MODULUS OF STRONG PROXIMINALITY AND CONTINUITY OF METRIC PROJECTION

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ABSTRACT. In this paper we initiate a quantitative study of strong proximinality. We define a quantity $\epsilon(x,t)$ which we call as modulus of strong proximinality and show that the metric projection onto a strongly proximinal subspace Y of a Banach space X is continuous at x if and only if $\epsilon(x,t)$ is continuous at x whenever $t > 0$. The best possible estimate of $\epsilon(x,t)$ characterizes spaces with $1\frac{1}{2}$ ball property. Estimates of $\epsilon(x,t)$ are obtained for subspaces of uniformly convex spaces and of strongly proximinal subspaces of finite codimension in $C(K)$.

1. INTRODUCTION

Let X be a Banach space and Y a closed subspace of X. The metric projection of X onto Y is the set valued map defined by $P_Y(x) = \{y \in Y : \|x - y\| = \text{dist}(x, Y)\}$ for $x \in X$. If for every $x \in X$, $P_Y(x) \neq \emptyset$, we say that Y is a proximinal subspace of X.

For a Banach space X, we denote the closed unit ball and the unit sphere by B_X and S_X respectively. In general the open ball and the closed ball of radius r around $x \in X$ will be denoted by $B(x,r)$ and $B[x,r]$ respectively. We restrict ourselves to real scalars. All subspaces we consider are assumed to be closed.

For $x \in X \setminus Y$ and given any $t > 0$, there exists $y \in Y$ such that $\|x - y\| < \text{dist}(x,Y) + t$. If we call such a y as a nearly best approximation to x in Y, a natural question is whether y is close to an actual best approximation of x in Y. Clearly we are demanding more than proximinality of Y in X and in [5] the authors called such a subspace as a strongly proximinal subspace.

Definition 1.1. Let Y be a closed subspace in a Banach space X and $x \in X$. For $t > 0$, consider the following set

$$P_Y(x,t) = \{y \in Y : \|x - y\| < \text{dist}(x,Y) + t\}.$$

2000 *Mathematics Subject Classification.* 41A65, 46B20.
Key words and phrases. Strong proximinality, metric projections, modulus of strong proximinality, Hausdorff metric continuous.

A proximinal subspace Y is said to be strongly proximinal at $x \in X$ if for given $\varepsilon > 0$ there exists a $t > 0$ such that

$$P_Y(x,t) \subseteq P_Y(x) + \varepsilon B_Y.$$

It is to mention here that Vlasov studied the same notion under the name H-set [12]. Also [11] considered the notion of *local U-proximinal subspaces* and in [1] it was established that the *local U-proximinality* is same as the strong proximinality.

One of the main motivations to study strongly proximinal subspaces is for many natural classes of Banach spaces X, if Y is a strongly proximinal subspace of X, then P_Y has nice continuity properties. We need the following definitions.

Definition 1.2.
(a) P_Y is called lower Hausdorff semi-continuous (henceforth lHsc) at x if given $\varepsilon > 0$, there exists $\delta > 0$ such that for every z satisfying $\|z - x\| < \delta$ we have $P_Y(x) \subseteq P_Y(z) + \varepsilon B_Y$.
(b) P_Y is called upper Hausdorff semi-continuous (henceforth uHsc) at x if given $\varepsilon > 0$, there exists $\delta > 0$ such that for every z satisfying $\|z - x\| < \delta$ we have $P_Y(z) \subseteq P_Y(x) + \varepsilon B_Y$.
(c) P_Y is called Hausdorff metric continuous at x if it is continuous as a single valued map from X to 2^Y with respect to the Hausdorff metric d_h defined as follows:

$$d_h(A,B) = \max\{\sup_{x \in A} \mathrm{dist}(x,B), \sup_{y \in B} \mathrm{dist}(y,A)\} \qquad A, B \in 2^Y.$$

Remark 1.3.
(a) It is a simple consequence of the definition that if Y is a strongly proximinal subspace then P_Y is uHsc.
(b) If Y is proximinal in X, then P_Y is Hausdorff metric continuous if and only if P_Y is both lHsc and uHsc.
(c) Sometimes in the literature (see [3, 8]) lower Hausdorff semi-continuity is referred as strongly lower semi-continuity.

Remark 1.4. For a subspace $Y \subseteq X$, let $D(Y)$ denote the set $\{x \in S_X : \mathrm{dist}(x,Y) = 1\}$. A simple normalization shows that to check the strong proximinality of Y and the continuity of P_Y, it is enough to verify them for $x \in D(Y)$.

In [8] it was shown that if $X \subseteq c_0$ and $Y \subseteq X$ is a strongly proximinal subspace of finite codimension in X, then P_Y is Hausdorff metric continuous. More general results were obtained in [3], where the authors showed that if X is a Banach space with Property $(*)$ (see [3] for the definition of Property $(*)$) and $Y \subseteq X$ is a proximinal subspace of finite codimension, then P_Y is lHsc. By [3], every separable polyhedral space has a renorming with Property $(*)$. In particular, if $1 \leq \alpha < \omega_1$ is a countable ordinal then the space $C(\omega^\alpha)$ is an ℓ_1-predual and hence isomorphically polyhedral space. Thus $C(\omega^\alpha)$ has a renorming with Property $(*)$ (see [7]).

