STRONG PROXIMINALITY OF CLOSED CONVEX SETS

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Abstract. We show that in a Banach space X every closed convex subset is strongly proximinal if and only if the dual norm is strongly sub differentiable and for each norm one functional f in the dual space X^*, $J_X(f)$ - the set of norm one elements in X where f attains its norm is compact. As a consequence, it is observed that if the dual norm is strongly sub differentiable then every closed convex subset of X is strongly proximinal if and only if the metric projection onto every closed convex subsets of X is upper semi-continuous.

1. Introduction

Let X be a Banach space and C a closed subset of X. The metric projection of X onto C is the set valued map defined by $P_C(x) = \{y \in C : \|x - y\| = d(x, C)\}$ for $x \in X$, where $d(x, C)$ denotes the distance of x from C. If for every $x \in X$, $P_C(x) \neq \emptyset$, we say that C is a proximinal subset of X.

For a Banach space X, we denote the closed unit ball and the unit sphere by B_X and S_X respectively. If $f \in S_{X^*}$ is a norm attaining functional, we define $J_X(f) = \{x \in S_X : f(x) = 1\}$.

For $x \in X \setminus C$ and given any $t > 0$, there exists $y \in C$ such that $\|x - y\| < d(x, C) + t$. If we call such a y a nearly best approximation to x in C, a natural question is whether y is close to an actual best approximation of x in C. Clearly we are demanding more than proximinality of C in X and in [5] the authors called such a subset as a strongly proximinal subset.

Definition 1.1. Let C be a closed subset in a Banach space X and $x \in X$. For $t > 0$, consider the following set

$$P_C(x, t) = \{y \in C : \|x - y\| < d(x, C) + t\}.$$

A proximinal set C is said to be *strongly proximinal* at $x \in X$ if for given $\varepsilon > 0$ there exists a $t > 0$ such that

$$P_C(x, t) \subseteq P_C(x) + \varepsilon B_X.$$

If C is strongly proximinal at all points of X we say that C is strongly proximinal.

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Some sufficient (and necessary) conditions for strong proximinality of certain subspaces of some classical Banach spaces are studied in the literature [2, 3, 6, 4, 5].

In this paper we are motivated by the following question:

Question 1.2. *Under what condition (necessary or sufficient)is every closed convex subset of X strongly proximinal?*

A known necessary condition for every closed convex subset of X to be strongly proximinal is that the norm of X^* is strongly sub-differentiable (see below for the definition). This follows from the fact that every closed hyperplane in X is strongly proximinal if and only if the norm of X^* is strongly sub-differentiable (this was noted as a corollary to a main theorem in [5]; however we will present a direct proof of this fact in Section 2 for completeness).

Our aim is to find an additional condition so that this necessary condition and the additional one become necessary and sufficient.

Definition 1.3. Let X be a Banach space. The norm $\|\cdot\|$ is said to be *strongly subdifferentiable* (in short *SSD*) at $x \in X$ if the one-sided limit

$$\lim_{t \to 0^+} \frac{\|x + th\| - \|x\|}{t}$$

exists uniformly for $h \in S_X$. If the norm $\|\cdot\|$ of X is SSD at all points of S_X, we say that $\|\cdot\|$ is SSD or the space X is SSD.

Recall that (see [7]) a closed set $C \subseteq X$ is said to be approximatively compact if every minimizing sequence in C has a convergent subsequence. It is easy to see that if C is approximatively compact then C is strongly proximinal. Also every closed convex subset of X is approximatively compact if and only if X is reflexive and (KK), where (KK) means that the relative weak and norm topologies coincide on the unit sphere S_X of X. This is implicit in [9] (see also [7]).

Therefore a sufficient condition for every closed convex subset of X to be strongly proximinal is that X is reflexive and (KK). Interestingly, this condition also turns out to be necessary.

