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PROJECTIONS IN THE CONVEX HULL OF THREE SURJECTIVE ISOMETRIES ON $C(\Omega)$

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Abstract. Let Ω be a compact connected Hausdorff space. We define generalized n-circular projection on $C(\Omega)$ as a natural analogue of generalized bi-circular projection and show that such a projection P can always be represented as $P = \frac{I+T+T^2+\cdots+T^{n-1}}{n}$ where I is the identity operator and T is a surjective isometry on $C(\Omega)$ such that $T^n = I$. We next show that if convex combination of three distinct surjective isometries on $C(\Omega)$ is a projection, then it is a generalized 3-circular projection.

1. Introduction

Let X be a complex Banach space and $\mathbb{T}$ denote the unit circle in the complex plane. A projection P on X is said to be a generalized bi-circular projection (hence forth GBP) if there exists a $\lambda \in \mathbb{T} \setminus \{1\}$ such that $P + \lambda(I - P)$ is a surjective isometry on X. Here I denotes the identity operator on X.

The notion of GBP was introduced in [7]. In [2] it was shown that a projection on $C(\Omega)$, where Ω is a compact connected Hausdorff space, is a GBP if and only if $P = \frac{I+T}{2}$, where T is a surjective involution of $C(\Omega)$, that is $T^2 = I$. Similar result was obtained for GBP in $C(\Omega, X)$ when X is a complex Banach space for which vector-valued Banach Stone Theorem holds true. In [4] it was shown that the set of GBP's on $C(\Omega)$ is algebraically reflexive and a description of the algebraic closure of GBP's in $C(\Omega, X)$ was also obtained.

In [1] an interesting characterization of GBP's on $C(\Omega)$ was obtained. It was shown that if P is any projection on $C(\Omega)$ such that $P = \alpha T_1 + (1-\alpha)T_2$, $\alpha \in (0,1)$, T_1, T_2 are two surjective isometries on $C(\Omega)$, then $\alpha = \frac{1}{2}$ and P can be written as $\frac{I+T}{2}$ for some surjective isometry T such and $T^2 = I$. This shows any projection which is convex combination of two surjective isometries on $C(\Omega)$ is indeed a GBP. Motivated by this, in the same paper, the author introduced the notion of generalized n-circular projection as follows. A projection P on a Banach space X is a generalized n-circular projection if there exists a surjective isometry L on X of order n, that is $L^n = I$, such that $P = \frac{I+L+L^2+\cdots+L^{n-1}}{n}$. It was suggested

2000 *Mathematics Subject Classification.* 47L05; 46B20.

Key words and phrases. Isometry, Generalized 3-circular projection.

in [1] that any projection which is in the convex hull of 3 surjective isometries on $C(\Omega)$ should be a generalized 3-circular projection. It was proved in [3] that if $P = \frac{T_1+T_2+T_3}{3}$, where $T_i,\ i = 1, 2, 3$ are surjective isometries on $C(\Omega)$ and P is a projection then there exists a surjective isometry T such that $P = \frac{I+T+T^2}{3}$ and $T^3 = I$, hence P is a generalized 3-circular projection.

In this paper we try to complete this circle of ideas on generalized 3-circular projections on $C(\Omega)$ as obtained in [1] for GBP's. We start with the following definition of a generalized n-circular projection which is a more natural one to start with if we want to put the definition of GBP in this general set up.

Definition 1.1. Let X be a complex Banach space. A projection P_0 on X is said to be a generalized n-circular projection, $n \geq 3$, if there exist $\lambda_1, \lambda_2, \cdots, \lambda_{n-1} \in \mathbb{T} \setminus \{\pm 1\}$, $\lambda_i,\ i = 1, 2, \cdots, n-1$ are of finite order and projections $P_1, P_2, \cdots, P_{n-1}$ on X such that

(a) If $i \neq j, i, j = 1, 2, \cdots, n-1$ then $\lambda_i \neq \pm\lambda_j$
(b) $P_0 \oplus P_1 \oplus \cdots \oplus P_{n-1} = I$
(c) $P_0 + \lambda_1 P_1 + \cdots + \lambda_{n-1} P_{n-1}$ is a surjective isometry.

Note that in the case of GBP, if $P + \lambda(I - P)$ is a surjective isometry and $\lambda \in \mathbb{T} \setminus \{1\}$ is of infinite order then P is a hermitian projection (see [8]). Such projections were called trivial in [4, 8]. Thus in Definition 1.1 it is natural to start with λ_i's which are of finite order.

If P is a projection on $C(\Omega)$ such that $P = \frac{I+T+T^2+\cdots+T^{n-1}}{n}$ for a surjective isometry T such that $T^n = I$ then it is easy to show that P is a generalized n-circular projection in the sense of Definition 1.1. To see this, let $\lambda_0 = 1, \lambda_1, \lambda_2, \cdots, \lambda_{n-1}$ be the n distinct roots of identity. For $i = 1, 2, \cdots, n-1$, we define $P_i = \frac{I+\overline{\lambda_i}T+\overline{\lambda_i}^2T^2+\cdots+\overline{\lambda_i}^{n-1}T^{n-1}}{n}$. Then each P_i is a projection, $P \oplus P_1 \oplus P_2 \oplus \cdots \oplus P_{n-1} = I$ and $P_0 + \lambda_1 P_1 + \lambda_2 P_2 + \cdots + \lambda_{n-1} P_{n-1} = T$.

Our first result shows that the definition of generalized n-circular projection given in Definition 1.1 is equivalent to the one considered in [1, 3] for the space $C(\Omega)$. We prove our result for $n = 3$ and the proof in the general case follows the same line of argument. In particular we show

Theorem 1.2. *Let Ω be a compact connected Hausdorff space and P_0 a generalized 3-circular projection on $C(\Omega)$. Then there exists an surjective isometry L on $C(\Omega)$ such that*

(a) $P_0 + \omega P_1 + \omega^2 P_2 = L$ where P_1 and P_2 are as in Definition 1.1 and ω is a cube root of identity,
(b) $L^3 = I$.

Hence $P_0 = \frac{I+L+L^2}{3}$.

Next we prove that a projection in the convex hull of 3 isometries is either a GBP or a generalized 3-circular projection.

Theorem 1.3. *Let Ω be a compact connected Hausdorff space. Let P be a projection on $C(\Omega)$ such that $P = \alpha_1 T_1 + \alpha_2 T_2 + \alpha_3 T_3$ where T_1, T_2, T_3 are surjective isometries of $C(\Omega)$, $\alpha_i > 0$, $i = 1, 2, 3$ $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Then either,*

(*a*) $\alpha_i = \frac{1}{2}$ *for some* $i = 1, 2, 3$ $\alpha_j + \alpha_k = \frac{1}{2}$, $j, k \neq i$ *and* $T_j = T_k$
or

(*b*) $\alpha_1 = \alpha_2 = \alpha_3 = \frac{1}{3}$ *and* T_1, T_2, T_3 *are distinct surjective isometries. Moreover in this case there exists a surjective isometry* L *on* $C(\Omega)$ *such that* $L^3 = I$ *and* $P = \frac{I+L+L^2}{3}$.

A few remarks are in order.

Remark 1.4. (a) If P is a proper projection which can be written as $P = \alpha T_1 + (1-\alpha) T_2$ where T_1, T_2 are surjective isometries on $C(\Omega)$, then $\alpha = \frac{1}{2}$. To see this, since P is proper, there exists $f \in C(\Omega)$, $f \neq 0$, such that $Pf = 0$. Thus $\alpha T_1 f = -(1-\alpha) T_2 f$. Since T_1, T_2 are isometries, taking norms on both sides we observe that $\alpha = \frac{1}{2}$.

(b) As mentioned above, in [3] it was already proved that if a projection P on $C(\Omega)$ can be written as $P = \frac{T_1+T_2+T_3}{3}$ for 3 distinct surjective isometries, then it is indeed a generalized 3-circular projection in the sense of definition in [1] and hence a generalized 3-circular projection by Theorem 1.2. Our proof for this part of Theorem 1.3 essentially follows the same idea as in [3].

(c) Throughout the next section where we present the proofs of Theorem 1.2 and Theorem 1.3 we will use standard Banach Stone Theorem, that is a surjective isometry T of $C(\Omega)$ is given by $Tf(\omega) = u(\omega) f(\phi(\omega))$, $f \in C(\Omega)$, where ϕ is a homeomorphism of Ω and u is a continuous function $u : \Omega \to \mathbb{T}$ (see [5]).

(d) For the case of $C(\Omega, X)$, X is a complex Banach space where vector-valued Banach stone Theorem holds true (see [6]), same proof with obvious modification will give us the corresponding results.

(e) The assumption of connectedness is essential. In [3], a GBP on ℓ_∞ was constructed which is not given by average of identity and a surjective isometry of order 2. For generalized 3-circular projections, a similar example can easily be constructed on ℓ_∞.

(f) Although the proof of Theorem 1.3 suggests that similar result should be true for $n \geq 4$ (and this is also mentioned in [1, 3]), the number of cases occurring in the proof becomes increasingly difficult to handle. It seems that one needs some other approach to prove Theorem 1.3 for general n.

2. Proof of main results

We will need the following lemma in the proof of Theorem 1.2.