In [2], it was shown that if Y is a strongly proximinal subspace of finite codimension in $C(K)$, then P_Y is continuous in Hausdorff metric.

However, a recent result by Indumathi in [9], shows that if Y is a proximinal subspace of X of finite codimension such that $Y^\perp$ is polyhedral, then P_Y is lHsc. Hence, by Remark 1.3(a), it follows that for a strongly proximinal subspace Y of X, if $Y^\perp$ is polyhedral then P_Y is continuous. Thus the results mentioned above now follow as corollaries to the result in [9].

In this paper we initiate a quantitative study of strong proximinality. Taking cue from [11] and a result from [1] (see below), we define a local modulus of strong proximinality and establish that the continuity of P_Y on strongly proximinal subspaces is equivalent to the continuity of this modulus. Further, we estimate the values of this modulus on certain spaces. Also, we show that the best possible estimate of modulus of strong proximinality characterizes spaces with $1\frac{1}{2}$-ball property.

In [11], *(local) U-proximinality* was defined through a function $\varepsilon : X \setminus Y \times \mathbb{R}^+ \to \mathbb{R}^+$. If Y is *locally U-proximinal*, by [11, Theorem 3.3], a sufficient condition for continuity of P_Y is that $\varepsilon(\cdot, t)$ is upper semi-continuous on $X \setminus Y$ for each $t > 0$.

However, in [1, Proposition 3.1] it was noted that the function ε defined by Lau coincides with $d_h(P_Y(x), P_Y(x,t))$ on $D(Y)$. As a consequence it follows that Y is *locally U-proximinal* if and only if Y is strongly proximinal. Taking cue from this observation and the above mentioned result of Lau, we define the following quantity which we call as *modulus of strong proximinality.*

Definition 1.5. Let $Y \subseteq X$ be a proximinal subspace. The *modulus of strong proximinality* $\varepsilon : X \setminus Y \times \mathbb{R}^+ \to \mathbb{R}^+$ is defined by

$$\varepsilon(x,t) = \inf\{r > 0 : \ P_Y(x,t) \subseteq P_Y(x) + rB_Y\}.$$

Intuitively, $\varepsilon(x,t)$ measures how close a nearly best approximation of x in Y is to an actual best approximation of x in Y.

In Section 2 we first show that for each x, $\varepsilon(x, \cdot)$ is continuous at $t > 0$ and it is continuous at $t = 0$ if and only if Y is strongly proximinal. Our main result in Section 2 is that if Y is a strongly proximinal subspace then P_Y is continuous at x if and only if $\varepsilon(\cdot, t)$ is continuous at x for each $t > 0$. This will be done in two parts: First we show that for any proximinal subspace $Y \subseteq X$, if P_Y is uHsc at x then $\varepsilon(\cdot, t)$ is lsc at x and if P_Y is lHsc at x then $\varepsilon(\cdot, t)$ is usc at x. Next we show that if $\varepsilon(x, \cdot)$ is continuous at 0 and $\varepsilon(\cdot, t)$ is usc at x for each $t > 0$, then P_Y is lHsc at x. Since continuity of $\varepsilon(x, \cdot)$ at 0 already implies P_Y is uHsc, we conclude our result. Note that in the process we also recover [11, Theorem 3.3] which, in combination of [1, Proposition 3.1], just gives the sufficient part.

Section 3 is devoted to study of quantitative estimates of $\varepsilon(\cdot,\cdot)$. In particular we consider subspaces with $1\frac{1}{2}$-ball property, subspaces of uniformly convex spaces with power type modulus of convexity and strongly proximinal subspaces of finite codimension in $C(K)$.

We show that the best possible estimates for ε is $\varepsilon(x,t) < t$, that is, $\varepsilon(x,t)$ is proportional to t for all x and this happens if and only if Y has $1\frac{1}{2}$-ball property in X.

Definition 1.6. $Y \subseteq X$ is said to have the $1\frac{1}{2}$-ball property in X if $x \in X$, $y \in Y$, $B[x,r] \cap Y \neq \emptyset$ and $\|x-y\| < r+s$, then the intersection $Y \cap B[x,r] \cap B[y,s] \neq \emptyset$.

From our result it follows, in particular, that if Y has $1\frac{1}{2}$-ball property in X then P_Y is 2-Lipschitz continuous which was observed in [10, Section 2.6].

For examples of subspaces with $1\frac{1}{2}$-ball property see [13] and [10, Section 2.6].

For uniformly convex spaces with modulus of convexity $\delta(t)$ having power type p, we show that there exists a function g such that $\varepsilon(x,t) \leq g(x,d)t^{\frac{1}{p}},\ d = \operatorname{dist}(x,Y)$, where g is Lipschitz in first variable and locally Lipschitz in the second variable. Since the local component depends only on $\operatorname{dist}(x,Y)$, in particular, on $D(Y)$, P_Y is $Lip_{\frac{1}{p}}$ function.