Our main result in this paper is the following:

Theorem 1.4. *Let X be a Banach space. Then the following statements are equivalent.*

- (a) *X^* is SSD and $J_X(f)$ is compact for every $f \in S_{X^*}$.*
- (b) *X is reflexive and (KK).*
- (c) *Every closed convex subset of X is approximatively compact.*
- (d) *Every closed convex subset of X is strongly proximinal.*

Compare this with the well known result that every closed convex subset in X is proximinal if and only if X is reflexive.

We relate our main result with the continuity of metric projection as follows.

Definition 1.5. Let $C \subseteq X$ and $x \in X$. P_C is said to be

(a) *upper semi-continuous* (in short usc) at x if for every open set $U \subseteq X$ such that $P_C(x) \subseteq U$, there exists $\delta > 0$ such that $P_C(z) \subseteq U$ for every z satisfying $\|z - x\| < \delta$

(b) *upper Hausdorff semi-continuous* (in short uHsc) at x if for every $\varepsilon > 0$ there exists a $\delta > 0$ such that $P_C(z) \subseteq P_C(x) + \varepsilon B_X$ for every z satisfying $\|z - x\| < \delta$.

It is well known [1] that if C is a subspace of X then P_C is usc at x if and only if P_C is uHsc at x and $P_C(x)$ is compact. Also it is straight forward to see that if C is a strongly proximinal subset then P_C is uHsc. We will show that if X^* is SSD then every closed convex subset of X is strongly proximinal if and only if P_C is usc for every closed convex subset of X. In this case we also get P_C is uHsc for every closed convex subset C of X and $P_C(x)$ is compact for every $x \in X$.

2. Main results

We first give a straight forward proof of the fact that the condition that X^* is SSD is necessary for every closed convex subset of X to be strongly proximinal. We will be using [5, Lemma 1.1] which we state as a fact.

Fact 1: Let X be a Banach space and $f \in S_{X^*}$. The following assertions are equivalent.

(a) The dual norm on X^* is SSD at f.

(b) f is norm attaining and for all $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$f(x) > 1 - \delta \Rightarrow d(x, J_X(f)) < \varepsilon.$$

Proposition 2.1. *Let X be a Banach space and $f \in S_{X^*}$. Then the following statements are equivalent.*

(a) *For every $c \in \mathbb{R}$, the hyperplane $H_c = \{x \in X : f(x) = c\}$ is strongly proximinal.*

(b) *The norm of X^* is SSD at f.*

Proof. $(a) \Rightarrow (b)$: Since the hyperplane $H = \{x \in X : f(x) = 1\}$ is proximinal, $J_X(f) \neq \emptyset$. Let (x_n) be a sequence in B_X such that $f(x_n) \to 1$. We will show that $d(x_n, J_X(f)) \to 0$. By **Fact 1** it follows that the norm of X^* is SSD at f. Note that $d(0, H) = 1$ and $P_H(0) = J_X(f)$. We put $y_n = \frac{x_n}{f(x_n)}$. Then $y_n \in H$ and $\|y_n\| \to 1$. Therefore $y_n \in P_H(0, \delta_n)$ for some $\delta_n \to 0$ and since H is strongly proximinal $d(y_n, P_H(0)) \to 0$. Since $\|x_n - y_n\| \to 0$, $d(x_n, P_H(0)) = d(x_n, J_X(f)) \to 0$.

$(b) \Rightarrow (a)$: Since the norm of X^* is SSD at f, by **Fact 1**, $J_X(f) \neq \emptyset$. Therefore for every $c \in \mathbb{R}$, the hyperplane $H_c = \{x \in X : f(x) = c\}$ is proximinal. To show that H_c is strongly proxminal, let $x_0 \in X \backslash H_c$. Without loss of generality we assume $f(x_0) > c$. For $n \geq 1$ let $x_n \in P_{H_c}(x_0, \frac{1}{n})$. We show that $d(x_n, P_{H_c}(x_0)) \to 0$ which completes the proof. We first note that $H_c = \{x \in X : f(x) = f(x_0) - d\}$ where