Lemma 2.1. *Let Ω be a compact connected Hausdorff space and P_0, P_1, P_2 are projections on $C(\Omega)$ such that $P_0 \oplus P_1 \oplus P_2 = I$. Let $\lambda_1, \lambda_2 \in \mathbb{T}$ be of finite order such that $P_0 + \lambda_1 P_1 + \lambda_2 P_2$ is a surjective isometry on $C(\Omega)$. Then λ_1 and λ_2 are of same order.*

Proof. Let $\lambda_1^m = \lambda_2^n = 1$ and $m \neq n$. Without loss of generality we assume that $m < n$. Let $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = L$ where L is a surjective isometry on $C(\Omega)$. Then $P_0 + \lambda_1^m P_1 + \lambda_2^m P_2 = (P_0 + P_1) + \lambda_2^m P_2 = L^m$. Since L^m is again a surjective isometry and $P_2 = I - (P_0 + P_1)$, by [2, Theorem 1] we have $\lambda_2^m = -1$. Hence n divides $2m$. Similarly we obtain $\lambda_1^n = -1$ and m divides $2n$. Thus $2n = mk_1, 2m = nk_2$. Thus, $k_1 k_2 = 4$. Since we have assumed $m < n$, this implies $k_1 = 4, k_2 = 1$. But then $-1 = \lambda_1^n = \lambda_1^{2m} = 1$ - A contradiction. Hence $m = n$. □

Proof of the Theorem 1.2:

Let $P_0 \oplus P_1 \oplus P_2 = I$ and $P_0 + \lambda_1 P_1 + \lambda_2 P_2 = L$ where L is a surjective isometry on $C(\Omega)$. Note that this implies $P_0 + \lambda_1^2 P_1 + \lambda_2^2 P_2 = L^2$. Thus eliminating P_1, P_2 we obtain

$$P_0 = \frac{(L^2 - \lambda_1^2 I) - (\lambda_1 + \lambda_2)(L - \lambda_1 I)}{(1 - \lambda_1)(1 - \lambda_2)}. \qquad (i)$$

By classical Banach Stone Theorem there exists a homeomorphism ϕ of Ω and a continuous function $u : \Omega \to \mathbb{T}$ such that for any $f \in C(\Omega), Lf(\omega) = u(\omega) f(\phi(\omega))$.

Next we observe that $(L - \lambda_2 I)(L - \lambda_1 I)(L - I) = 0$. Taking $\lambda_1 + \lambda_2 = a$ and $\lambda_1 \lambda_2 = b$ this implies,

$$L^3 - (1 + a)L^2 + (a + b)L - bI = 0. \qquad (*)$$

We consider the following cases:

(I) $\omega = \phi^2(\omega),\ \omega \neq \phi(\omega)$. Then we have $\phi(\omega) = \phi^3(\omega)$. We consider a function $f \in C(\Omega)$ such that $f(\omega) = 1,\ f(\phi(\omega)) = 0$. Then Equation $(*)$ becomes $-(1 + a)u(\omega)u(\phi(\omega)) - b = 0$, hence $u(\omega)u(\phi(\omega)) = -\frac{b}{1+a}$. Similarly considering a $f \in C(\Omega)$ such that $f(\omega) = 0,\ f(\phi(\omega)) = 1$, the Equation $(*)$ gives $u(\omega)u(\phi(\omega)) = -(a + b)$. Thus we have $\frac{b}{1+a} = a + b$.

That is, $(1 + \lambda_1 + \lambda_2)(\lambda_1 + \lambda_2 + \lambda_1 \lambda_2) = \lambda_1 \lambda_2$,

or

$2 + \lambda_1 + \lambda_2 + \frac{1}{\lambda_1} + \frac{1}{\lambda_2} + \frac{\lambda_1}{\lambda_2} + \frac{\lambda_2}{\lambda_1} = 0.$

By Lemma 2.1, there exists an n such that both λ_1 and λ_2 are nth roots of identity. Hence we may assume $\lambda_2 = \lambda_1^m$ for some m.

Thus the above equation can written as,

$\lambda_1^{2m} + \lambda_1^{2m-1} + \lambda_1^{m+1} + 2\lambda_1^m + \lambda_1^{m-1} + \lambda_1 + 1 = 0,$

or

$(\lambda_1 + 1)(\lambda_1^{m-1} + 1)(\lambda_1^m + 1) = 0.$

Since $\lambda_1 \neq -1$, we will have $\lambda_1^m = -1$ or $\lambda_1^{m-1} = -1$. If $\lambda_1^m = -1$ then $\lambda_2 = -1$ which is a contradiction on the assumptions on λ_2 and if $\lambda_1^{m-1} = -1$ then $\lambda_2 = \lambda_1^m = -\lambda_1$ - A contradiction again.

Thus this case is not possible.

(II) $\omega = \phi^3(\omega),\ \omega \neq \phi(\omega) \neq \phi^2(\omega) \neq \omega$. We choose respectively, $f \in C(\Omega)$ such that $f(\omega) = 1,\ f(\phi(\omega)) = 0, f(\phi^2(\omega)) = 0,\ f \in C(\Omega)$ such that $f(\omega) = 0,\ f(\phi(\omega)) = 1, f(\phi^2(\omega)) = 0$ and $f \in C(\Omega)$ such that $f(\omega) = 0,\ f(\phi(\omega)) = 0, f(\phi^2(\omega)) = 1$ to get $a = -1$ and $b = 1$. Also we have $u(\omega)u(\phi(\omega))u(\phi^2(\omega)) = 1$. Thus λ_1 and λ_2 are the cube roots of identity and $u(\omega)u(\phi(\omega))u(\phi^2(\omega)) = 1$.

(III) $\omega = \phi(\omega)$. In this case Equation $(*)$ gives $u^3(\omega) - (1 + a)u^2(\omega) + (a + b)u(\omega) - b = 0$. Thus for each $\omega \in \Omega$, $u(\omega)$ has 3 possible values. Now if $\omega = \phi(\omega)$ is the entire set then from connectedness of Ω it follows that u is a constant function. By Equation (i), in this case P_0 is constant multiple of the identity operator and since P_0 is a projection, it is either I or 0 operator.

In conclusion we have λ_1 and λ_2 are cube roots of identity and $L^3 = I$.

It is now straight forward to see that $P_0 = \frac{I+L+L^2}{3}$.

This completes the proof of Theorem 1.2.

Proof of Theorem 1.3: We start by observing the following fact. If P is a proper projection, then $\exists\ f \in C(\Omega),\ f \neq 0$ such that $Pf = 0$. Hence, $\alpha_1 T_1 f + \alpha_2 T_2 f = -\alpha_3 T_3 f$. Since T_1, T_2, T_3 are isometries, by taking norms we have $\alpha_1 + \alpha_2 \geq \alpha_3$. Similarly, $\alpha_2 + \alpha_3 \geq \alpha_1$ and $\alpha_1 + \alpha_3 \geq \alpha_2$. Thus, if P is a proper projection then $\alpha_1, \alpha_2, \alpha_3$ are the lengths of sides of a triangle. It is also evident that $\alpha_i \leq 1/2,\ i = 1, 2, 3$.

Let $T_i f(\omega) = u_i(\omega) f(\phi_i(\omega)),\ i = 1, 2, 3$, where u_i and ϕ_i are given by the Banach Stone Theorem.

P is a projection if and only if

$$\alpha_1 u_1(\omega)[\alpha_1 u_1(\phi_1(\omega)) f(\phi_1^2(\omega)) + \alpha_2 u_2(\phi_1(\omega)) f(\phi_2 \circ \phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega)) f(\phi_3 \circ \phi_1(\omega))] +$$

$$\alpha_2 u_2(\omega)[\alpha_1 u_1(\phi_2(\omega)) f(\phi_1 \circ \phi_2(\omega)) + \alpha_2 u_2(\phi_2(\omega)) f(\phi_2^2(\omega)) + \alpha_3 u_3(\phi_2(\omega)) f(\phi_3 \circ \phi_2(\omega))] +$$

$\alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega))f(\phi_1\circ\phi_3(\omega))+\alpha_2 u_2(\phi_3(\omega))f(\phi_2\circ\phi_3(\omega))+\alpha_3 u_3(\phi_3(\omega))f(\phi_3^2(\omega))]$
$= \alpha_1 u_1(\omega)f(\phi_1(\omega))+\alpha_2 u_2(\omega)f(\phi_2(\omega))+\alpha_3 u_3(\omega)f(\phi_3(\omega)).$ $(**)$

We partition Ω as follows:

$A = \{\omega \in \Omega : \ \phi_1(\omega) = \phi_2(\omega) = \phi_3(\omega)\}$,
$B_i = \{\omega \in \Omega : \ \omega = \phi_j(\omega) = \phi_k(\omega) \neq \phi_i(\omega)\}$,
$C_i = \{\omega \in \Omega : \ \omega = \phi_i(\omega) \neq \phi_j(\omega) = \phi_k(\omega)\}$,
$D_i = \{\omega \in \Omega : \ \omega = \phi_i(\omega) \neq \phi_j(\omega) \neq \phi_k(\omega) \neq \omega\}$,
$E_i = \{\omega \in \Omega : \ \omega \neq \phi_i(\omega) \neq \phi_j(\omega) = \phi_k(\omega) \neq \omega\}$ and
$F = \{\omega \in \Omega : \text{none of } \omega, \phi_1(\omega), \phi_2(\omega), \phi_3(\omega) \text{ are equal }\}$,
where $i, j, k = 1, 2, 3$.