We next consider strongly proximinal subspaces of finite codimension in $C(K)$. Here also we show that for $f \in C(K) \setminus Y$, there exists a constant C and t_0, both depending on f such that $\varepsilon(f,t) < Ct$ for all $t < t_0$. However, we do not know if in this case P_Y is locally Lipschitz.

For a closed subspace $X \subseteq c_0$ and proximinal subspaces of finite codimension in X, the same technique as in the case of $C(K)$ applies.

2. Continuity properties of $\varepsilon(\cdot,\cdot)$

We start with the following lemma.

Lemma 2.1. *Let $Y \subseteq X$ be a proximinal subspaces. Then for each fixed $x \in X \setminus Y$, $\varepsilon(x,\cdot)$ is a continuous and increasing function of t whenever $t > 0$. Moreover if* $\operatorname{dist}(x,Y) = d$ and $t > s$, *then* $\varepsilon(x,t) - \varepsilon(x,s) \leq (t-s)\frac{2d+t}{t}$.

Proof. That $\varepsilon(x,\cdot)$ is an increasing function of t follows from definition. Let $s > 0$ and $t = s+\eta, \eta > 0$. For any $\varepsilon_1 > \varepsilon(x,s)$, $P_Y(x,s) \subseteq P_Y(x) + \varepsilon_1 B_Y$. We will show $P_Y(x,t) \subseteq P_Y(x,s) + (t-s)\frac{2d+t}{t}B_Y$ and hence $P_Y(x,t) \subseteq P_Y(x) + (\varepsilon_1 + (t-s)\frac{2d+t}{t})B_Y$. It follows that $\varepsilon(x,t) \leq \varepsilon(x,s) + (t-s)\frac{2d+t}{t}$.

Let $y \in P_Y(x,t)$. Since $t = s+\eta$, $\|x-y\| \leq d+s+\eta$. Choose $y_0 \in P_Y(x)$ and consider $\bar{y} = (1-\lambda)y + \lambda y_0$, where $\lambda = \frac{\eta}{s+\eta}$. Then $\|x-\bar{y}\| < (1-\lambda)(d+s+\eta) + \lambda d = d+s+\eta-\lambda s-\lambda\eta = d+s$. Hence $\bar{y} \in P_Y(x,s)$.

Also, $\|y-\bar{y}\| = \lambda\|y-y_0\| = \frac{\eta}{s+\eta}(\|y-x\|+\|x-y_0\|) < \frac{\eta}{s+\eta}(d+s+\eta+d) = \eta\frac{2d+s+\eta}{s+\eta}$. ☐

The following lemma is an easy consequence of the definition of strongly proximinal subspaces.

Lemma 2.2. *A subspace Y in a Banach space X is strongly proximinal if and only if for each $x \in X \setminus Y$, $\varepsilon(x,t) \to 0$ as $t \to 0$.*

The next two propositions determine the continuity of $\varepsilon(\cdot,t)$ from the continuity of P_Y.

Proposition 2.3. *Suppose Y is a proximinal subspace of X and P_Y is lHsc at $x \in X \setminus Y$. Then for every $t > 0$, $\varepsilon(\cdot,t)$ is usc at x.*

Proof. Let $t > 0$ be fixed and $x_n \to x$. We need to show that $\limsup \varepsilon(x_n,t) \leq \varepsilon(x,t)$. Let $d_n = \text{dist}(x_n, Y)$.

Let $\alpha > \varepsilon(x,t)$. Choose $r > 0$ such that $\alpha > \alpha - r > \varepsilon(x,t)$. Hence $P_Y(x,t) \subseteq P_Y(x) + (\alpha - r)B_Y$. For any β, with $t > \beta > 0$ and n large enough we have $P_Y(x_n, t-\beta) \subseteq P_Y(x,t) \subseteq P_Y(x) + (\alpha-r)B_Y$. Using the fact that P_Y is lHsc at x, we have, for large n, $P_Y(x) \subseteq P_Y(x_n) + \frac{r}{2}B_Y$. Hence we have $\alpha - \frac{r}{2} \geq \varepsilon(x_n, t-\beta) \geq \varepsilon(x_n,t) - \beta\frac{2d_n+t}{t}$ - where the last inequality follows from Lemma 2.1. Note that $d_n \to d$ and taking lim sup as $n \to \infty$ we have $\alpha - \frac{r}{2} > \limsup \varepsilon(x_n,t) - \beta\frac{2d+t}{t}$. But β is arbitrary and hence the result follows. ☐

Proposition 2.4. *Suppose Y is a proximinal subspace of X and P_Y is uHsc at $x \in X \setminus Y$. Then for every $t > 0$, $\varepsilon(\cdot,t)$ is lsc at x.*

Proof. Let $t > 0$ be fixed and $x_n \to x$. We need to show that $\liminf \varepsilon(x_n,t) \geq \varepsilon(x,t)$.

