Session 2003-04 | **PHY 452** | Semester II

Electromagnetic Theory I

END-SEMESTER EXAMINATION

BRIEF ANSWERS & SOLUTIONS

1. Answer the following in a few lines, Your answers must be precise and unambiguous. Detailed derivations are unnecessary, but no credit will be given for mere assertions without any explanation.

[25 × **2**]

(a) If the force between charges is given by Coulomb's law, why do we need the concept of a field of force?

Ans: Coulomb's law is an action-at-a distance law. We need the concept of a field because physics, being an empirical science, requires some intermediate agency to carry information about one charge to another, and vice versa. Moreover, in special relativity, signals have a maximum velocity, which means that this information propagates at a finite speed. The spreading-out of such information is completely equivalent to setting up a field of force.

(b) Under what condition(s) is the principle of linear superposition applicable?

Ans: If two fields $\vec{E}_1$ and $\vec{E}_2$ superpose to give a field $\vec{E} = f(\vec{E}_1, \vec{E}_2)$, them in the limit of weak fields, we can make a Taylor expansion

$$\vec{E} = f(\vec{E}_1, \vec{E}_2) = \vec{f}(\vec{0}, \vec{0}) + \vec{E}_1 . \left.\frac{\partial \vec{f}}{\partial \vec{E}_1}\right|_{\vec{E}_i=\vec{0}} + \vec{E}_2 . \left.\frac{\partial \vec{f}}{\partial \vec{E}_2}\right|_{\vec{E}_i=\vec{0}} + \ldots$$

If we make the very reasonable demand that $\vec{E} \to \vec{0}$ when $\vec{E}_1 = \vec{E}_2 \to \vec{0}$, then $f(\vec{0}, \vec{0}) = \vec{0}$ and hence

$$\vec{E} = \vec{E}_1 . \left.\frac{\partial \vec{f}}{\partial \vec{E}_1}\right|_{\vec{E}_i=\vec{0}} + \vec{E}_2 . \left.\frac{\partial \vec{f}}{\partial \vec{E}_2}\right|_{\vec{E}_i=\vec{0}} + \ldots \simeq \lambda_1 \vec{E}_1 + \lambda_2 \vec{E}_2 + \ldots$$

where $\lambda_{1,2}$ are, in general, tensors. In the weak field limit, we can neglect the higher order terms, leading to the principle of linear superposition.

Note: Those who have written that Coulomb's law in linear in charge get no credit.

(c) What geometric information is contained in the inverse square law?

Ans: If we write down, in d dimensions, Gauss' Law of constancy of electric flux over concentric spheres enclosing a point charge Q, we get a Coulomb-like Law

$$\vec{E} \propto \frac{Q}{r^{d-1}}$$

The fact that empirically we get an inverse square law is a manifestaion of the fact that $d = 3$, i.e. we live in three dimensions.

Note: Those who have written that the inverse square law is an expression of spherical symmetry get no credit.

(d) Relate 1 statCoulomb of charge to 1 Coulomb of charge (need not be numerical).

Ans: The Coulomb is defined as 0.1 abCoulomb, which is the electromagnetic unit of charge. The relation between statCoulomb and abCoulomb is simply that c statCoulombs = 1 abCoulomb, so that the $\frac{1}{c}$ factor is absorbed in the Biot-Savart law, if written in e.m.u. Hence,

$$1 \text{ Coulomb} = \frac{c}{10} \text{ statCoulomb} .$$

Note: Those who have tried a numerical relation have been marked accordingly. The correct answer is

$$1 \text{ Coulomb} \simeq 3 \times 10^9 \text{ statCoulomb} .$$

(e) Electrostatic potential is not a measurable quantity. What, then, is measured in a voltmeter?

Ans: All voltmeters are essentially galvanometers, i.e. the needle is moved by a magnetic force set up by a current. Hence, it is essentially current which is measured by all galvanometers. If, now, we assume that Ohm's law holds, we can calibrate the voltmeter to measure potential *difference*. To the extent that Ohm's law may be assumed, the potential *difference* is a measurable quantity, unlike the potential itself.

Note: Those who have just writen that a voltmeter measures potential difference get partial credit only.

(f) The electrostatic force is a conservative force. Is this still true when we allow time-variation of fields? If not, does this violate the principle of conservation of energy?

Ans: The integral of electrostatic force around a closed path C, enclosing a surface $S(C)$ is

$$\oint_C q\vec{E}.\vec{d\ell} = q \int_S ds\ \hat{n}.(\vec{\nabla} \times \vec{E}) = -\frac{q}{c} \int_S ds\ \hat{n}.\partial_t \vec{B} = -\frac{q}{c}\partial_t \int_S ds\ \hat{n}.\vec{B} = -\frac{q}{c}\frac{\partial \Phi_B}{\partial t}$$

where Φ_B is the magnetic flux. Thus, the electrostatic force is not conservative if we allow the magnetic field to have a time-variation. This does not violate the principle of conservation of energy because the dissipated energy is carried off by the electromagnetic field in the form of radiation.