Suppose $A \neq \emptyset$. If $\omega \in A$, i.e, $\phi_1(\omega) = \phi_2(\omega) = \phi_3(\omega)$, then Equation $(**)$ is reduced to

$$[\alpha_1 u_1(\omega) + \alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)][\alpha_1 u_1(\phi_1(\omega))f(\phi_1^2(\omega)) + \alpha_2 u_2(\phi_1(\omega))f(\phi_2^2(\omega))+$$
$$\alpha_3 u_3(\phi_1(\omega))f(\phi_3^2(\omega))] = [\alpha_1 u_1(\omega) + \alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)]f(\phi_1(\omega)). \qquad (A)$$

Let $A_1 = \{\omega \in A : \alpha_1 u_1(\omega) + \alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) \neq 0\}$ and $A_2 = A \setminus A_1$. If $\omega \in A_1$, then

$$\alpha_1 u_1(\phi_1(\omega))f(\phi_1^2(\omega)) + \alpha_2 u_2(\phi_1(\omega))f(\phi_2^2(\omega)) + \alpha_3 u_3(\phi_1(\omega))f(\phi_3^2(\omega)) = f(\phi_1(\omega)).$$

First evaluating at constant function 1 we observe that $\alpha_1 u_1(\phi_1(\omega)) + \alpha_2 u_2(\phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega)) = 1$. Hence $u_i(\phi_i(\omega)) = 1, \ i = 1, 2, 3$. Thus we obtain, $\alpha_1 f(\phi_1^2(\omega)) + \alpha_2 f(\phi_2^2(\omega)) + \alpha_3 f(\phi_3^2(\omega)) = f(\phi_1(\omega))$. Now if, $\phi_1(\omega)$ is not equal to any of $\phi_i^2(\omega), \ i = 1, 2, 3$, then choosing an $f \in C(\Omega)$ such that $f(\phi_1(\omega)) = 1$ and $f(\phi_i^2(\omega) = 0$, we get a contradiction. Similarly if $\phi_1(\omega)$ is equal to one or two among $\phi_i^2(\omega) \ i = 1, 2, 3$ then choosing an appropriate f we get either $\alpha_i = 1$ or $\alpha_j + \alpha_k = 1$, both contradicting the choices of $\alpha_1, \alpha_2, \alpha_3$.

Thus in this case, we must have, $\phi_1^2(\omega) = \phi_2^2(\omega) = \phi_3^2(\omega) = \phi_1(\omega)$ or $\omega = \phi_1(\omega) = \phi_2(\omega) = \phi_3(\omega)$. Hence, $Pf(\omega) = f(\omega)$ if $\omega \in A_1$ and $Pf(\omega) = 0$ if $\omega \in A_2$. In particular, for the constant function $1, P1$ is a $0, 1$ valued function. By the connectedness of Ω we have $\Omega \neq A$.

Lemma 2.2. *If P is a projection, then for $i = 1, 2, 3$, $E_i = \emptyset$ and $F = \emptyset$.*

Proof. We show $E_1 = \emptyset$. For the case of E_2 and E_3 the proof is exactly the same. Let $\omega \in E_1$, i.e $\omega \neq \phi_1(\omega) \neq \phi_2(\omega) = \phi_3(\omega) \neq \omega$.

Then Equation $(**)$ reduces to

$$\alpha_1 u_1(\omega)[\alpha_1 u_1(\phi_1(\omega))f(\phi_1^2(\omega))+\alpha_2 u_2(\phi_1(\omega))f(\phi_2\circ\phi_1(\omega))+\alpha_3 u_3(\phi_1(\omega))f(\phi_3\circ\phi_1(\omega))]$$
$$+[\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)][\alpha_1 u_1(\phi_2(\omega))f(\phi_1 \circ \phi_2(\omega)) + \alpha_2 u_2(\phi_2(\omega))f(\phi_2^2(\omega))+$$

$$\alpha_3 u_3(\phi_2(\omega))f(\phi_3^2(\omega))] = \alpha_1 u_1(\omega)f(\phi_1(\omega)) + [\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)]f(\phi_2(\omega)). \quad (E1)$$

We claim $\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) \neq 0$. To see the claim, if $\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) = 0$, then Equation (E1) further reduces to

$$\alpha_1 u_1(\phi_1(\omega))f(\phi_1^2(\omega)) + \alpha_2 u_2(\phi_1(\omega))f(\phi_2 \circ \phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega))f(\phi_3 \circ \phi_1(\omega))$$
$$= f(\phi_1(\omega)).$$

An argument similar to case (A) above shows that $\phi_1(\omega) = \phi_3 \circ \phi_1(\omega) = \phi_2 \circ \phi_1(\omega) = \phi_1^2(\omega)$, which is clearly a contradiction to the choice of $w \in E_1$.

We choose a continuous function $f \in C(\Omega)$ such that $f(\phi_1(\omega)) = 1$ and $f(\phi_2(\omega)) = f(\phi_1 \circ \phi_2(\omega)) = f(\phi_1^2(\omega)) = 0$. Equation (E1) now reduces to

$$\alpha_1 u_1(\omega)[\alpha_2 u_2(\phi_1(\omega))f(\phi_2 \circ \phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega))f(\phi_3 \circ \phi_1(\omega))] + [\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)]$$
$$[\alpha_2 u_2(\phi_2(\omega))f(\phi_2^2(\omega)) + \alpha_3 u_3(\phi_2(\omega))f(\phi_3^2(\omega))] = \alpha_1 u_1(\omega) \qquad (E2)$$

If $\phi_1(\omega)$ is not equal to any of the points $\phi_2 \circ \phi_1(\omega), \phi_3 \circ \phi_1(\omega), \phi_2^2(\omega)$ and $\phi_3^2(\omega)$, then we could have chosen our f to have value 0 at these points and this would have lead us to a contradiction. If $\phi_1(\omega) = \phi_2 \circ \phi_1(\omega)$ then clearly we could choose $f(\phi_2^2(\omega)) = 0$. If both $\phi_3 \circ \phi_1(\omega)$ and $\phi_3^2(\omega)$ are not equal to $\phi_1(\omega)$, then choosing f to take value 0 at $\phi_3 \circ \phi_1(\omega)$ and $\phi_3^2(\omega)$ we have

$$\alpha_1 \alpha_2 u_1(\omega) u_2(\phi_1(\omega)) = \alpha_1 u_1(\omega)$$

and hence $\alpha_2 = 1$, a contradiction again. Thus either of $\phi_3 \circ \phi_1(\omega)$ and $\phi_3^2(\omega)$ is equal to $\phi_1(\omega)$. Similar consideration with $\phi_1(\omega) = \phi_3 \circ \phi_1(\omega), \phi_1(\omega) = \phi_2^2(\omega)$ and $\phi_1(\omega) = \phi_3^2(\omega)$ lead us to the conclusion that $\phi_1(\omega)$ will be equal to exactly two elements of the set

$$\{\phi_2 \circ \phi_1(\omega), \phi_3 \circ \phi_1(\omega), \phi_2^2(\omega), \phi_3^2(\omega)\}.$$

If $\phi_1(\omega) = \phi_2 \circ \phi_1(\omega) = \phi_3 \circ \phi_1(\omega)$ then $(E2)$ will imply that $\alpha_2 u_2(\phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega)) = 1$ - A contradiction. Now, suppose that $\phi_1(\omega) = \phi_2 \circ \phi_i(\omega) = \phi_3 \circ \phi_j(\omega)$ where $i, j \in \{1, 2, 3\}$. Choose f such that $f(\phi_2(\omega)) = 1$ and $f(\phi_1(\omega)) = f(\phi_2 \circ \phi_{i_1}(\omega)) = f(\phi_2 \circ \phi_{j_1}(\omega)) = 0$, where $i_1 \neq i$, $j_1 \neq j$, and $i_1, j_1 = 1, 2, 3$. So, Equation $(E1)$ becomes

$$\alpha_1^2 u_1(\omega) u_1(\phi_1(\omega)) f(\phi_1^2(\omega)) + \alpha_1 u_1(\phi_2(\omega)) f(\phi_1 \circ \phi_2(\omega))[\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)]$$
$$= \alpha_2 u_2(\omega) + \alpha_3 u_3(\omega). \qquad (E3)$$

If $\phi_2(\omega)$ is not equal to any one of $\phi_1^2(\omega)$ or $\phi_1 \circ \phi_2(\omega)$, then we can choose f to be 0 at $\phi_1^2(\omega)$ and $\phi_1 \circ \phi_2(\omega)$, thereby getting $\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) = 0$, a contradiction. If $\phi_1(\omega) = \phi_1 \circ \phi_2(\omega)$, then by choosing f to be 0 at $\phi_1^2(\omega)$ we will get $\alpha_1 = 1$ which is a contradiction. Therefore, we have $\phi_2(\omega) = \phi_1^2(\omega)$. Similarly, $\phi_1 \circ \phi_2(\omega)$ must be equal to atleast one of $\phi_2 \circ \phi_{i_1}(\omega)$ or $\phi_2 \circ \phi_{j_1}(\omega)$. But in this case we will be left with 3 or 4 distinct points in Equation $(E1)$. By choosing f to be 0 at $\phi_1(\omega)$ and $\phi_2(\omega)$ and large enough at other points on the right hand side we will get a contradiction.