Let $\alpha < \varepsilon(x,t)$. Choose $r > 0$ such that $\alpha < \alpha + r < \varepsilon(x,t)$. If along some subsequence, $\varepsilon(x_n,t) \leq \alpha$ then $P_Y(x_n,t) \subseteq P_Y(x_n) + (\alpha + \frac{r}{4})B_Y$. Since P_Y is uHsc at x, for large n, $P_Y(x_n) \subseteq P_Y(x) + \frac{r}{4}$ and hence $P_Y(x_n,t) \subseteq P_Y(x) + (\alpha + \frac{r}{2})B_Y$. Now for any β satisfying $t > \beta > 0$ and large n, $P_Y(x, t-\beta) \subseteq P_Y(x_n,t)$. Thus we have $\alpha + \frac{r}{2} \geq \varepsilon(x, t-\beta) > \varepsilon(x,t) - \beta\frac{2d+t}{t}$. Since the choice of β is arbitrary, this contradicts the choice of α. ☐

Combining Proposition 2.3 and Proposition 2.4 we have the following result.

Theorem 2.5. *Let Y be a proximinal subspace of X. If P_Y is continuous at x then for every $t > 0$, $\varepsilon(\cdot,t)$ is continuous at x.*

We now state our main result of this section.

Theorem 2.6. *Let Y be a strongly proximinal subspace of X. Then P_Y is continuous at x if and only if for every $t > 0$, $\varepsilon(\cdot, t)$ is continuous at x.*

Proof. Only if part follows from Theorem 2.5. To show the if part, let Y be strongly proximinal and $\varepsilon > 0$. By continuity of $\varepsilon(x, \cdot)$ at 0, there exists $t_0 > 0$ such that $\varepsilon(x, t_0) < \frac{\varepsilon}{4}$. Now let $x_n \to x$. Since $\varepsilon(\cdot, t_0)$ is continuous at x, we have $\lim \varepsilon(x_n, t_0) = \varepsilon(x, t_0)$ and hence for n large $\varepsilon(x_n, t_0) \leq \varepsilon(x, t_0) + \frac{\varepsilon}{4}$.

Also for n large,

$$P_Y(x_n) \subseteq P_Y(x, t_0) \subseteq P_Y(x) + (\varepsilon(x, t_0) + \frac{\varepsilon}{4})B_Y$$

and

$$P_Y(x) \subseteq P_Y(x_n, t_0) \subseteq P_Y(x_n) + \varepsilon(x_n, t_0)(1 + \frac{1}{n})B_Y.$$

Thus, $d_h(P_Y(x), P_Y(x_n)) \leq \max\{\varepsilon(x, t_0) + \frac{\varepsilon}{4}, \varepsilon(x_n, t_0)(1 + \frac{1}{n})\} \leq (\varepsilon(x, t_0) + \frac{\varepsilon}{4})(1 + \frac{1}{n}) \leq \varepsilon$.

This shows that P_Y is continuous at x. □

Remark 2.7. Note that for the if part, we only need to use that for every $t > 0$, $\varepsilon(\cdot, t)$ is usc at x. A similar result was obtained in [11, Theorem 3.3].

3. Estimates for $\varepsilon(x, t)$

We first take up the subspaces with $1\frac{1}{2}$-ball property. We need the following known result [6].

Theorem 3.1. *Let Y be a proximinal subspace of X. Then the following statements are equivalent.*

- (a) *Y has the $1\frac{1}{2}$- ball property.*
- (b) *For each $x \in X$ and $y \in Y$ we have $\|x-y\| = dist(x, Y) + dist(y, P_Y(x))$.*
- (c) *For each $x \in X$ we have $\|x\| = dist(x, Y) + dist(0, P_Y(x))$.*

In the following theorem we characterize subspaces with $1\frac{1}{2}$-ball property in terms of the modulus of strong proximinality.

Theorem 3.2. *Let $Y \subseteq X$ be a proximinal subspace. Then Y has $1\frac{1}{2}$-ball property in X if and only if for all $x \in X \setminus Y$, $\varepsilon(x, t) < t$.*

Proof. Suppose Y has the $1\frac{1}{2}$ - ball property. Let $d = dist(x, Y)$ and $y \in P_Y(x, t)$. Then, by Theorem 3.1, $d + dist(y, P_Y(x)) = \|x-y\| < d+t$. Therefore $d(y, P_Y(x)) < t$ and hence $\epsilon(x, t) < t$.