$d = d(x_0, H_c)$. Let $y_n = \frac{x_0 - x_n}{d+2/n}$. Then $y_n \in B_X$ and $f(y_n) \to 1$. Therefore, by the assumption and **Fact 1** there exists a sequence (z_n) from $J_X(f)$ such that $\|y_n - z_n\| \to 0$. Now the sequence $(x_0 - dz_n)$ is in $P_{H_c}(x_0)$ and $\|x_n - x_0 + dz_n\| = \|x_0 - y_n(d + \frac{2}{n}) - x_0 + dz_n\| \to 0$. □

From the proof of $(b) \Rightarrow (a)$ of Proposition 2.1 it is clear that if the norm of X^* is SSD at all norm attaining functionals of S_{X^*} then all proximinal hyperplanes of X are strongly proximinal. However, the condition that the norm of X^* is SSD at all norm attaining functionals of S_{X^*} is not sufficient for every proximinal convex subset to be strongly proximinal. To show this we construct an example of a proximinal convex subset of c_0 which is not strongly proximinal. Note that the norm of ℓ_1 is SSD at every norm one norm attaining functional on c_0.

We need the following result from [8] which will be used in the sequel.

Lemma 2.2. [8, Propostion 5] *Let $H = \{x : f(x) = c\}$ be a closed hyperplane in X and (x_n) be a sequence in X such that $x_n \to x_0$ weakly for some x_0. Suppose $f(x_n) > c$ for all n . Then $(\overline{co}\{x_n : n \in \mathbb{N}\}) \cap H \neq \emptyset$ if and only if $x_0 \in H$, and in this case $(\overline{co}\{x_n : n \in \mathbb{N}\}) \cap H = \{x_0\}$.*

Example 2.3. *Consider the sequence (x_n) in c_0 where $x_n = (-\frac{1}{n}, 0, \cdots, \frac{1}{2}, 0, \cdots)$, where $\frac{1}{2}$ occurs at the nth place. It is easy to see that $x_n \to 0$ weakly. Consider $C = \overline{co}\{x_n : n \in \mathbb{N}\}$. Then C is weakly compact hence proximinal. We show that C is not strongly proximinal.*

Let $x = (1, 0, 0, \cdots) \in c_0$ and $f = (-1, 0, 0, \cdots) \in \ell_1$. Then $f(x_n) > 0$ for all n. Thus by Lemma 2.2, $(\overline{co}\{x_n : n \in \mathbb{N}\}) \cap \ker f = \{0\}$. Note that (x_n) is a minimizing sequence for x in C. Also $P_C(x) = (\overline{co}\{x_n : n \in \mathbb{N}\}) \cap \ker f = \{0\}$. But surely (x_n) does not converge to 0 in norm. Thus P_C is not strongly proximinal at x.

We now introduce the following property for a Banach space X.

Definition 2.4. We say that a Banach space X has the property *Strong HR* (in short SHR) if for any $f \in S_{X^*}$ such that $J_X(f) \neq \emptyset$, any sequence (y_n) in X such that $f(y_n) \geq 1$ and $d(y_n, J_X(f)) \to 0$ we have

$$d(y_n, (\overline{co}\{y_n : n \in \mathbb{N}\}) \cap S_X) \to 0.$$

The property (SHR) is a stronger version of the property (HR) defined in [8] as follows: X has the property *HR* if for any $f \in S_{X^*}$ such that $J_X(f) \neq \emptyset$, any sequence (y_n) in X such that $f(y_n) = 1$ and $d(y_n, J_X(f)) \to 0$ we have $d(y_n, (\overline{co}\{y_n : n \in \mathbb{N}\}) \cap S_X) \to 0$.

We first show that in a reflexive space the property (SHR) is equivalent to that $J_X(f)$ is compact for every $f \in S_{X^*}$.