Now, suppose that $\omega \in F$, i.e all $\omega, \phi_1(\omega), \phi_2(\omega), \phi_3(\omega)$ are distinct. Consider the following matrix:

$$\begin{pmatrix} \phi_1(\omega) & \phi_2(\omega) & \phi_3(\omega) \\ \phi_1^2(\omega) & \phi_2 \circ \phi_1(\omega) & \phi_3 \circ \phi_1(\omega) \\ \phi_1 \circ \phi_2(\omega) & \phi_2^2(\omega) & \phi_3 \circ \phi_2(\omega) \\ \phi_1 \circ \phi_3(\omega) & \phi_2 \circ \phi_3(\omega) & \phi_3^2(\omega) \end{pmatrix}$$

Observe that points belonging to any column are all non equal. Choose first f such that $f(\phi_1(\omega)) = 1$ and $f(\phi_2(\omega)) = f(\phi_3(\omega)) = f(\phi_1^2(\omega)) = f(\phi_1 \circ \phi_2(\omega)) = f(\phi_1 \circ \phi_3(\omega)) = 0$. Equation $(**)$ becomes

$$\begin{aligned} &\alpha_1 u_1(\omega)[\alpha_2 u_2(\phi_1(\omega)) f(\phi_2 \circ \phi_1(\omega)) + \alpha_3 u_3(\phi_1(\omega)) f(\phi_3 \circ \phi_1(\omega))] + \\ &\alpha_2 u_2(\omega)[\alpha_2 u_2(\phi_2(\omega)) f(\phi_2^2(\omega)) + \alpha_3 u_3(\phi_2(\omega)) f(\phi_3 \circ \phi_2(\omega))] + \\ &\alpha_3 u_3(\omega)[\alpha_2 u_2(\phi_3(\omega)) f(\phi_2 \circ \phi_3(\omega)) + \alpha_3 u_3(\phi_3(\omega)) f(\phi_3^2(\omega))] \\ &= \alpha_1 u_1(\omega) f(\phi_1(\omega)). \qquad (F1) \end{aligned}$$

Equation $(F1)$ implies that $\phi_1(\omega)$ must be equal to at least 2 elements from the set

$$\{\phi_2 \circ \phi_1(\omega), \phi_3 \circ \phi_1(\omega), \phi_2^2(\omega), \phi_3 \circ \phi_2(\omega), \phi_2 \circ \phi_3(\omega), \phi_3^2(\omega)\}.$$

Since this set does not contain three equal elements, it follows that $\phi_1(\omega)$ is equal to exactly two; say $\phi_2 \circ \phi_{i_1}(\omega)$ and $\phi_2 \circ \phi_{j_1}(\omega)$ with $i_1, j_1 \in \{1, 2, 3\}$. Therefore,

$$\alpha_{i_1} \alpha_2 u_{i_1}(\omega) u_2(\phi_{i_1}(\omega)) + \alpha_{j_1} \alpha_3 u_{j_1}(\omega) u_3(\phi_{j_1}(\omega)) = \alpha_1 u_1(\omega).$$

This implies that

$$\alpha_1 \leq \alpha_2 \alpha_{i_1} + \alpha_3 \alpha_{j_1}.$$

Similar arguments applied to $\phi_2(\omega)$ and $\phi_3(\omega)$ implies the inequalities:

$\alpha_2 \leq \alpha_1 \alpha_{i_2} + \alpha_3 \alpha_{j_2}$ and $\alpha_3 \leq \alpha_1 \alpha_{i_3} + \alpha_2 \alpha_{j_3}$.

Adding these three inequalities we get

$$\begin{aligned} 1 = \alpha_1 + \alpha_2 + \alpha_3 &\leq \alpha_1(\alpha_{i_2} + \alpha_{i_3}) + \alpha_2(\alpha_{i_1} + \alpha_{j_3}) + \alpha_3(\alpha_{j_1} + \alpha_{j_2}) \\ &\leq \max\{\alpha_{i_2} + \alpha_{i_3}, \alpha_{i_1} + \alpha_{j_3}, \alpha_{j_1} + \alpha_{j_2}\}. \end{aligned}$$

This is impossible.

□

Now we set ourselves to show the following:

Lemma 2.3. *If* $\omega \in C_i$, $i = 1, 2, 3$ *then* $\alpha_i = 1/2$ *and* $u_i(\omega) = u_i(\phi_j(\omega)) = u_j(\omega) = u_k(\omega) = u_j(\phi_j(\omega)) = u_k(\phi_j(\omega)) = 1$ *for* $j = 1, 2, 3$ *and* $j \neq i$. *If* $\omega \in D_i$, $i = 1, 2, 3$ *then* $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$.

Proof. We prove the result for $i = 1$. For $i = 2$ and 3 similar argument is true. Let $\omega \in C_1$, i.e $\omega = \phi_1(\omega) \neq \phi_2(\omega) = \phi_3(\omega)$, then equation $(**)$ reduces to

$$\alpha_1 u_1(\omega)[\alpha_1 u_1(\omega))f(\omega)+\alpha_2 u_2(\omega)f(\phi_2(\omega))+\alpha_3 u_3(\omega)f(\phi_2(\omega)]+[\alpha_2 u_2(\omega)+\alpha_3 u_3(\omega)]$$

$$[\alpha_1 u_1(\phi_2(\omega)f(\phi_1 \circ \phi_2(\omega)) + \alpha_2 u_2(\phi_2(\omega)f(\phi_2^2(\omega)) + \alpha_3 u_3(\phi_2(\omega))f(\phi_3^2(\omega))] =$$

$$\alpha_1 u_1(\omega)f(\omega) + [\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)]f(\phi_2(\omega)). \qquad (C1)$$

Note that in this case we must have $\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) \neq 0$; otherwise (C1) will give us $\alpha_1 = 1$.

We choose a function $f \in C(\Omega)$ such that $f(\phi_2(\omega)) = 1$, $f(\omega) = f(\phi_2^2(\omega)) = f(\phi_3^2(\omega)) = 0$ which will reduce (C1) to

$$\alpha_1 u_1(\omega)[\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)] + \alpha_1 u_1(\phi_2(\omega))f(\phi_1 o \phi_2(\omega))[\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)]$$

$$= \alpha_2 u_2(\omega) + \alpha_3 u_3(\omega). \qquad (C2)$$

Since $\alpha_2 u_2(\omega)+\alpha_3 u_3(\omega) \neq 0$ we obtain $\alpha_1 u_1(\omega)+\alpha_1 u_1(\phi_2(\omega))f(\phi_1 \circ \phi_2(\omega)) = 1$. Thus, $\phi_1 \circ \phi_2(\omega) = \phi_2(\omega)$ and $\alpha_1 \geq 1/2$. Since $\alpha_i \leq 1/2$, $\forall i$ we conclude $\alpha_1 = 1/2$ and $u_1(\omega) = u_1(\phi_2(\omega)) = 1$. Using a function f such that $f(\omega) = 0, f(\phi_2(\omega)) = 1$ Equation (C1) becomes

$$\alpha_2 u_2(\phi_2(\omega))f(\phi_2^2(\omega)) + \alpha_3 u_3(\phi_2(\omega))f(\phi_3^2(\omega)) = 0.$$

The points $\phi_2^2(\omega)$ and $\phi_3^2(\omega)$ must be equal to one of ω or $\phi_2(\omega)$. Since $\phi_2^2(\omega)$ and $\phi_3^2(\omega)$ cannot be equal to $\phi_2(\omega)$ we have $\phi_2^2(\omega) = \phi_3^2(\omega) = \omega$. Now choose a function f such that $f(\omega) = 1$, $f(\phi_2(\omega) = 0$, Equation (C1) is reduced to

$$[\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega)][\alpha_2 u_2(\phi_2(\omega)) + \alpha_3 u_3(\phi_2(\omega))] = 1/4.$$

Since $\alpha_2 + \alpha_3 = 1/2$, we have $\alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) = \alpha_2 u_2(\phi_2(\omega)) + \alpha_3 u_3(\phi_2(\omega)) = 1/2$. This will imply that $u_2(\omega) = u_3(\omega) = u_2(\phi_2(\omega)) = u_3(\phi_2(\omega)) = 1$.

We show that if $\omega \in D_1$ then $\alpha_1 = \alpha_2 = \alpha_3 = 1/3$. $\omega \in D_1 \Rightarrow \omega = \phi_1(\omega) \neq \phi_2(\omega) \neq \phi_3(\omega) \neq \omega$. Equation $(**)$ reduces to

$$\alpha_1 u_1(\omega)[\alpha_1 u_1(\omega)f(\omega) + \alpha_2 u_2(\omega)f(\phi_2(\omega)) + \alpha_3 u_3(\omega)f(\phi_3(\omega))] + \alpha_2 u_2(\omega)$$

$$[\alpha_1 u_1(\phi_2(\omega))f(\phi_1 \circ \phi_2(\omega)) + \alpha_2 u_2(\phi_2(\omega))f(\phi_2^2(\omega)) + \alpha_3 u_3(\phi_2(\omega))f(\phi_3 \circ \phi_2(\omega))]+$$

$$\alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega))f(\phi_1 \circ \phi_3(\omega))+\alpha_2 u_2(\phi_3(\omega))f(\phi_2 \circ \phi_3(\omega))+\alpha_3 u_3(\phi_3(\omega))f(\phi_3^2(\omega))]$$

$$= \alpha_1 u_1(\omega)f(\omega) + \alpha_2 u_2(\omega)f(\phi_2(\omega)) + \alpha_3 u_3(\omega)f(\phi_3(\omega)). \qquad (D1)$$

We can choose a function $f \in C(\Omega)$ satisfying $f(\omega) = 1,\ f(\phi_2(\omega)) = f(\phi_3(\omega)) = f(\phi_1 \circ \phi_2(\omega)) = f(\phi_1 \circ \phi_3(\omega)) = 0$. Then (D1) reduces to

$$\alpha_1^2 u_1^2(\omega) + \alpha_2 u_2(\omega)[\alpha_2 u_2(\phi_2(\omega)) f(\phi_2^2(\omega)) + \alpha_3 u_3(\phi_2(\omega)) f(\phi_3 \circ \phi_2(\omega))] + \alpha_3 u_3(\omega)$$
$$[\alpha_2 u_2(\phi_3(\omega)) f(\phi_2 \circ \phi_3(\omega)) + \alpha_3 u_3(\phi_3(\omega)) f(\phi_3^2(\omega))] = \alpha_1 u_1(\omega). \qquad (D2)$$

If $\phi_2^2(\omega),\ \phi_3 \circ \phi_2(\omega),\ \phi_2 \circ \phi_3(\omega)$ and $\phi_3^2(\omega)$ are all different from ω, by choosing our function f to take value 0 at all these points we will have $\alpha_1^2 u_1^2(\omega) = \alpha_1 u_1(\omega)$ and hence $\alpha_1 = 1$. Thus not all these points are different from ω.

