Session 2003-04 **PHY 452** Semester II

Electromagnetic Theory I

Problem Set I

Try and investigate all aspects of the problems posed below. Calculate numerical results wherever possible.

1. At the atomic scale, the electromagnetic force is incomparably greater than the gravitational force. Convince yourself of this by the following excercises.

   (a) Calculate the electrostatic force between a proton and an electron separated by a distance a_0, where a_0 is the Bohr radius. Then calculate the gravitational force between the same objects. Find the ratio.

   (b) Consider two protons separated by a distance of 1 fm and calculate the electrostatic repulsion between them. What would be the mass of the proton if this were exactly balanced by their mutual gravitational attraction? (This hypothetical mass is, in fact, known as the *Planck mass*.)

   (c) Estimate the electrostatic repulsion between two surface layers of atoms, separated by a distance of 10 Å and assuming the bulk contains N_A atoms, where N_A is Avogadro's number. How large a mass would you have to place on such a surface to crush the stoms?

2. Many vector identities can be quickly derived/proved by writing them as rank-one (Cartesian) tensors. Use this method to establish the following well-known identities. Use the notation $\vec{A} = \hat{e}_i A_i$.

   (a) Box product: $\vec{A}.(\vec{B} \times \vec{C}) = \vec{B}.(\vec{C} \times \vec{A}) = \vec{C}.(\vec{A} \times \vec{B})$,

   (b) BAC-CAB rule: $\vec{A} \times (\vec{B} \times \vec{C}) = (\vec{B}.\vec{A})\vec{C} - (\vec{C}.\vec{A})\vec{B}$,

   (c) Jacobi identity: $\vec{A} \times (\vec{B} \times \vec{C}) + \vec{B} \times (\vec{C} \times \vec{A}) + \vec{C} \times (\vec{A} \times \vec{B}) = \vec{0}$.

3. The same method is also good for establishing vector identities involving derivatives. Use the notation $\vec{\nabla} = \hat{e}_i \partial_i$.

   (a) $\vec{\nabla}.\vec{\nabla}\phi(\vec{x}) = \nabla^2\phi(\vec{x})$, $\quad \vec{\nabla} \times \vec{\nabla}\phi(\vec{x}) = 0$,

   (b) $\vec{\nabla}.(\vec{\nabla} \times \vec{A}(\vec{x})) = 0$, $\quad \vec{\nabla} \times (\vec{\nabla} \times \vec{A}(\vec{x})) = \vec{\nabla}(\vec{\nabla}.\vec{A}(\vec{x})) - \nabla^2\vec{A}(\vec{x})$,

   (c) $\vec{\nabla}(\phi_1\phi_2) = \phi_1\vec{\nabla}\phi_2 + \phi_2\vec{\nabla}\phi_1$, $\quad \vec{\nabla}.(\vec{A} \times \vec{B}) = \vec{B}.(\vec{\nabla} \times \vec{A}) - \vec{A}.(\vec{\nabla} \times \vec{B})$,

   (d) $\vec{\nabla}.(\phi\vec{A}) = (\vec{\nabla}\phi).\vec{A} + \phi(\vec{\nabla}.\vec{A})$, $\quad \vec{\nabla} \times (\phi\vec{A}) = (\vec{\nabla}\phi) \times \vec{A} + \phi(\vec{\nabla} \times \vec{A})$,

(e) $\vec{\nabla}(\vec{A}.\vec{B}) = (\vec{B}.\vec{\nabla})\vec{A} + (\vec{A}.\vec{\nabla})\vec{B} + \vec{B} \times (\vec{\nabla} \times \vec{A}) + \vec{A} \times (\vec{\nabla} \times \vec{B})$,
$\vec{\nabla} \times (\vec{A} \times \vec{B}) = (\vec{B}.\vec{\nabla})\vec{A} - (\vec{A}.\vec{\nabla})\vec{B} - \vec{B}(\vec{\nabla}.\vec{A}) + \vec{A}(\vec{\nabla}.\vec{B})$.

4. Coulomb's law can be used directly to calculate the electrostatic force due to simple systems of point charges. Try the following examples.

   (a) Two point charges $[+Q, a\hat{\imath}]$ and $[+Q, -a\hat{\imath}]$.

   (b) Two point charges $[+Q, a\hat{\imath}]$ and $[-Q, -a\hat{\imath}]$.

   (c) Four point charges $[+Q, a\hat{\imath}]$, $[+Q, -a\hat{\imath}]$, $[+Q, a\hat{\jmath}]$, $[+Q, -a\hat{\jmath}]$.

   (d) Prove that the field at the centre of a uniformly charged ring vanishes. Is this a point of stability or instability for a (positive) test charge?