Proposition 2.5. *Let X be such that $J_X(f)$ is compact for every $f \in S_{X^*}$. Then X has the property (SHR). Conversely, if X is reflexive, then (SHR) implies that $J_X(f)$ is compact for every $f \in S_{X^*}$.*

Proof. Let $f \in S_{X^*}$ be such that $J_X(f) \neq \emptyset$ and $J_X(f)$ be compact. Let (y_n) be such that $f(y_n) \geq 1$, $d(y_n, J_X(f)) \to 0$ and $d(y_{n_k}, (\overline{co}\{y_n : n \in \mathbb{N}\}) \cap S_X) \geq \epsilon$ for some subsequence (y_{n_k}) and for some $\epsilon > 0$. Then by the compactness of $J_X(f)$ there is a subsequence of (y_{n_k}) which converges to some $x \in J_X(f)$. Note that $x \in (\overline{co}\{y_n : n \in \mathbb{N}\}) \cap S_X$ and this contradicts the assumption.

Now let X be reflexive. Then for every $f \in S_{X^*}$, $J_X(f)$ is nonempty and weakly compact. Suppose X has the property (SHR). Let (x_n) be a sequence in $J_X(f)$. Define $y_n = (1 + \frac{3}{n})x_n$ for every n. Then $f(y_n) > 1$ for every n and $d(y_n, J_X(f)) \to 0$. Since $J_X(f)$ is weakly compact, we may choose a subsequence (y_{n_k}) of (y_n) such that $y_{n_k} \to y_0$ weakly for some $y_0 \in J_X(f)$. By Lemma 2.2,

$$(\overline{co}\{y_{n_k} : k \in \mathbb{N}\}) \cap H = \{y_0\}$$

where $H = \{x \in X : f(x) = 1\}$. Since $y_0 \in S_X$ and

$$(\overline{co}\{y_{n_k} : k \in \mathbb{N}\}) \cap S_X \subseteq (\overline{co}\{y_{n_k} : k \in \mathbb{N}\}) \cap H,$$

we have $\overline{co}\{y_{n_k} : k \in \mathbb{N}\}) \cap S_X = \{y_0\}$. By the property (SHR) we get that $d(y_{n_k}, (\overline{co}\{y_{n_k} : k \in \mathbb{N}\}) \cap S_X) \to 0$. That is $y_{n_k} \to y_0 \in J_X(f)$. This implies that $x_{n_k} \to y_0$ and the proof is complete. □

Before we prove our main result, we give an example to show that the property (SHR) is strictly stronger than the property (HR) considered in [8]

Example 2.6. *In ℓ_2 consider the following set*

$$B' = \{x = (x(1), x(2), ...) \in l_2 : \|x\|_2 \leq 1,\ |x(1)| \leq \frac{1}{2}\}.$$

Let $|||\cdot|||$ be the Minkowski's functional of B'. Then $|||\cdot|||$ is an equivalent norm on ℓ_2 and let $X = (\ell_2, |||\cdot|||)$. It is shown by Osman [8] *that the space X has the property (HR). We will show that the space does not have the property (SHR).*

Consider $f \in X^*$ defined by $f((x(1), x(2), ...)) = 2x(1)$. It is clear that $f \in S_{X^*}$. Let $H = \{x = (x(1), x(2), ...) \in X : f(x) = 1\} = \{x \in X : x(1) = \frac{1}{2}\}$. Then the closed hyperplane H supports the unit ball B' and $J_X(f) = H \cap B'$ is not compact. Hence by Proposition 2.5, the space cannot have the property (SHR).

Proof of Theorem 1.4:

$(a) \Rightarrow (b)$: If the norm of X^* is SSD at some $f \in S_{X^*}$ then f is norm attaining on S_X [5, Lemma 1.1]. Hence if X^* is *SSD* then X is reflexive.