Claim: If $\omega = \phi_2 \circ \phi_i(\omega)$, $i = 2$ or 3 then $\omega = \phi_3 \circ \phi_j(\omega)$, $j = 2$ or 3.

First we assume the claim and complete the proof then establish the claim. Choosing a function $f \in C(\Omega)$ such that $f(\phi_2(\omega)) = 1,\ f((\omega)) = f(\phi_3(\omega)) = f(\phi_2^2(\omega)) = f(\phi_2 \circ \phi_3(\omega)) = 0$ and then a function f such that $f(\phi_3(\omega)) = 1,\ f((\omega)) = f(\phi_2(\omega)) = f(\phi_3^2(\omega)) = f(\phi_3 \circ \phi_2(\omega)) = 0$ in Equation (D1) we will get the following two equations.

$$\alpha_1\alpha_2 u_1(\omega) u_2(\omega) f(\phi_2(\omega)) + \alpha_2 u_2(\omega)[\alpha_1 u_1(\phi_2(\omega)) f(\phi_1 \circ \phi_2(\omega)) + \alpha_3 u_3(\phi_2(\omega))$$
$$f(\phi_3 \circ \phi_2(\omega))] + \alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega)) f(\phi_1 \circ \phi_3(\omega)) + \alpha_3 u_3(\phi_3(\omega)) f(\phi_3^2(\omega))]$$
$$= \alpha_2 u_2(\omega) f(\phi_2(\omega)). \qquad (D3)$$
$$\alpha_1\alpha_3 u_1(\omega) u_3(\omega) f(\phi_3(\omega)) + \alpha_2 u_2(\omega)[\alpha_1 u_1(\phi_2(\omega)) f(\phi_1 \circ \phi_2(\omega)) + \alpha_2 u_2(\phi_2(\omega))$$
$$f(\phi_2^2(\omega))] + \alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega)) f(\phi_1 \circ \phi_3(\omega)) + \alpha_2 u_2(\phi_3(\omega)) f(\phi_2 \circ \phi_3(\omega))]$$
$$= \alpha_3 u_3(\omega) f(\phi_3(\omega)). \qquad (D4)$$

From the above claim we have the following disjoint and exhaustive cases which may occur.

$D_{11} = \{\omega \in D_1 :\ \omega = \phi_2^2(\omega) = \phi_3 \circ \phi_2(\omega),\ \phi_2(\omega) = \phi_3^2(\omega) = \phi_1 \circ \phi_2(\omega),\ \phi_3(\omega) = \phi_1 \circ \phi_3(\omega) = \phi_2 \circ \phi_3(\omega)\}$.

$D_{12} = \{\omega \in D_1 :\ \omega = \phi_2^2(\omega) = \phi_3 \circ \phi_2(\omega),\ \phi_2(\omega) = \phi_3^2(\omega) = \phi_1 \circ \phi_3(\omega),\ \phi_3(\omega) = \phi_1 \circ \phi_2(\omega) = \phi_2 \circ \phi_3(\omega)\}$.

$D_{13} = \{\omega \in D_1 :\ \omega = \phi_2 \circ \phi_3(\omega) = \phi_3 \circ \phi_2(\omega),\ \phi_2(\omega) = \phi_3^2(\omega) = \phi_1 \circ \phi_2(\omega),\ \phi_3(\omega) = \phi_1 \circ \phi_3(\omega) = \phi_2^2(\omega)\}$.

$D_{14} = \{\omega \in D_1 :\ \omega = \phi_2 \circ \phi_3(\omega) = \phi_3 \circ \phi_2(\omega),\ \phi_2(\omega) = \phi_3^2(\omega) = \phi_1 \circ \phi_3(\omega),\ \phi_3(\omega) = \phi_1 \circ \phi_2(\omega) = \phi_2^2(\omega)\}$.

$D_{15} = \{\omega \in D_1 :\ \omega = \phi_2^2(\omega) = \phi_3^2(\omega),\ \phi_2(\omega) = \phi_1 \circ \phi_2(\omega) = \phi_3 \circ \phi_2(\omega),\ \phi_3(\omega) = \phi_1 \circ \phi_3(\omega) = \phi_2 \circ \phi_3(\omega)\}$.

$D_{16} = \{\omega \in D_1 :\ \omega = \phi_2^2(\omega) = \phi_3^2(\omega),\ \phi_2(\omega) = \phi_1 \circ \phi_3(\omega) = \phi_3 \circ \phi_2(\omega),\ \phi_3(\omega) = \phi_1 \circ \phi_2(\omega) = \phi_2 \circ \phi_3(\omega)\}$.

Now for any $\omega \in D_{11}$, Equation (D1) is reduced to

$$\{\alpha_1^2 u_1^2(\omega) + \alpha_2 u_2(\omega)[\alpha_2 u_2(\phi_2(\omega)) + \alpha_3 u_3(\phi_2(\omega))]\}f(\omega)+$$
$$[\alpha_1\alpha_2 u_1(\omega)u_2(\omega) + \alpha_1\alpha_2 u_1(\phi_2(\omega))u_2(\omega) + \alpha_3^2 u_3(\omega)u_3(\phi_3(\omega))]f(\phi_2(\omega))$$
$$+\{\alpha_1\alpha_3 u_1(\omega)u_3(\omega) + \alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega)) + \alpha_2 u_2(\phi_3(\omega))]\}f(\phi_3(\omega))$$
$$= \alpha_1 u_1(\omega)f(\omega) + \alpha_2 u_2(\omega)f(\phi_2(\omega)) + \alpha_3 u_3(\omega)f(\phi_3(\omega)). \qquad (D11)$$

Since $\omega \neq \phi_2(\omega) \neq \phi_3(\omega)$, choosing appropriate functions we have

$$\alpha_1 \leq \alpha_1^2 + \alpha_2(\alpha_2 + \alpha_3), \alpha_2 \leq 2\alpha_1\alpha_2 + \alpha_3^2 \text{ and } 1 \leq 2\alpha_1 + \alpha_2. \qquad (D11)'$$

For $\omega \in D_{12}$, we have

$$\{\alpha_1^2 u_1^2(\omega) + \alpha_2 u_2(\omega)[\alpha_2 u_2(\phi_2(\omega)) + \alpha_3 u_3(\phi_2(\omega))]\}f(\omega)+$$
$$[\alpha_1\alpha_2 u_1(\omega)u_2(\omega) + \alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega)) + \alpha_3 u_3(\phi_3(\omega))]f(\phi_2(\omega))+$$
$$\{\alpha_1\alpha_3 u_1(\omega)u_3(\omega) + \alpha_1\alpha_2 u_2(\omega)u_1(\phi_2(\omega)) + \alpha_2\alpha_3 u_3(\omega)u_2(\phi_3(\omega))\}f(\phi_3(\omega))$$
$$= \alpha_1 u_1(\omega)f(\omega) + \alpha_2 u_2(\omega)f(\phi_2(\omega)) + \alpha_3 u_3(\omega)f(\phi_3(\omega)). \qquad (D12)$$

This implies that

$$\alpha_1 \leq \alpha_1^2 + \alpha_2(\alpha_2 + \alpha_3), \alpha_2 \leq \alpha_1\alpha_2 + \alpha_3(\alpha_1 + \alpha_3) \text{ and}$$
$$\alpha_3 \leq \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1. \qquad (D12)'$$

For $\omega \in D_{13}$, we have

$$\{\alpha_1^2 u_1^2(\omega) + \alpha_2\alpha_3[u_2(\omega)u_3(\phi_2(\omega)) + u_3(\omega)u_2(\phi_3(\omega))]\}f(\omega)+$$
$$[\alpha_1\alpha_2 u_1(\omega)u_2(\omega) + \alpha_1\alpha_2 u_2(\omega)u_1(\phi_2(\omega)) + \alpha_3^2 u_3(\omega)u_3(\phi_3(\omega))]f(\phi_2(\omega))$$
$$+\{\alpha_1\alpha_3 u_1(\omega)u_3(\omega) + \alpha_2^2 u_2(\omega)u_2(\phi_2(\omega)) + \alpha_1\alpha_3 u_3(\omega)u_1(\phi_3(\omega))\}f(\phi_3(\omega))$$
$$= \alpha_1 u_1(\omega)f(\omega) + \alpha_2 u_2(\omega)f(\phi_2(\omega)) + \alpha_3 u_3(\omega)f(\phi_3(\omega)). \qquad (D13)$$