   Sketch the lines of force and equipotential surfaces in each case.

5. For each of the following charge distributions, try calculating the electrostatic field or potential, both by direct integration of Coulomb's law and by application of Gauss' law.

   (a) A straight infinite line with uniform charge distribution λ per unit length.

   (b) An infinite plane with uniform charge distribution σ per unit area.

   (c) A spherical shell of radius a with a constant surface density of charge σ per unit area.

   (d) A charged disc of radius a and surface density of charge σ per unit area. Consider points both on and off the axis of symmetry.

   Sketch the lines of force and equipotential surfaces in each case.

6. (a) Given scalar fields $\phi(\vec{x})$ and vector fields $\vec{A}(\vec{x})$, which vanish at infinity at least as fast as r^{-1}, establish the following relations:

$$\int d^3\vec{x}\ \phi_1\vec{\nabla}\phi_2 = -\int d^3\vec{x}\ \phi_2\vec{\nabla}\phi_1\ ,$$

$$\int d^3\vec{x}\ \phi\vec{\nabla}.\vec{A} = -\int d^3\vec{x}\ \vec{\nabla}\phi.\vec{A}\ ,$$

$$\int d^3\vec{x}\ \phi\vec{\nabla} \times \vec{A} = -\int d^3\vec{x}\ \vec{\nabla}\phi \times \vec{A}\ .$$

   These formulae are known as *Integration by parts.*

   (b) Show that for a volume V bounded by a closed surface S, the electrostatic field and potential obey the relation:

$$\int_V d^3\vec{x}\ \vec{E}(\vec{x}) = -\oint_S ds\ \phi\hat{n}\ .$$

(c) Show that for a surface S bounded by a closed contour C, the electrostatic field and potential obey the relation:

$$\int_S ds\ \vec{E} \times \hat{n} = \oint_C \phi d\vec{\ell}\ .$$

7. (a) Consider the electric field

$$\vec{E}(\vec{x}) = (yz - 2x)\hat{\imath} + xz\hat{\jmath} + xy\hat{k}\ .$$

Is this acceptable by the laws of electrostatics? If so, find the potential and charge density at the point $\vec{x}$.

(b) Consider a hollow sphere with internal radius a and external radius b. The space between these surfaces is filled with a charge density $\rho(r) = \rho_0 r^k$, where k is an integer. Find the electric field $\vec{E}$ everywhere. What happens in the limit $a \to 0$?

(c) Consider a spherical shell of radius a. On the outside of this shell, there exists a surface charge density

$$\sigma = \sigma_0 \cos\theta\ ,$$

where θ is the polar angle measured from the north pole of the sphere. Calculate the electrostatic field and potential created by this charge distribution along the axis of symmetry. How would you calculate these at off-axis points?

8. (a) The electrostatic potential created by a hydrogen atom is given by

$$\phi(r) = \frac{e_0}{r}\left(1 + \frac{r}{a_0}\right) e^{-2r/a_0}\ ,$$

where e_0 is the electronic charge and a_0 is the Bohr radius. Calculate the charge density $\rho(\vec{x})$ and interpret your result physically.

(b) A uniform surface charge density σ exists on a hollow cone of height a and radius a. Calculate the potential difference between the apex of the cone and the centre of its base.

(c) Two identical discs of radius r are placed parallel to each other with centres on the same vertical line at a distance d apart. Each carries a uniform surface charge density σ. Calculate the electrostatic field and the electrostatic potential. Discuss the limits $r \gg a$ and $r \ll a$.

9. Find the monopole, dipole and quadrupole moments of the following charge distributions.

(a) Two line charges, each of length a and charge density λ, in the form of a Greek cross, about the point of intersection.

(b) A charged ring of radius a and charge density λ, about the centre of the ring.

(c) A sphere of radius r and uniform charge density ρ, about its own north pole.

(d) A spherical shell of radius a and a surface charge density $\sigma = \sigma_0 \cos\theta$ (see above), about the centre of the sphere.

10. (a) A point charge Q is placed at each of the corners of a cube of side a. Calculate the electrostatic energy of this system.

(b) A test charge $+q$ of mass m is placed at the centre between two fixed point charges $+Q$ each. Show that for small displacements from the centre, the test charge will undergo simple harmonic motion. Then investigate how the electrostatic energy of the system changes with time.

(c) Find the electrostatic energy of a sphere having radius r and filled with a uniform charge density ρ.

(d) Find the self energy of a charge distribution $\rho(r) = \rho_0 e^{-2r/a_0}$, which is clearly the $1s$ state of hydrogen. Relate ρ_0 to the electronic charge and calculate the self-energy numerically. How does this compare with the rest energy m_0c^2 of the electron?