To show (KK), let (x_n) be a sequence in S_X such that $x_n \to x$ weakly for some $x \in S_X$. Suppose $f(x) = 1$ for some $f \in S_{X^*}$. Then $f(x_n) \to 1$. Thus by [5, Lemma 1.1], $d(x_n, J_X(f)) \to 0$. By the compactness of $J_X(f)$, there exists a norm convergent subsequence (x_{n_k}). Since $x_n \to x$ weakly, $x_{n_k} \to x$ in norm. Starting with any subsequence of (x_n) we can produce, by the above argument, a further subsequence which is norm convergent to x. Hence (x_n) converges to x in norm.

$(b) \Rightarrow (c)$: This is essentially proved in [9]. See also [7].

$(c) \Rightarrow (d)$ is easy.

$(d) \Rightarrow (a)$: Since every closed convex subset of X is strongly proximinal, by Proposition 2.1, X^* is SSD. We will show that X has the property (SHR). This, with Proposition 2.5, will show that $J_X(f)$ is compact for every $f \in S_{X^*}$.

Let (y_n) be any sequence in X such that $f(y_n) \geq 1$ for all n and $d(y_n, J_X(f)) \to 0$ for some $f \in S_{X^*}$. Take $C = \overline{co}\{y_n : n \in \mathbb{N}\}$. Since $\|y_n\| \to 1$, $d(0, C) = 1$ and $y_n \in P_C(0, \delta_n)$ for some $\delta_n \to 0$. By the strong proximinality of C, we have $d(y_n, P_C(0)) \to 0$. Since

$$P_C(0) = (\overline{co}\{y_n : n \in \mathbb{N}\}) \cap S_X,$$

we have $d(y_n, \overline{co}\{y_n : n \in \mathbb{N}\} \cap S_X) \to 0$. □

Remark 2.7. In Question 1.2 we demand that every closed convex set is strongly proximinal in X. This automatically forces the space X to be reflexive. However, in a non-reflexive set up one may ask when every proximinal convex subset is strongly proximinal. From Proposition 2.1, we observe that the condition that the norm of X^* is SSD at all norm attaining functionals of S_{X^*} is necessary for every proximinal convex subset to be strongly proximinal. In the following proposition we show that the conditions that the norm of X^* is SSD at all norm attaining functionals of S_{X^*} and X has the property (SHR) are sufficient for this.

Proposition 2.8. *Let X be a Banach space such that the norm of X^* is SSD at every norm attaining functional of S_{X^*} and X has the property (SHR). Then every proximinal convex subset in X is strongly proximinal.*

Proof. Suppose C is a proximinal convex subset such that $d(0, C) = 1$ and $y_n \in P_C(0, \frac{1}{n})$ for every n. Let $H = \{x \in X : f(x) = 1\}$, $f \in S_{X^*}$. Then H separates C and B_X. Choose $y'_n \in [0, y_n] \cap H$. Then

$$1 = d(0, H) \leq \|y'_n\| \leq 1 + \frac{1}{n} \to 1.$$

This implies that $\|y'_n\| \to 1$. Since f is norm attaining on S_X, the norm of X^* is SSD at f. By Proposition 2.1, H is strongly proximinal and therefore $d(y'_n, P_H(0)) \to 0$. Note that $\|y_n - y'_n\| \to 0$ and $P_H(0) = J_X(f)$. Therefore $d(y_n, J_X(f)) \to 0$. Since $f(y_n) \geq 1$, by the property of (SHR), $d(y_n, (\overline{co}\{y_n : n \in \mathbb{N}\}) \cap S_X) \to 0$. This implies that $d(y_n, P_C(0)) \to 0$ because $(\overline{co}\{y_n : n \in \mathbb{N}\}) \cap S_X \subseteq P_C(0)$. This proves that C is strongly proximinal. □

Remark 2.9. Note that in case $X = c_0$, the dual norm is SSD at all norm attaining functionals of S_{X^*}. Proposition 2.8 and Example 2.3 together show that c_0 does not have the property (SHR). It is also evident that for every norm attaining $f \in S_{\ell_1}$, $J_{c_0}(f)$ is never compact. We do not know an example of a non-reflexive Banach space X satisfying the condition of Proposition 2.8 but $J_X(f)$ is not compact for a norm attaining functional $f \in S_{X^*}$.