This implies that

$$\alpha_1 \leq \alpha_1^2 + 2\alpha_2\alpha_3), \alpha_2 \leq 2\alpha_1\alpha_2 + \alpha_3^2 \text{ and } \alpha_3 \leq 2\alpha_1\alpha_3 + \alpha_2^2. \qquad (D13)'$$

For $\omega \in D_{14}$, we have

$$\{\alpha_1^2 u_1^2(\omega) + \alpha_2\alpha_3[u_2(\omega)u_3(\phi_2(\omega)) + u_3(\omega)u_2(\phi_3(\omega))]\}f(\omega)+$$
$$\{[\alpha_1\alpha_2 u_1(\omega)u_2(\omega) + \alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega)) + \alpha_3 u_3(\phi_3(\omega))]\}f(\phi_2(\omega))$$
$$+\{\{\alpha_1\alpha_3 u_1(\omega)u_3(\omega) + \alpha_2 u_2(\omega)[\alpha_1 u_1(\phi_2(\omega)) + \alpha_2 u_2(\phi_2(\omega))]\}f(\phi_3(\omega))$$
$$= \alpha_1 u_1(\omega)f(\omega) + \alpha_2 u_2(\omega)f(\phi_2(\omega)) + \alpha_3 u_3(\omega)f(\phi_3(\omega)). \qquad (D14)$$

This implies that

$$\alpha_1 \leq \alpha_1^2 + 2\alpha_2\alpha_3), \alpha_2 \leq \alpha_1\alpha_2 + \alpha_3(\alpha_1 + \alpha_3) \text{ and}$$
$$\alpha_3 \leq \alpha_1\alpha_3 + \alpha_2(\alpha_1 + \alpha_2). \qquad (D14)'$$

For $\omega \in D_{15}$, we have

$$\{\alpha_1^2 u_1^2(\omega) + \alpha_2^2 u_2(\omega) u_2(\phi_2(\omega)) + \alpha_3^2 u_3(\omega) u_3(\phi_3(\omega))\} f(\omega) +$$
$$\{[\alpha_1\alpha_2 u_1(\omega) u_2(\omega) + \alpha_2 u_2(\omega)[\alpha_1 u_1(\phi_2(\omega)) + \alpha_3 u_3(\phi_2(\omega))]\} f(\phi_2(\omega))$$
$$+\{\{\alpha_1\alpha_3 u_1(\omega) u_3(\omega) + \alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega)) + \alpha_2 u_2(\phi_3(\omega))]\} f(\phi_3(\omega))$$
$$= \alpha_1 u_1(\omega) f(\omega) + \alpha_2 u_2(\omega) f(\phi_2(\omega)) + \alpha_3 u_3(\omega) f(\phi_3(\omega)). \qquad (D15)$$

This implies that

$$\alpha_1 \leq \alpha_1^2 + \alpha_2^2 + \alpha_3^2, 1 \leq 2\alpha_1 + \alpha_3 \text{ and } 1 \leq 2\alpha_1 + \alpha_2. \qquad (D15)'$$

For $\omega \in D_{16}$, we have

$$\{\alpha_1^2 u_1^2(\omega) + \alpha_2^2 u_2(\omega) u_2(\phi_2(\omega)) + \alpha_3^2 u_3(\omega) u_3(\phi_3(\omega))\} f(\omega) +$$
$$\{\alpha_1\alpha_2 u_1(\omega) u_2(\omega) + \alpha_2\alpha_3 u_2(\omega) u_3(\phi_2(\omega)) + \alpha_1\alpha_3 u_3(\omega) u_1(\phi_3(\omega))\} f(\phi_2(\omega))$$
$$+\{\alpha_1\alpha_3 u_1(\omega) u_3(\omega) + \alpha_1\alpha_2 u_2(\omega) u_1(\phi_2(\omega)) + \alpha_2\alpha_3 u_3(\omega) u_2(\phi_3(\omega))\} f(\phi_3(\omega))$$
$$= \alpha_1 u_1(\omega) f(\omega) + \alpha_2 u_2(\omega) f(\phi_2(\omega)) + \alpha_3 u_3(\omega) f(\phi_3(\omega)). \qquad (D16)$$

This implies that

$$\alpha_1 \leq \alpha_1^2 + \alpha_2^2 + \alpha_3^2 \text{ and } \alpha_2 \leq \alpha_1\alpha_2 + \alpha_2\alpha_3 + \alpha_3\alpha_1. \qquad (D16)'$$

For Equations $(D1i)', i = 1, ..., 6$ it is easy to observe that $\alpha_i = 1/3,\ i = 1, 2, 3$ is the only solution.

We now need to find the condition on $u_i(\omega)$ and $u_i(\phi_j(\omega))$ where $i, j = 1, 2, 3$. We substitute $\alpha_i = 1/3$ in Equations $(D1i)$, $i = 1, ..., 6$ and we choose three sets of functions for each Equation. Firstly, a function $f \in C(\Omega)$ such that $f(\omega) = 1$, $f(\phi_2(\omega)) = f(\phi_3(\omega)) = 0$. Then, a function $f \in C(\Omega)$ such that $f(\phi_2(\omega)) = 1$, $f(\omega) = f(\phi_3(\omega)) = 0$ and finally a function $f \in C(\Omega)$ such that $f(\phi_3(\omega)) = 1$, $f(\omega) = f(\phi_2(\omega)) = 0$. Moreover, by observing that $u_i(\omega)$ and $u_i(\phi_j(\omega))$ lie on the unit circle and all the points on the circle are extreme points we get the following conditions on $u_i(\omega)$ and $u_i(\phi_j(\omega))$ where $i, j = 1, 2, 3$:

For $\omega \in D_{11}$ we get

$$u_1(\omega) = u_2(\omega) u_2(\phi_2(\omega)) = u_2(\omega) u_3(\phi_2(\omega)) = 1, u_1(\phi_2(\omega)) = 1,$$
$$u_3(\omega) u_3(\phi_3(\omega)) = u_2(\omega) \text{ and } u_1(\phi_3(\omega)) = u_2(\phi_3(\omega)) = 1.$$

For $\omega \in D_{12}$ we get

$$u_1(\omega) = u_2(\omega) u_2(\phi_2(\omega)) = u_2(\omega) u_3(\phi_2(\omega)) = 1, u_2(\omega) u_1(\phi_2(\omega)) = u_3(\omega),$$
$$u_2(\omega) = u_3(\omega) u_1(\phi_3(\omega)) = u_2(\omega) u_3(\omega) u_3(\phi_3(\omega)) \text{ and } u_2(\phi_3(\omega)) = 1.$$

For $\omega \in D_{13}$ we get

$$u_1(\omega) = u_2(\omega) u_3(\phi_2(\omega)) = u_3(\omega) u_2(\phi_3(\omega)) = 1, u_1(\phi_2(\omega)) = u_1(\phi_3(\omega)) = 1,$$

$$u_2(\omega) = u_3(\omega)u_3(\phi_3(\omega)) \text{ and } u_3(\omega) = u_2(\omega)u_2(\phi_2(\omega)).$$

For $\omega \in D_{14}$ we get

$$u_1(\omega) = u_2(\omega)u_3(\phi_2(\omega)) = u_3(\omega)u_2(\phi_3(\omega)) = 1, u_2(\omega) = u_3(\omega)u_1(\phi_3(\omega)) =$$

$$u_3(\omega)u_3(\phi_3(\omega)) \text{ and } u_3(\omega) = u_2(\omega)u_2(\phi_2(\omega)) = u_2(\omega)u_1(\phi_2(\omega)).$$

For $\omega \in D_{15}$ we get

$$u_1(\omega) = u_2(\omega)u_2(\phi_2(\omega)) = u_3(\omega)u_3(\phi_3(\omega)) = 1 \text{ and } u_1(\phi_2(\omega)) = u_1(\phi_3(\omega)) =$$

$$u_3(\phi_2(\omega)) = u_2(\phi_3(\omega)) = 1.$$

For $\omega \in D_{16}$ we get

$$u_1(\omega) = u_2(\omega)u_2(\phi_2(\omega)) = u_3(\omega)u_3(\phi_3(\omega)) = 1, u_2(\omega) = u_3(\omega)u_1(\phi_3(\omega)),$$

$$u_3(\omega) = u_2(\omega)u_1(\phi_2(\omega)) \text{ and } u_3(\phi_2(\omega)) = u_2(\phi_3(\omega)) = 1.$$

□

Proof of the claim. Let $\omega = \phi_2 \circ \phi_i(\omega)$, $i = 2$ or 3 then in Equation $(D2)$ $f(\phi_2 \circ \phi_j(\omega)) = 0$, $j = 2$ or 3 and $j \neq i$. Suppose to the contrary that $\omega \neq \phi_3 \circ \phi_k(\omega)$ for $k = 2, 3$ then by choosing our f to be 0 at these points we get from $(D2)$

$$\alpha_1^2 u_1^2(\omega) + \alpha_2^2 u_2(\omega)u_2(\phi_2(\omega)) = \alpha_1 u_1(\omega). \qquad (D1.1)$$

This will imply that $\alpha_1 \leq \alpha_1^2 + \alpha_2^2$. We now choose a function $f \in C(\Omega)$ such that $f(\phi_2(\omega)) = 1$ and $f(\omega) = f(\phi_3(\omega)) = f(\phi_2^2(\omega)) = f(\phi_2 \circ \phi_3(\omega)) = 0$. Then Equation (D1) is reduced to

$$\alpha_1\alpha_2 u_1(\omega)u_2(\omega) + \alpha_2 u_2(\omega)[\alpha_1 u_1(\phi_2(\omega))f(\phi_1 \circ \phi_2(\omega)) + \alpha_3 u_3(\phi_2(\omega))f(\phi_3 \circ \phi_2(\omega))] +$$

$$\alpha_3 u_3(\omega)[\alpha_1 u_1(\phi_3(\omega))f(\phi_1 \circ \phi_3(\omega)) + \alpha_3 u_3(\phi_3(\omega))f(\phi_3^2(\omega))] = \alpha_2 u_2(\omega). \qquad (D1.2)$$