Conversely, suppose $\epsilon(x, t) < t$ for any $t > 0$. If $0 < \beta = \|x\| - d$, then for any $\epsilon > 0$, we have $\|x\| = \|x - 0\| < d + \beta + \epsilon$. By the assumption there exists $z_\epsilon \in P_Y(x)$

such that $\|z_\epsilon\| < \beta + \epsilon$. This implies that

$$\|x\| = d + \beta > d + \|z_\epsilon\| - \epsilon \geq d + d(0, P_Y(x)) - \epsilon.$$

Since ϵ is arbitrary, this implies that $\|x\| = d + d(0, P_Y(x))$. By Theorem 3.1, X has the $1\frac{1}{2}$ - ball property. □

Corollary 3.3. *Suppose $Y \subseteq X$ and Y has $1\frac{1}{2}$-ball property in X. Then P_Y is 2-Lipschitz continuous.*

Proof. Let $x, z \in X \setminus Y$ and $\|x - z\| < t$ for some t. If $y \in P_Y(z)$ then $\|x - y\| \leq \|x - z\| + \|z - y\| < t + \text{dist}(z, Y) \leq t + \text{dist}(x, Y) + t$. Hence $y \in P_Y(x, 2t)$ and by Theorem 3.2, $P_Y(z) \subseteq P_Y(x) + 2tB_Y$. Similarly, $P_Y(x) \subseteq P_Y(z) + 2tB_Y$. Hence $d_h(P_Y(x), P_Y(z)) \leq 2t$. □

Remark 3.4. In [1, Corollary 3.6] it was shown that if $Y \subseteq X$ has $1\frac{1}{2}$-ball property in X then $C(K, Y)$ is strongly proximinal in $C(K, X)$ and the metric projection is 2-Lipschitz continuous.

We now give estimate for $\varepsilon(x, t)$ for uniformly convex spaces.

Proposition 3.5. *Let X be a uniformly convex space with modulus of uniform convexity satisfying $\delta_X(t) \geq Ct^p,\ 2 \leq p < \infty$. Then for any subspace $Y \subseteq X$, there exits a function $g : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for $x \in X \setminus Y$ and t sufficiently small, we have $\varepsilon(x, t) \leq g(x, d)t^{\frac{1}{p}}$. The function g depends on Y only through the distance of x from Y. In particular, there exists a constant K depending only on X such that for any subspace $Y \subseteq X$, and $x \in D(Y)$ one has $\varepsilon(x, t) \leq Kt^{\frac{1}{p}}$ for sufficiently small t.*

Proof. Let $x \in X$ and $Y \subseteq X$ a subspace. Let $\text{dist}(x, Y) = d$ and $y_0 \in P_Y(x)$. There exists a unique $f \in S_{X^*},\ f|_Y = 0$ such that $f(x + y_0) = f(x) = d$. Let $y \in P_Y(x, t)$. Then

$$\left\|\frac{x + y_0}{d} + \frac{x + y}{\|x + y\|}\right\| \geq f\left(\frac{x + y_0}{d} + \frac{x + y}{\|x + y\|}\right) = 1 + \frac{d}{\|x + y\|} > 2 - \frac{t}{d},$$

for sufficiently small t.

Hence, $1 - \frac{1}{2}\left\|\frac{x+y_0}{d} + \frac{x+y}{\|x+y\|}\right\| < \frac{t}{2d}$. By definition of δ_X, it follows $\left\|\frac{x+y_0}{d} - \frac{x+y}{\|x+y\|}\right\| < \delta_X^{-1}(t/2d)$. That is,

$$\frac{x + y}{\|x + y\|} \in \frac{x + y_0}{d} + \delta_X^{-1}(t/2d)B_X,$$

hence,

$$y \in \frac{\|x + y\|}{d} y_0 + \frac{x(\|x + y\| - d)}{d} + \|x + y\|\delta_X^{-1}(t/2d)B_X.$$

Since $\|x + y\| < d + t$ and $\|y_0\| \leq \|x\| + d$, we have

$$\|y - y_0\| \leq t + \frac{2t}{d}\|x\| + (d + t)\delta_X^{-1}(t/2d).$$

Therefore, $\varepsilon(x,t) \leq (1 + \frac{2}{d}\|x\|)t + (d+t)\delta_X^{-1}(t/2d)$.

Now if δ_X is of power type $p \in [2,\infty)$ then for $K' = \frac{1}{C(2d)^{1/p}}$, we have $\delta_X^{-1}(\frac{t}{2d}) \leq K't^{\frac{1}{p}}$. Since for $t < 1$ one has $t < t^{\frac{1}{p}}$ we conclude $\varepsilon(x,t) \leq g(x,d)t^{\frac{1}{p}}$ where $g(x,t) = 1 + \frac{2}{d}\|x\| + K'(d+1)$. On $D(Y)$, g is constant and hence there exists a constant K such that $\varepsilon(x,t) \leq Kt^{\frac{1}{p}}$. □

Remark 3.6. (a) The function g defined in the proof of the previous theorem is Lipschitz in x (in fact depends only through $\|x\|$) and locally Lipschitz in d.

(b) Same calculation as in the case of Corollary 3.3 now shows that if $Y \subseteq X$ where X is a uniformly convex space with modulus of convexity of power type p, then P_Y is a locally $Lip_{\frac{1}{p}}$ function. In particular if $x, z \in D(Y)$, $\|x - z\| < 1$ one has $\|P_Y(x) - P_Y(z)\| \leq (6 + \frac{3}{C})\|x - z\|^{\frac{1}{p}}$.

