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ALMOST CONSTRAINED SUBSPACES OF BANACH SPACES

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ABSTRACT. In this paper, we obtain some sufficient conditions for an almost constrained subspace to be constrained (in fact, by a *unique* norm 1 projection), which improves significantly upon all existing conditions of similar type with significantly simpler proofs.

1. INTRODUCTION

Let X be a *real* Banach space. We will denote by $B_X[x, r]$ the closed ball of radius $r > 0$ around $x \in X$. We will identify any element $x \in X$ with its canonical image in X^{**}. Unless otherwise specified, all subspaces we consider are norm closed. Our notations are otherwise standard. Any unexplained terminology can be found in either [4] or [9].

Recall that a subspace Y of X is called 1-complemented or constrained if there is a norm 1 projection on X with range Y.

Definition 1.1 ([7])**.** A Banach space X is said to have the finite-infinite intersection property ($IP_{f,\infty}$) if every family of closed balls in X with empty intersection contains a finite subfamily with empty intersection.

It is well known that dual spaces and their constrained subspaces have $IP_{f,\infty}$. By w*-compactness of the dual ball and the Principle of Local Reflexivity, it can be shown (see e.g., [7]) that X has the $IP_{f,\infty}$ if and only if any family of closed balls centred at points of X that intersects in X^{**} also intersects in X. With this in mind, we define

Definition 1.2 ([1])**.** A subspace Y of X is said to be an almost constrained (AC) subspace of X if any family of closed balls centred at points of Y that intersects in X also intersects in Y.

Thus, X has the $IP_{f,\infty}$ if and only if X is an AC-subspace of X^{**}. Clearly, any constrained subspace is an AC-subspace. In the case of $IP_{f,\infty}$, whether the converse is also true remains an open question (see [12, Remark 2, page 60], also [6, X(10)]). However, we will give an example to show that an AC-subspace need not, in general, be constrained.

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In addition, we apply some tools and techniques developed in [1] to obtain sufficient conditions for an AC-subspace to be constrained, much in the spirit of [6, 7]. Our condition is in terms of functionals with "locally unique" Hahn-Banach (i.e., norm-preserving) extensions, which improves significantly upon all existing conditions of similar type, as noted in [3, 8], and has significantly simpler proof. As in [6, 7], these conditions actually imply the existence of a *unique* norm 1 projection.

Definition 1.3. Let Y be subspace of X.

(a) For $y^* \in Y^*$, $\mathrm{HB}(y^*) = \{x^* \in X^* : x^*|_Y = y^* \text{ and } \|x^*\| = \|y^*\|\}$.
(b) Y is a U-subspace of X if for any $y^* \in Y^*$, $\mathrm{HB}(y^*)$ is a singleton. X is said to be Hahn-Banach smooth if X is a U-subspace of X^{**}.
(c) The duality mapping D for X is the set-valued map from $S(X)$ to $S(X^*)$ defined by
$$D(x) = \{x^* \in S(X^*) : x^*(x) = 1\}, \quad x \in S(X).$$
(d) $x \in S(X)$ is a smooth point of $B(X)$ if $D(x)$ is a singleton.
(e) Y is a weakly U-subspace of X if for every $y^* \in D(S(Y))$, $\mathrm{HB}(y^*)$ is a singleton.
X is weakly Hahn-Banach smooth if X is a weakly U-subspace of X^{**}.

If Y is a U-subspace, or even a weakly U-subspace of X, then it satisfies our sufficient condition. It is shown in [8, Theorem 2] that an AC-subspace Y is constrained in X if every point of $S(Y)$ is a smooth point of $B(X)$. We show that this happens if and only if every subspace Z of Y is a weakly U-subspace of X. Thus, our condition is weaker.

It follows from our result that X is smooth if and only if every subspace of X is a weakly U-subspace. This parallels the classical result of Taylor-Foguel [15, 5] that X^* is strictly convex if and only if every subspace of X is a U-subspace.

2. Some characterizations and a counterexample

We will use the following notation:

Notation. Let Y be a subspace of X. For all $x \in X$,
$$\mathfrak{P}(x) = \bigcap_{y \in Y} B_Y[y, \|x - y\|].$$
Clearly, $\mathfrak{P}(y) = \{y\}$ for all $y \in Y$. Also, Y is an AC-subspace of X if and only if $\mathfrak{P}(x) \neq \emptyset$ for all $x \in X$.

We recall a definition from [1].

