x}\sin(\beta x)$, $xe^{\alpha x}\cos(\beta x), xe^{\alpha x}\sin(\beta x), x^2e^{\alpha x}\cos(\beta x), x^2e^{\alpha x}\sin(\beta x), \cdots\cdots, x^{k-1}e^{\alpha x}\cos(\beta x)$ and $x^{k-1}e^{\alpha x}\sin(\beta x)$ to $\mathcal{B}$.*

Example 4. *Solve* $y^{(5)}(x) + y^{(4)}(x) - 2y^{(3)}(x) - 2y^{(2)}(x) + y^{(1)}(x) + y = 0$

Solution: The characteristic equation is $m^5 + m^4 - 2m^3 - 2m^2 + m + 1 = 0 \Rightarrow (m+1)^3(m-1)^2 = 0 \Rightarrow m = -1, -1, -1, 1, 1$. The general solution is $y = e^{-x}(C_1 + C_2 x + C_3 x^2) + e^x(C_4 + C_5 x)$

Example 5. *Solve* $y^{(6)}(x) + 8y^{(5)}(x) + 25y^{(4)}(x) + 32y^{(3)}(x) - y^{(2)}(x) - 40y^{(1)}(x) - 25y = 0$

The characteristic equation is $m^6 + 8m^5 + 25m^4 + 32m^3 - m^2 - 40m - 25 = 0 \Rightarrow (m+1)(m-1)(m^2+4m+5)^2 = 0 \Rightarrow m = -1, 1, -2 \pm i, -2 \pm i$. The general solution is $y = C_1 e^{-x} + C_2 e^x + e^{-2x}\Big((C_3 + C_4 x)\cos x + (C_5 + C_6 x)\sin x\Big)$