We now proceed to estimate $\varepsilon(x,t)$ for finite codimensional strongly proximinal subspaces of $C(K)$.

It was noted in [2, Corollary 2.3] that a subspace Y of finite codimension in $C(K)$ is strongly proximinal if and only if $Y^\perp$ is contained in the set of so called 'quasi-polyhedral' (see [2] for definition) points of $C(K)^*$. Such points of $C(K)^*$ are completely described in [2, Theorem 2.1] as the finitely supported measures on K. Also, it is known that (see [5]) if $Y^\perp$ is contained in the 'quasi-polyhedral' points of X^*, then $Y^\perp$ and hence X/Y, are both finite dimensional polyhedral spaces.

Let E be an n-dimensional polyhedral space. If $f \in S_{E^*}$, we define $J_E(f) = \{e \in S_E : f(e) = 1\}$. For $\Phi \in S_E$, consider the following sets:

$$A_\Phi = \{f \in B_{E^*} \ : f(\Phi) = 1\}$$

$$C_\Phi = \{f \in \text{ext}B_{E^*} : f(\Phi) = 1\}.$$

Then C_Φ is a finite set and $\cap_{f \in A_\Phi} J_E(f) = \cap_{f \in C_\Phi} J_E(f)$. Let $\{f_1, f_2, \cdots, f_k\}$, $1 \leq k \leq n$ be a maximal linearly independent subset of C_Φ. Then the set $\cap_{i=1}^k J_E(f_i)$ is a minimal face of B_E containing Φ. It is easy to deduce that if Φ is an extreme point of B_E, then $k = n$ and $\{f_1, f_2, \cdots, f_n\}$ forms a basis of E^*.

We recall the following proposition from [5].

Proposition 3.7. [5, Proposition 2.4] *Let E be an n-dimensional normed linear space and $\Phi \in S_E \setminus \text{ext}(B_E)$. Let F be the minimal face of S_E to which Φ belongs. If $F = \cap_{i=1}^k J_E(f_i)$, $k < n$ for some linearly independent set $\{f_1, f_2, \cdots, f_k\}$ in S_{E^*}, then the set $\{f_1, f_2, \cdots, f_k\}$ can be extended to a basis $\{f_1, f_2, \cdots, f_n\}$ of E^*, $\|f_i\| = 1$, $i = 1, \cdots, n$ such that*

$\inf\{f_i(\Theta):\ f_j(\Theta) = f_j(\Phi) \text{ for } 1 \le j \le i-1\} < f_i(\Phi) <$
$\sup\{f_i(\Theta):\ f_j(\Theta) = f_j(\Phi) \text{ for } 1 \le j \le i-1\} \text{ for } i = k+1, \cdots, n.$

We will also use the following two simple lemmas. The proofs are routine and hence we omit them.

Lemma 3.8. *Let E be an n dimensional normed linear space and $\{e_1^*, e_2^*, \cdots, e_n^*\} \subseteq S_{E^*}$ be a basis for E^*. If $e \in S_E$ satisfies $e_i^*(e) = 1,\ i = 1, \cdots, n$, then e is an extreme point of B_E.*

Lemma 3.9. *Let $\mu \in C(K)^*$ be such that $\mu = \sum_{i=1}^m \alpha_i \delta_{k_i}$ where $k_i \in K$ and $\sum_{i=1}^m |\alpha_i| = 1$.*

(*a*) *Let $f \in S_{C(K)}$. Then $\mu(f) = 1$ if and only if $f(k_i) = sgn\alpha_i,\ i = 1, \cdots, m$.*

(*b*) *Let $\alpha = \min\ \{|\alpha_i| :\ i = 1, \cdots, m\}$ and $0 < t < 1$. If $f \in B_{C(K)}$ is such that $\mu(f) > 1 - \alpha t$ then $|f(k_i) - sgn\alpha_i| < t,\ i = 1, \cdots, m$.*

We now have all the preparation to prove the following result.

Theorem 3.10. *Let $Y \subseteq C(K)$ be a strongly proximinal subspace of finite codimension. Then given $f \in C(K) \setminus Y$, there exist a constant $C > 0$ and t_0 (both depending on f) such that for $t < t_0$ we have $\varepsilon(f, t) < Ct$.*

Proof. Let codim $Y = n,\ f \in C(K) \setminus Y$ and $\text{dist}(f, Y) = d$. Let $f_0 = f/d$. We consider the following two cases:

CASE 1: $f_0|_{Y^\perp}$ is an extreme point of $B_{Y^{\perp *}}$.