Again, if all $\phi_1 \circ \phi_2(\omega)$, $\phi_3 \circ \phi_2(\omega)$, $\phi_1 \circ \phi_3(\omega)$ and $\phi_3^2(\omega)$ are different from $\phi_2(\omega)$, by choosing f initially to take value 0 at all these points we could have $\alpha_1 = 1$. Suppose $\phi_2(\omega) = \phi_1 \circ \phi_{i_1}(\omega)$ where $i_1 = 2$ or 3. Then we could choose f in (D1.2) such that $f(\phi_1 \circ \phi_{i_2}(\omega)) = 0$, $i_2 = 2$ or 3 and $i_2 \neq i_1$. If $\phi_2(\omega) \neq \phi_3 \circ \phi_{i_3}(\omega)$, $i_3 = 2, 3$. Then by the same argument we get from $(D1.2)$

$$\alpha_1\alpha_2 u_1(\omega)u_2(\omega) + \alpha_1\alpha_{i_1} u_{i_1}(\omega)u_1(\phi_{i_1}(\omega)) = \alpha_2 u_2(\omega). \qquad (D1.3)$$

This implies that $\alpha_2 \leq \alpha_1(\alpha_2 + \alpha_{i_1})$. For $i_1 = 2$ we get $\alpha_1 = 1/2$ and $(D1.1)$ implies that $\alpha_2 = 1/2$ and for $i_1 = 3$ we will have $\alpha_2 = 1$, a contradiction in both the cases.

Now, if $\phi_2(\omega) = \phi_3 \circ \phi_{i_4}(\omega)$, $i_4 = 2$ or 3. So, by choosing a function f such that $f(\omega) = f(\phi_1(\omega)) = f(\phi_3(\omega)) = 0$ in Equation $(D1)$we will be left with three points, i.e., $\phi_1 \circ \phi_{i_5}(\omega)$ $(i_5 \neq i_1)$, $\phi_2 \circ \phi_{i_6}(\omega)$ $(i_6 \neq i)$, $\phi_3 \circ \phi_{i_7}(\omega)$ $(i_7 \neq i_4)$ and we have 0 on the right hand side. It is also clear that $\phi_3 \circ \phi_{i_7}(\omega)$ is not equal to any of

$\omega, \phi_2(\omega)$, or $\phi_3(\omega)$. So, it has to be equal to at least one of $\phi_1 \circ \phi_{i_5}(\omega)$ or $\phi_2 \circ \phi_{i_6}(\omega)$. But in all these cases we can choose f large enough to get a contradiction.

We will need one more lemma to complete the proof of Theorem 1.3.

Lemma 2.4. *With the assumption in Theorem 1.3, one and only one of the following conditions is possible: (In all the cases $i, j, k = 1, 2, 3$)*

(*i*) $\Omega = A \bigcup B_i$.
(*ii*) $\Omega = B_i$.
(*iii*) $\Omega = A \bigcup B_i \bigcup C_i$.
(*iv*) $\Omega = C_i$.
(*v*) $\Omega = A \bigcup C_i$.
(*vi*) $\Omega = D_{ij}$.
(*vii*) $\Omega = A \bigcup D_{ij}$.
(*viii*) $\Omega = A \bigcup D_{ij} \bigcup D_{kl}$, $l = 1, ..., 6$.
(*ix*) $\Omega = A \bigcup D_{1i} \bigcup D_{2j} \bigcup D_{3k}$.

Proof. We have seen in the beginning of proof of Theorem 1.3 that $\Omega \neq A$. Suppose $\Omega = A \bigcup B_1 \bigcup B_2 \bigcup B_3$. Let us consider any $w \in B_1$, i.e $w = \phi_3(w) = \phi_2(\omega) \neq \phi_1(\omega)$. The case $\omega \in B_2$ or B_3 are similar. Equation($**$) is reduced to

$$[\alpha_3 u_3(\omega) + \alpha_2 u_2(\omega)][\alpha_3 u_3(\omega) f(\omega) + \alpha_2 u_2(\omega) f(\omega) + \alpha_1 u_1(\omega) f(\phi_1(\omega))] + \alpha_1 u_1(\omega)$$

$$[\alpha_3 u_3(\phi_1(\omega)) f(\phi_3 \circ \phi_1(\omega)) + \alpha_2 u_2(\phi_1(\omega)) f(\phi_2 \circ \phi_1(\omega)) + \alpha_1 u_1(\phi_1(\omega)) f(\phi_1^2(\omega))]$$

$$= [\alpha_3 u_3(\omega) + \alpha_2 u_2(\omega)] f(\omega) + \alpha_1 u_1(\omega) f(\phi_1(\omega)). \qquad (B1)$$

First we claim that $\alpha_3 u_3(\omega) + \alpha_2 u_2(\omega) \neq 0$. Suppose on the contrary that $\alpha_3 u_3(\omega) + \alpha_2 u_2(\omega) = 0$. Then, $\alpha_3 = \alpha_2,\ u_3(\omega) + u_2(\omega) = 0$ and Equation (B1) becomes

$$\alpha_2 u_3(\phi_3(\omega)) f(\phi_3 \circ \phi_1(\omega)) + \alpha_2 u_2(\phi_1(\omega)) f(\phi_2 \circ \phi_1(\omega)) + \alpha_1 u_1(\phi_1(\omega)) f(\phi_1^2(\omega))$$

$$= f(\phi_1(\omega)).$$

As $\phi_1(\omega) \neq \phi_1^2(\omega)$, $\phi_1(\omega)$ must be equal to only one of $\phi_3 \circ \phi_1(\omega)$ and $\phi_2 \circ \phi_1(\omega)$, because if not then one can choose a function f to assume value 0 at $\phi_1^2(\omega),\ \phi_3 \circ \phi_1(\omega),\ \phi_2 \circ \phi_1(\omega)$ and 1 at $\phi_1(\omega)$ to get a contradiction. By same argument we see that $\phi_1(\omega)$ cannot be equal to both $\phi_3 \circ \phi_1(\omega)$ and $\phi_2 \circ \phi_1(\omega)$. Moreover, if $\phi_1(\omega) = \phi_3 \circ \phi_1(\omega)$, then $\phi_2 \circ \phi_1(\omega)$ must be equal to $\phi_1^2(\omega)$. Therefore, suppose that $\phi_1(\omega) = \phi_3 \circ \phi_1(\omega),\ \phi_1^2(\omega) = \phi_2 \circ \phi_1(\omega)$. The case $\phi_1(\omega) = \phi_2 \circ \phi_1(\omega),\ \phi_1^2(\omega) = \phi_3 \circ \phi_1(\omega)$ is similar. Take a function f so that $f(\phi_1(\omega)) = 1,\ f(\phi_1^2(\omega)) = 0$ we will get $\alpha_3 = 1$, a contradiction. Now for a continuous function f such that

$f(\omega) = 1,\ f(\phi_1(\omega)) = f(\phi_3 \circ \phi_1(\omega)) = f(\phi_2 \circ \phi_1(\omega)) = 0$, then Equation (B1) becomes

$$[\alpha_3 u_3(\omega) + \alpha_2 u_2(\omega)]^2 + \alpha_1^2 u_1(\omega) u_1(\phi_1(\omega)) f(\phi_1^2(\omega)) = \alpha_3 u_3(\omega) + \alpha_2 u_2(\omega). \qquad (B2)$$

$\phi_1^2(\omega)$ must be equal to one of $\omega, \phi_3 \circ \phi_1(\omega)$ and $\phi_2 \circ \phi_1(\omega)$. If $\phi_1^2(\omega) = \phi_3 \circ \phi_1(\omega)$ or $\phi_2 \circ \phi_1(\omega)$, then $f(\phi_1^2(\omega)) = 0$. This implies that $\alpha_3 u_3(\omega) + \alpha_2 u_2(\omega) = 1$ as $\alpha_3 u_3(\omega) + \alpha_2 u_2(\omega) \neq 0$. Thus, $1 \leq \alpha_2 + \alpha_3$, a contradiction to the fact that $\alpha_1 + \alpha_2 + \alpha_3 = 1$. Therefore, $\phi_1^2(\omega) = \omega$ and (B2) is reduced to

$$[\alpha_3 u_3(\omega) + \alpha_2 u_2(\omega)]^2 + \alpha_1^2 u_1(\omega) u_1(\phi_1(\omega)) = \alpha_3 u_3(\omega) + \alpha_2 u_2(\omega). \qquad (B2')$$

Now, for a continuous function f such that $f(\omega) = 0, f(\phi_1(\omega)) = 1$, Equation (B1) reduces to

$$\alpha_3 u_3(\omega) + \alpha_2 u_2(\omega) + \alpha_3 u_3(\phi_1(\omega)) f(\phi_3 \circ \phi_1(\omega)) + \alpha_2 u_2(\phi_1(\omega)) f(\phi_2 \circ \phi_1(\omega)) = 1. \qquad (B3)$$