(g) The electrostatic potential is always continuous at the interface of two media, but may not be a smooth function at the interface. Is this statement correct?

Ans: The statement is correct because

(i) the electric potential must be continuous at the interface to prevent its derivativem the electric field, from having a singularity. Such a singularity would represent an infinite force on a charge crossing the surface, which is clearly unphysical.

(ii) the normal component of the electric field may have a discontinuity of $4\pi\sigma$ at the interface. This means that the potential cannot be smooth, unless the surface charge-density σ vanishes.

Note: To get full credit you must explain both why the potential is continuous and why its derivative may be discontinuous.

(h) Correct this statement: In order to determine a solution of Laplace's equation for the electrostatic potential, it is necessary to know the electric field over the bounding surface.

Ans: Corrected statement: In order to determine a solution of Laplace's equation for the electrostatic potential, it is necessary to know the normal component of the electric field over the bounding surface, or the potential itself over the bounding surface.

Note: For full credit you must mention both the Dirichlet and Neumann boundary conditions.

(i) A point charge is clearly a fiction and does not represent a natural object. Why, then, do we use it?

Ans: A point charge is used for the following reasons:
(*i*) an infinitesimal volume element containing a charge can be treated as a point charge, and finite distributions built out of such elements,
(*ii*) a bounded charge distribution, at large distances, behaves like a point charge,
(*iii*) spherically symmetric charge distributins behave like point charges concentrated at the centre of symmetry,

Note: Full credit as been given for mentioning any two of these (related) points.

(j) The multipole expansion is useful in practical computations. Why?

Ans: Coulomb's law tells us that the electrostatic potetial at a point $\vec{x}$ due to a charge distribution $\rho(\vec{x}')$ confined in a volume $V'(S')$ is given by

$$\phi(\vec{x}) = \int_{V'} d^3\vec{x}' \ \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

Using this formula to map the potential means that we have to do an integral (sum an infinite series) for every point $\vec{x}$ which we map. This is cumbersome and slow, even numerically on a computer.

By making a multipole expansion, we replace

$$\phi(\vec{x}) = \frac{Q}{r} + \frac{\vec{\mu}.\vec{x}}{r^3} + \ldots \quad (|\vec{x}| = r)$$

which is an algebraic relation, involving $\vec{x}$ only, and easy to compute. Since the series converges very fast for large values of r, it is enough, for most practical purposes, to calculate just the first few moments Q, $\vec{\mu}$, etc., which are admittedly integrals. Thus, we replace an integral at every point $\vec{x}$ by just a few integrals and a simple algebraic relation.

(k) We know that the ionic bond between atoms e.g. Na^+ and Cl^- is a purely electrostatic bond. According to Earnshaw's theorem, then, no ionic molecule would be stable. If this is true, why doesn't the salt on your dining table disappear?

Ans: When the two ions try to come close together, as required by Earnshaw's theorem, the electron clouds begin to overlap. By the Pauli exclusion principle, no two electrons can have exactly the same configuration, so a repulsive force develops. This appears in the effective interaction as an *exchange term*, which has the opposite sign to the usual Coulomb term. The repulsive force is called the *degeneracy force*, and is of non-electrostatic origin.

Note: The nuclear interactions have no part to play in molecular stability.

(l) According to Thomson's theorem, some work must be done to pull an uncharged conductor away from a set of other (charged) conductors. Why do you think this happens?

Ans: What Thomson's theorem is telling us is that an uncharged conductor is attracted to the other (charged) conductors, and hence work must be done against this attractive force to pull it away. This attractive force is just the usual electrostatic attraction between the charges and the induced charge on the conductor we are trying to remove. As usual, this is induced so as to cause maximum attraction.

Note: No credit has been given for explanations based on the formula for electrostatic energy density.

(m) A conducting globe is being charged by a friction belt. Can this process continue indefinitely?

Ans: Since the globe has a finite capacitance equal to its radius, the increase in charge will have two consequences.
(*i*) the charge density on the surface will keep on growing, and hence the electrostatic pressure outwards will keep growing;
(*ii*) the potential of the conductor will also keep on growing.
This process cannot continue indefinitely. Either the pressure will become so much that the globe will burst apart, or the potential will become so much that discharges in the form of sparks will occur in the medium surrounding the globe. Which will happen first depends on the rigidity of the globe and the nature of the dielectric surrounding it.