By the remark before Proposition 3.7, there exist $\mu_1, \mu_2, \cdots, \mu_n \in S_{Y^\perp}$, a basis for $Y^\perp$ such that $\mu_i(f_0) = 1,\ i = 1, \cdots, n$. Since Y is strongly proximinal in $C(K)$, μ_i's are all finitely supported. We take $\cup_{j=1}^n \text{supp}(\mu_j) = \{k_1, k_2, \cdots, k_l\}$ and $\alpha = \min\{|\mu_j(k_i)| :\ k_i \in \text{supp}\mu_j,\ j = 1, \cdots, n\}$

We choose $\theta_i,\ i = 1, \cdots, l$, a neighborhood of k_i such that $\theta_i \cap \theta_j = \emptyset,\ i \neq j$.

Note that, by Lemma 3.9, if $k_i \in \text{supp}\mu_j$ then $f_0(k_i) = sgn\mu_j(k_i)$.

We choose $t < d$. If $g \in P_Y(f, t)$ then $\|f - g\| < d + t$. We put $h = \frac{f-g}{\|f-g\|}$. For each i, $\mu_i(h) = \frac{d}{\|f-g\|} > 1 - t/d$. Hence by Lemma 3.9 we have $|h(k_i) - sgn\mu_j(k_i)| < \frac{t}{d\alpha}$ whenever $k_i \in \text{supp}\mu_j$.

We define further neighborhoods B_i of k_i as follows.

If $h(k_i) > 0$ take $B_i = \theta_i \cap \{s \in K :\ h(s) > 1 - \frac{t}{d\alpha}\}$.

If $h(k_i) < 0$ take $B_i = \theta_i \cap \{s \in K :\ h(s) < -1 + \frac{t}{d\alpha}\}$.

Define a continuous functions $z' \in B_{C(K)}$ such that

$$z'(k) = \begin{cases} f_0(k) & \text{if } k \in \{k_1, k_2, \cdots, k_l\} \\ h(k) & \text{if } k \in K \setminus \cup_{i=1}^n B_i \end{cases}$$

Then take,

$$z'' = z' \wedge (h + \frac{t}{d\alpha})$$

and

$$z = z'' \vee (h - \frac{t}{d\alpha}).$$

It is straightforward to check that $\|z\| = 1, z(k_i) = f_0(k_i)$ and $\|z - h\| < \frac{t}{d\alpha}$. Note that $\mu_j(z) = 1,\ j = 1, \cdots, n$.

Let us now put $g_1 = f - dz$. Then $g_1 \in P_Y(f)$. Also $\|g - g_1\| = \|f - \|f - g\|h - f + dz\| < d\|z - h\| + |\|f - g\| - d| < \frac{t}{\alpha} + t = (1 + \frac{1}{\alpha})t$.

CASE 2: $f_0|_{Y^\perp}$ is not an extreme point of $B_{Y^{\perp *}}$.

By Proposition 3.7 there exists $k < n$ and $\mu_1, \mu_2, \cdots, \mu_k \in S_{Y^\perp}$ such that $\mu_i(f_0) = 1$ and the set $\{\mu_1, \mu_2, \cdots, \mu_k\}$ can be extended to a basis $\{\mu_1, \mu_2, \cdots, \mu_n\}$ of $Y^\perp$ such that $\|\mu_i\| = 1,\ i = 1, \cdots, n$ and

$$\inf_{z \in C_{i-1}} \mu_i(z) < \mu_i(f_0) < \sup_{z \in C_{i-1}} \mu_i(z)$$

for $i = k + 1, \cdots, n$ where $C_{i-1} = \{z \in S_{C(K)} :\ \mu_j(z) = \mu_j(f_0) \text{ for } 1 \le j \le i - 1\}$. For $k + 1 \le i \le n$ we set

$$\beta_i = \min\{\sup_{z \in C_{i-1}} \mu_i(z) - \mu_i(f_0),\ \mu_i(f_0) - \inf_{z \in C_{i-1}} \mu_i(z)\}.$$

and choose $\beta > 0$ such that $2\beta < \min\{\beta_i : i = k + 1, \cdots, n\}$.

Since the supports of $\mu_1 \cdots, \mu_k$ are all finite, as in CASE 1, let $\mu_i = \sum_{j=1}^{m_i} \alpha_j^i \delta_{k_j^i}$ and $\alpha = \min\{|\mu_j(k_i)| :\ k_i \in \text{supp}\mu_j,\ j = 1, \cdots, k\}$. Choose $t_0 < \frac{\alpha d}{1+\alpha} \frac{\beta^{n-k}}{3^n}$.

Let $t < t_0$ and $g \in P_Y(f, t)$.

STEP 1: Let $Y_1 = \cap_{i=1}^k \ker \mu_i$. We will find $g_1 \in P_{Y_1}(f)$ such that $\|g - g_1\| < (1 + \frac{1}{\alpha})t$.

We first note that $\text{dist}(f, Y_1) = d$ and hence $g \in P_{Y_1}(f, t)$. Since $\mu_i(f_0) = 1,\ i = 1, \cdots, k$, by Lemma 3.8, $f_0|_{Y_1^\perp}$ is an extreme point of $B_{Y_1^{\perp *}}$. Since $t < t_0$, as in CASE 1, there exists $g_1 \in P_{Y_1}(f)$ such that $\|g - g_1\| < (1 + \frac{1}{\alpha})t$.