By a similar line of arguments we conclude that $\phi_1(\omega) = \phi_3 \circ \phi_1(\omega) = \phi_2 \circ \phi_1(w)$. So, (B3) becomes

$$\alpha_3 u_3(\omega) + \alpha_2 u_2(\omega) + \alpha_3 u_3(\phi_1(\omega)) + \alpha_2 u_2(\phi_1(\omega)) = 1. \qquad (B3')$$

This implies that $\alpha_3 + \alpha_2 \geq 1/2$. Now $Pf(\omega) = [\alpha_3 u_3(\omega) + \alpha_2 u_2(\omega)] f(\omega) + \alpha_1 u_1(\omega) f(\phi_1(\omega))$, which implies that $|Pf(\omega)| \leq |\alpha_3 u_3(\omega) + \alpha_2 u_2(\omega)||f(\omega)| + \alpha_1 |f(\phi_1(\omega))|$. Now, consider the following cases:

(a) If all B_i's are closed, then as A is closed, by connectedness of Ω we have $\Omega = B_1$, $\Omega = B_2$ or $\Omega = B_3$. If $\Omega = B_1$, then $\exists\ \omega_0 \in \Omega$ and f such that $||f|| = 1 = |Pf(\omega_0)|$, which shows that $|\alpha_3 u_3(\omega_0) + \alpha_2 u_2(\omega_0)| = \alpha_3 + \alpha_2$. Thus, $u_3(\omega_0) = u_2(\omega_0) = 1$. From Equation $(B2')$ we get $\alpha_1 \geq 1/2$. Since, $\alpha_1 \leq 1/2$ we conclude, $\alpha_3 + \alpha_2 = \alpha_1 = 1/2$. From $(B3')$ we get $u_2(\omega) = u_3(\omega) = u_2(\phi_1(\omega)) = u_3(\phi_1(\omega)) = 1$. Similarly is the case when $\Omega = B_2$ or $\Omega = B_3$.

(b) If only one B_i is closed, then as any limit point of B_i can belong to either B_i or A we get $A \bigcup B_j \bigcup B_k$ is closed and hence either $\Omega = B_i$ or $\Omega = A \bigcup B_j \bigcup B_k$. Suppose that B_3 is closed and $\Omega = A \bigcup B_1 \bigcup B_2$. The other cases are similar. Since B_2 is not closed there exists $\omega_n \in B_1$ such that $\omega_n \to \omega$ and $\omega \in A$. Note that $\phi_1(\omega) = \phi_2(\omega) = \phi_3(\omega) = \omega$. If $\omega \in A_1$, then $u_1(\omega) = u_2(\omega) = u_3(\omega) = 1$ and from Equation $(B2')$ we have $[\alpha_2 + \alpha_3]^2 + \alpha_2^2 = \alpha_2 + \alpha_3$, which implies that $\alpha_1 = 1/2$. If $\omega \in A_2$, then $\alpha_1 u_1(\omega) + \alpha_2 u_2(\omega) + \alpha_3 u_3(\omega) = 0$ and Equation $(B3')$ implies that $-\alpha_1 u_1(\omega) = 1/2$ and hence $\alpha_1 = 1/2$. Similar argument for B_2 will give us $\alpha_2 = 1/2$ - a contradiction.

Thus, $\Omega \neq A \bigcup B_1 \bigcup B_2$.

(c) If two B_i's are closed then we will have $\Omega = A \bigcup B_i$, for some i or $\Omega = B_j,\ i \neq j$. Suppose $\Omega = A \bigcup B_1$, B_1 is not closed. Considering a sequence in B_1

and proceeding as above we conclude that $\alpha_1 = \alpha_2 + \alpha_3 = 1/2$ and from Equation $(B3')$ we get $u_2(\omega) = u_3(\omega) = u_2(\phi_1(\omega)) = u_3(\phi_1(\omega)) = 1$.

(d) If no B_i's are closed then $\Omega = A \bigcup B_1 \bigcup B_2 \bigcup B_3$. Proceeding in the same way as in case (b), we can see that this case is also not possible.

From previous lemma one can see that none of C_1, C_2, C_3 can occur together. Suppose $\Omega = A \bigcup B_1 \bigcup B_2 \bigcup B_3 \bigcup C_1$. The cases in which $\Omega = A \bigcup B_1 \bigcup B_2 \bigcup B_3 \bigcup C_i$, $i = 2, 3$ are similar. Now, a sequential argument will show that B_2, B_3 and $A \bigcup B_1 \bigcup C_1$ are closed. From connectedness of Ω we get that $\Omega = B_2$ or $\Omega = B_3$ or $A \bigcup B_1 \bigcup C_1$.

Let $\Omega = A \bigcup B_1 \bigcup C_1$. If B_1 and C_1 are closed then $\Omega = B_1$ or $\Omega = C_1$. If one of B_1 is closed and C_1 is not, then $\Omega = B_1$ or $\Omega = A \bigcup C_1$. If C_1 is closed and B_1 is not, then $\Omega = C_1$ or $\Omega = A \bigcup B_1$. This proves assertions (i)-(v).

It is also clear from previous lemma that for $i = 1, 2, 3$, C_i cannot occur with D_i. Also, for fixed $i = 1, 2, 3$, no two or more D_{ij}, $j = 1, ..., 6$ can occur simultaneously.

Suppose that $\Omega = A \bigcup B_i \bigcup D_{jk}$. Then $\alpha_i = 1/3$ for $i = 1, 2, 3$. So, if B_i and D_{jk} are not closed then by a sequential argument as in case (b) above we will get $\alpha_i = 1/2$, a contradiction. Thus, no B_i can occur with D_{jk}. Assume $\Omega = A \bigcup D_{1i} \bigcup D_{2j} \bigcup D_{3k}$. If some of D_{ij}'s are closed, then by arguing in a similar way we will get cases (vi)-(ix).

This completes the proof of Lemma 2.4

Completion of proof of Theorem 1.3: For any $\omega \in B_1$ we have $u_2(\omega) = u_3(\omega) = u_2(\phi_1(\omega)) = u_3(\phi_1(\omega)) = 1$ and for $\omega \in C_1$; $u_2(\omega) = u_3(\omega) = u_2(\phi_2(\omega)) = u_3(\phi_2(\omega)) = 1$. Therefore, $T_2 f(\omega) = T_3 f(\omega)$ for all $f \in C(\Omega)$, $\omega \in B_1 \bigcup C_1$. So, if $\Omega = B_1$, C_1,, $A \bigcup B_1$, $A \bigcup C_1$, or $A \bigcup B_1 \bigcup C_1$ we have $P = \frac{T_1+T_2}{2}$. Similarly is the case when any one of conditions (i)-(v) holds.

Thus the proof of Theorem 1.3 (a) is complete.

It remains to consider the case when $\Omega = A \bigcup D_{1i} \bigcup D_{2j} \bigcup D_{3k}$. We further assume that $i, k \leq 4, j \geq 5$. The remaining cases and conditions (vi)-(viii) are similar. Our aim is to show that there exists a surjective isometry on $C(\Omega)$ such that $L^3 = I$ and $P = \frac{(I+L+L^2)}{3}$. Since $P = 1/3(T_1 + T_2 + T_3)$ is a projection we have $P = \frac{1}{9}(T_1^2 + T_2^2 + T_3^2 + T_1T_2 + T_2T_1 + T_1T_3 + T_3T_1 + T_2T_3 + T_3T_2)$.

Using the conditions obtained earlier on $u_i(\omega)$'s and $u_i(\phi_j(\omega))$ we see that for any $\omega \in D_{11}$; $T_1^2 f(\omega) = T_2^2 f(\omega) = f(\omega)$, $T_3^2 f(\omega) = T_2 f(\omega)$, $T_1T_2 f(\omega) = T_2T_1 f(\omega) = T_2 f(\omega)$, $T_1T_3 f(\omega) = T_3T_1 f(\omega) = T_3T_2 f(\omega) = T_3 f(\omega)$, $T_2T_3 f(\omega) = f(\omega)$. That is, $P = \frac{I+T_3+T_3^2}{3}$ and $T_3^3 = I$. Similarly if $\omega \in D_{12}, D_{13}$ or D_{14} we have $P = \frac{I+T_3+T_3^2}{3}$

and $T_3^3 = I$. If $w \in D_{15}$ or D_{16}, then we get $P = \frac{I+T_2+T_3}{3} = \frac{I+T_2T_3+(T_2T_3)^2}{3}$ and $(T_2T_3)^3 = I$. Similar considerations can be done for D_2 and D_3. We now define

$$u(w) = \begin{cases} u_1(\omega), \text{ if } \omega \in A_1 \\ \\ u_3(\omega), \text{ if } \omega \in D_{1i} \\ \\ u_1(\omega)u_3(\phi_1(\omega)), \text{ if } \omega \in D_{2j} \\ \\ u_1(\omega), \text{ if } \omega \in D_{3k} \end{cases} \quad \text{and } \phi(\omega) = \begin{cases} \phi_1(\omega), \text{ if } \omega \in A_1 \\ \\ \phi_3(\omega), \text{ if } \omega \in D_{1i} \\ \\ \phi_3 o \phi_1(\omega), \text{ if } \omega \in D_{2j} \\ \\ \phi_1(\omega), \text{ if } \omega \in D_{3k} \end{cases}$$

Let $Lf(\omega) = u(\omega)f(\phi(\omega))$. Observe that the limit point of any sequence in D_{ij} can go only to D_{ij} or A. So, it follows that u is continuous and ϕ is a homeomorphism. Hence the proof of Theorem 1.3 (b) is complete.

□

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