We now relate our main result to continuity property of the metric projection.

It is easy to see that if C is a strongly proximinal subset then P_C is uHsc. In the next result we see that if every closed convex subset of X is strongly proximinal then the metric projection becomes usc for every closed convex subset of X.

It is well known [1] that if C is a subspace of X then P_C is usc at x if and only if P_C is uHsc at x and $P_C(x)$ is compact. The following result characterizes the upper semi continuity of $P_C(\cdot)$ for every closed convex subset C of X.

Proposition 2.10. *Let X be a Banach space such that X^* is SSD. Then the following statements are equivalent.*

(a) *For every $f \in S_{X^*}$, $J_X(f)$ is compact.*
(b) *Every closed convex subset of X is strongly proximinal.*
(c) *For every closed convex subset C of X, the metric projection $P_C(\cdot)$ is uHsc and $P_C(x)$ is compact for every $x \in X$.*
(d) *For every closed convex subset C of X, the metric projection $P_C(\cdot)$ is usc .*

Proof. $(a) \Rightarrow (b)$: This follows from Theorem 1.4.

$(b) \Rightarrow (c)$: By Theorem 1.4, every closed convex subset of X is approximatively compact. The compactness of $P_C(x)$ follows from the approximative compactness of C.

$(c) \Rightarrow (d)$: This is known [1].

$(d) \Rightarrow (a)$: For given $f \in S_{X^*}$ consider the hyperspace $G = \{x \in X : f(x) = 0\}$. Since G is a subspace, by (d) and a result of [1], $P_G(x)$ is compact for every $x \in X$. This implies that $J_X(f)$ is compact because $P_G(x) = \{x - f(x)z : z \in J_X(f)\}$. □

References

[1] F. Deutsch, W. Pollul and I. Singer, *On set-valued metric projections, Hahn-Banach extension maps, and spherical image maps*, Duke Math. J. **40** (1973), 355–370. MR0348450 (50:948).

[2] S. Dutta and D. Narayana, *Strongly proximinal subspaces in Banach spaces*, Contemp. Math., Amer. Math. Soc. **435**, (2007), 143—152. MR2359424 (2008m:41029).

[3] S. Dutta and D. Narayana, *Strongly proximinal subspaces of finite codimension in $C(K)$*, Colloquium Math. **109** No. 1, (2007), 119—128. MR2308830 (2009e:41067).

[4] G. Godefroy, V. Indumathi and F. Lust-Piquard, *Strong subdifferentiability of convex functionals and proximinality*, J. Approx. Theory **116** (2002), no. 2, 397—415. MR1911087 (2003d:41034).

[5] G. Godefroy and V. Indumathi, *Strong proximinality and polyhedral spaces*, Rev. Mat. Complut. **14** (2001), no. 1, 105—125. MR1851725 (2002f:46016).

[6] V.Indumathi, *Metric projections of closed subspaces of c_0 onto subspaces of finite codimension*, Collq. Math. 99, (2004), no. 2, 231—252. MR2079329 (2005f:46025).

[7] R. E. Megginson, *An introduction to Banach space theory,* Graduate Texts in Mathematics, 183. Springer-Verlag, New York, 1998. MR1650235 (99k:46002).

[8] E. V. Osman, *On the continuity of metric projections in Banach spaces*, Math. USSR Sbornik, **9** (1969), no. 2, 171—182. doi: 10.1070/SM1969v009n02ABEH001283.

[9] L. P. Vlasov, *Approximative properties of sets in normed linear spaces*, (Russian) Uspehi Mat. Nauk **28** (1973), no. 6(174), 3—66, English translation: Russian Math. Surveys 28 (1973), no. 6, 1—66. MR0404963 (53:8761).

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