Note: You have to mention both possibilities to get full credit.

(n) If electrostatic energy resides where the charges reside, does the electrostatic energy *density* $|\vec{E}|^2/8\pi$ have any physical significance at all?

Ans: The physical significance of the electrostatic energy density $u = |\vec{E}|^2/8\pi$ lies in its connection with Maxwell's stress tensor. The eigenvalues of the stress tensor are proportional to the energy density u. These are measurable quantities (since force is measurable) and hence u has a physical significance. Treated purely as an energy density, we can never measure u.

Note: Full credit has been given if any mention of electrostatic pressure or the stress tensor was made.

(o) The electrostatic energy density $|\vec{E}|^2/8\pi$ is sometimes divergent. Does this mean that the conservation of energy principle is violated?

Ans: The electrostatic energy density becomes divergent if we include singular sources like point charges, dipoles, etc. However, these contribute constant, though divergent terms to the total energy and do not affect *changes* in the total energy. The principle of conservation of energy is formulated only for energy changes and does not place any constraints on the total energy of the system.

(p) The fact that the charge on a conductor is proportional to its potential is ultimately a consequence of which laws or assumptions?

Ans: The relation between potential and charge is linear because Laplace's equation is a linear differential equation. This is ultimately a consequence of the principle of linear superposition.

Note: This is *not* a consequence of Coulomb's law, because that merely fixes the form of the linear differenmtial operator, and not its linear property.

(q) The dielectric polarisation $\vec{p}(\vec{x})$ depends on the applied electric field $\vec{E}(\vec{x})$. Discuss the nature of this dependence in different media.

Ans: The relation between dielectric polarisation and electric field has the form

$$\vec{p}(\vec{x}) = \chi[\vec{x}, \vec{E}(\vec{x})]\vec{E}(\vec{x})$$

where the rank-2 tensor $\chi[\vec{x}, \vec{E}(\vec{x})]$ remains finite in the limit $\vec{E} \to \vec{0}$.
If the medium in *linear*, $\chi[\vec{x}, \vec{E}(\vec{x})] = \chi(\vec{x})$.
If the medium is also *homogeneous*, $\chi(\vec{x}) = \chi$.
If the medium is also *isotropic*, $\chi(\vec{x}) = \chi\mathbf{1}$, where χ is a scalar and $\mathbf{1}$ is the unit tensor.

(r) The force between two point charges in a dielectric medium with $\epsilon > 1$ is diminished, but the electrostatic energy is increased. Isn't this a paradox?

Ans: The force between two point charges in a dielectric is of the form $F = q_1q_2/\epsilon r^2$, which is diminished for $\epsilon > 1$ (usual case). The electrostatic energy density is $u = \vec{E}.\vec{D}/8\pi = \epsilon|\vec{E}|^2/8\pi$, which increases for $\epsilon > 1$. If the force decreases but energy increases, it looks paradoxical. But this is not really so, since the presence of dielectric polarisation effectively increases the charge density in the medium and hence the electrostatic energy includes the energies of these bound charges.

(s) The Clausius-Mossotti relation is valid only in the limit of perfectly random or perfectly regular distribution of molecular dipoles. In the intermediate case, qualitatively explain how the equation would change.

Ans: The internal electric field in a dielectric is given by

$$\vec{E}_{int} = \frac{4\pi}{3}\vec{p} + \vec{E}_N = \frac{4\pi}{3}(1+\lambda)\vec{p}$$

assuming that $\vec{E}_N = \frac{3}{4\pi}\lambda\vec{p}$, with an unknown numerical constant λ depending on the medium. If the rest of the derivation is pursued, we get a modified Clausius-Mossotti equation

$$\alpha = \frac{3}{4\pi N}\left(\frac{1}{1+\lambda}\right)\left(\frac{\epsilon-1}{\epsilon+2}\right)$$

(t) Drude's classical theory of resistance requires a charge-carrier to collide with the molecules of the conducting medium and give up momentum. Explain why this idea does not work in quantum mechanics.

Ans: In quantum mechanics, the electron is considered as a matter wave propagating through a three-dimensional lattice. Collisions with the ionic cores still occur, but the electron density waves can now diffract around these. Treated as a three-dimensional diffraction greeting, it can be shown that the electron waves build up everywhere as Bloch functions, which are eigenstates of momentum, i.e. momentum is conserved. Thus there is no resistance in a perfect crystal.

Note: No credit has been gven for answers which say that trajectories do not exist in quantum mechanics. Some have written that momentum is not well-defined in quantum mechanics because of the uncertainty principle. This is simply wrong.

(u) All magnetic fields are due to currents. Justify this statement.