STEP 2: Let $Y_2 = \cap_{i=1}^{k+1} \ker \mu_i$. Having obtained g_1, we define $h_1 = \frac{f - g_1}{d}$. We will find $g_2 \in P_{Y_2}(f)$ such that $\|g - g_2\| < (1 + \frac{1}{\alpha}) \frac{3t}{\beta}$.

We note that $\|h_1\| = 1$ and $\mu_i(h_1) = \mu_i(f_0) = 1$ for $i = 1, \cdots, k$. Therefore $h_1 \in C_k$. Since $\frac{t}{d} < \beta$ we have

$$|\mu_{k+1}(h_1 - f_0)| = |\mu_{k+1}(g_1/d)| = \frac{1}{d}|\mu_{k+1}(g_1 - g)| < (1 + \frac{1}{\alpha})\frac{t}{d} < \beta < \frac{\beta_{k+1}}{2}.$$

If $\mu_{k+1}(f_0) \geq \mu_{k+1}(h_1)$ we choose $w \in C_k$ such that $\mu_{k+1}(w) > \mu_{k+1}(f_0) \geq \mu_{k+1}(h_1)$ and $|\mu_{k+1}(h_1 - w)| > \beta$. If $\mu_{k+1}(f_0) \leq \mu_{k+1}(h_1)$ we choose $w \in C_k$ such that $\mu_{k+1}(w) < \mu_{k+1}(f_0) \leq \mu_{k+1}(h_1)$ and $|\mu_{k+1}(h_1 - w)| > \beta$. In any case, we can find a $\lambda \in (0,1)$ and $h_2 = \lambda h_1 + (1-\lambda)w$ such that $\mu_{k+1}(h_2) = \mu_{k+1}(f_0)$. This shows that $\mu_i(h_2) = \mu_i(f_0)$ for $i = 1, \cdots, k+1$ and hence $h_2 \in C_{k+1}$.

Define $g_2 = f - dh_2$. Since $\text{dist}(f, Y_2) = d, g_2 \in P_{Y_2}(f)$. Note that

$$\|g - g_2\| \leq \|g - g_1\| + \|g_1 - g_2\| \leq (1 + \frac{1}{\alpha})t + d\|h_1 - h_2\|.$$

Let us now calculate $\|h_1 - h_2\|$. Since

$$(1 + \frac{1}{\alpha})\frac{t}{d} > |\mu_{k+1}(h_1 - f_0)| = |\mu_{k+1}(h_1 - h_2)| = (1-\lambda)|\mu_{k+1}(h_1 - w)|$$

we have $1 - \lambda < (1 + \frac{1}{\alpha})\frac{t}{d|\mu_{k+1}(h_1 - w)|} < (1 + \frac{1}{\alpha})\frac{t}{d\beta}$. Hence

$$\|h_1 - h_2\| = (1-\lambda)\|h_1 - w\| < 2(1-\lambda) < (1 + \frac{1}{\alpha})\frac{2t}{d\beta}.$$

Therefore $\|g - g_2\| < (1 + \frac{1}{\alpha})(t + \frac{2t}{\beta}) < (1 + \frac{1}{\alpha})\frac{3t}{\beta}$. This proves STEP 2.

Note that $h_2 = \frac{f - g_2}{d}$. By proceeding inductively as in step 2 we get $h_0 = h_{n-k+1} \in C_n$. If $g_0 = f - h_0$ then $g_0 \in P_Y(f)$ and $\|g - g_0\| < Ct$ for some suitable constant C. □

Note that in Theorem 3.10, though we get $\varepsilon(f,t) < Ct$, the choice of C depends on f. An affirmative answer to the following question will be more satisfactory.

Question 3.11. *Let Y be a strongly proximinal subspace of finite codimension in $C(K)$ and $f \in C(K) \setminus Y$. Is P_Y locally Lipschitz at f?*

Remark 3.12.
(a) From the proof of Case 1 in Theorem 3.10 it follows that for a strongly proximinal hyperplane Y in $C(K)$, the metric projection is locally Lipschitz on $D(Y)$.
(b) If $X \subseteq c_0$ and $Y \subseteq X$ is a proximinal subspace of finite codimension in X, then from [8] it follows that $Y^\perp$ is contained in the 'quasi-polyhedral' points of X^* and hence Y is strongly proximinal. Such points in X^* in turn extend to 'quasi-polyhedral' points of ℓ_1. But 'quasi-polyhedral' points of ℓ_1 are precisely the elements in ℓ_1 which are finitely supported (see [2, 8]). Thus as in the case of $C(K)$ above, same estimate for $\varepsilon(\cdot,\cdot)$ will hold for Y.

Acknowledgements. *In the earlier version of the paper a different and longer proof of Theorem 3.2 was given. The referee pointed out our attention to the result quoted in Theorem 3.1 and also suggested the present line of argument. We thank the referee for this and also for suggesting other necessary modifications of the earlier version.*

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