Ans: Since Maxwell's equation $\vec{\nabla}.\vec{B} = 0$ is of universal applicability, this means that all sources of magnetic fields are such as to generate dpole moments but no monopole moments. Current sources following the Biot-Savart law satisfy this criterion. Hence we replace every other magnetic source, whatever its nature, by an *effective* current. The above statement is true only to this extent.

Note: Some have written that atomic sources of magnetic field have circulating currents. This is a semi-classical idea and does not work in quantum mechanics because probability/charge densities for electron clouds are time-independent. Moreover, some magnetic fields are due to spin, which is a purely quantum mechanical effect. No credit has been given for such answers.

(v) The magnetic field is known to be gauge invariant, i.e. we can always write it as $\vec{B} = \vec{\nabla} \times \vec{A}$ where $\vec{A} \to \vec{A} + \vec{\nabla}\psi$. Can we take $\psi = \log r$ and hence create a magnetic monopole term?

Ans: Gauge invariance of the magnetic field arises from the Maxwell equation $\vec{\nabla}.\vec{B} = 0$, and any gauge function must be compatible with this. If we change $\vec{A} \to \vec{A} + \vec{\nabla}\log r = \vec{A} + \frac{\hat{r}}{r}$, we get a monopole term with a pole-type singularity at $r \to 0$. Taking a judicious limit, this would mean that $\vec{\nabla}.\vec{B} = 4\pi \neq 0$ (just as in electrostatics), which contradicts the original assumption. Hence $\psi = \log r$ is not an acceptable gauge function.

Note: It is not enough to just say that the gauge function leads to a singular term in the vector potential, because there is no problem with such terms in general.

(w) Ampére's law can be derived from the Lorentz force but not vice versa. Why is this?

Ans: If we consider a steady stream of point charges building up a steady currrent and then apply the Lorentz force relation to each of them, it is easy to derive Ampére's law. The converse is not possible since a single charge, even if it moves with a uniform velocity, does not constitute a steady current.

(x) A superconducting medium exhibits a *Meissner effect*, i.e. the magnetic induction $\vec{B}$ inside the medium vanishes even if there is a finite magnetic field $\vec{H}$ outside. Calculate the magnetic susceptibility χ_m of the medium.

Ans: The applied magnetic field $\vec{H}$ and the magnetic induction $\vec{B}$ inside the superconductor are related by

$$\vec{B} = (1 + 4\pi\chi_m)\vec{H} \ ,$$

In a superconductor, the Meissner effect ensures that $\vec{B} = \vec{0}$ even when $\vec{H} \neq \vec{0}$. Hence $\mu = 0$ and hence $\chi_m = -1/4\pi$.

Note: If the formulae for $\vec{B}$ and $\vec{H}$ are reversed, so must be their roles in a superconductor, leading to the same result. Those who did one, but not the other, got $\chi_m \to \infty$, which is wrong.

(y) Why is the displacement current not a true current?

Ans: The displacement current is not a true current because it is not due to motion of charges but is due to time-variation of the electric field. It is, thus, another example of an effective current.

Note: No credit was given for those who wrote that displacement current cannot be measured. Consider the simplecase of a capacitor being charged by a cell: the charging current crosses the space between the capacitor plates as displacement current, and it is very much measurable.

2. Work out the following problems. You need not explain notations if they are standard ones or correspond to those used in the lectures. However, numerical results must be accompanied by the appropriate units.

[6 × **5**]

(a) A point charge Q is placed midway between two infinite parallel conducting planes, each of which is grounded. Calculate the electrostatic energy of the system.

(b) A cube of silver, of side 2 cm, is given a total charge 6 statCoulombs. Calculate the traceless quadrupole moment tensor of this charge about the centre of the cube as origin.

(c) Find the magnetic vector potential corresponding to a constant magnetic field $\vec{B}$. Is this unique? Give reasons for your answer.

(d) A bar magnet is allowed to drop under the influence of gravity through a horizontal coil of wire. Describe its motion qualitatively with the help of one or two graphs.

(e) A particle of mass m and charge q is rotating in a circular orbit under the influence of a magnetic field B along the axis of the circle. What is the frequency of the rotation?

3. (a) A sphere of radius a and dielectric constant ϵ is placed in vacuum (air) in a constant electric field $\vec{E}$. Solve Laplace's equation in the region exterior to the sphere and calculate the electric field. Hence determine the surface-density of charge on the surface of the sphere.

(b) Write down Maxwell's equations in a medium in the absence of sources. Show that they lead to wave equations for the electric and magnetic fields. Write down solutions for these equations and show that the waves are of transverse nature. Under what conditions would these waves travel at speeds grater than c? If this indeed happens, is this a violation of Einstein's principle of special relativity?

[2 × **10**]