Definition 2.1. Let Y be a subspace of X. We define
$$O(Y, X) = \{x \in X : \|x - y\| \geq \|y\| \text{ for all } y \in Y\}.$$
$O(X, X^{**})$ is denoted by $O(X)$.

The following proposition characterizes AC-subspaces.

Proposition 2.2. *For a subspace Y of X, the following are equivalent:*

(a) *Y is an AC-subspace of X.*
(b) *For all $x \in X$, there exists $y \in Y$ and $z \in O(Y, X)$ such that $x = y + z$.*
(c) *For every subspace Z of X such that $Y \subseteq Z$ and $\dim(Z/Y) = 1$, Y is constrained in Z.*

Proof. (a) $\Rightarrow$ (b). Let $x_0 \in X$. By (a), there exists $y_0 \in \mathfrak{P}(x_0)$. This implies $\|y_0 - y\| \leq \|x_0 - y\|$ for all $y \in Y$. Or, putting $u = y_0 - y$, $\|u\| \leq \|x_0 - y_0 + u\|$ for all $u \in Y$. That is, $z_0 = x_0 - y_0 \in O(Y, X)$ and $x_0 = y_0 + z_0$.

(b) $\Rightarrow$ (c). Let Z be as in (c). Then one can write $Z = \overline{\text{span}}[Y \cup \{x_0\}]$ for some $x_0 \in X$. By (b), there exists $y_0 \in Y$ and $z_0 \in O(Y, X)$ such that $x_0 = y_0 + z_0$. It follows that $Z = Y \oplus \mathbb{R}z_0$. But then, by definition of $O(Y, X)$, $\alpha z_0 + y \mapsto y$ is a norm 1 projection from Z onto Y.

(c) $\Rightarrow$ (a). By (c), for every $x \in X$, there is a norm 1 projection P_x from $Z_x = \overline{\text{span}}[Y \cup \{x\}]$ onto Y. Clearly, $P_x(x) \in \mathfrak{P}(x)$. □

Recall that a hyperplane H in X is a subspace such that $H = \ker(x^*)$ for some $x^* \in S(X^*)$. Since $\dim(X/H) = 1$, we get

Corollary 2.3. *Suppose H is a hyperplane in X. Then H is an AC-subspace if and only if H is constrained in X.*

Corollary 2.4. *A subspace Y is an AC-subspace of X if and only if there is a (not necessarily linear) map P from X onto Y satisfying the following properties:*

(a) $P^2 = P$;
(b) $P(\lambda x) = \lambda P(x)$ *for all* $x \in X$, $\lambda \in \mathbb{R}$;
(c) $P(x + y) = P(x) + y$ *for all* $x \in X$, $y \in Y$;
(d) $\|P(x)\| \leq \|x\|$ *for all* $x \in X$.

Proof. If P is as above, then clearly for any $x \in X$, $P(x) \in \mathfrak{P}(x)$. Thus, Y is an AC-subspace of X.

Conversely, let Y be an AC-subspace of X. For $z \in O(Y, X)$, let $Y_z = Y \oplus \mathbb{R}z$ and P_z be a norm 1 projection from Y_z onto Y. Observe that for $z_1, z_2 \in O(Y, X)$, either $Y_{z_1} \cap Y_{z_2} = Y$ or $Y_{z_1} = Y_{z_2}$. By Proposition 2.2(b), $\bigcup_{z \in O(Y,X)} Y_z = X$. Define $P : X \to Y$ by $P(x) = P_z(x)$, if $x \in Y_z$. Then P is well-defined and satisfies all the listed properties. □

Remark 2.5. Proposition 2.2(a) $\Leftrightarrow$ (c) for the case of $IP_{f,\infty}$ was noted in [12, Theorem 5.9]. Corollary 2.3 was also noted in [1]. Corollary 2.4 for the case of $IP_{f,\infty}$ was noted in [8, Theorem 2]. In all these cases, our proof is simpler.

Let us note that in Proposition 2.2(b), the representation $x = y + z$ with $y \in Y$ and $z \in O(Y, X)$ need not be unique.

Example 2.6. We now give an example to show that an AC-subspace need not, in general, be constrained. We need the following result (we thank Professor T.S.S.R.K. Rao of ISI, Bangalore, for drawing our attention to this result).

Theorem 2.7 ([11])**.** *There exist Banach spaces $Z \supseteq X$ with $dim(Z/X) = 2$ satisfying*

(i) *There is no projection with norm* 1 *from Z onto X.*
(ii) *For every $\varepsilon > 0$, there is a projection with norm $\leq 1 + \varepsilon$ from Z onto X.*
(iii) *For every Y with $Z \supseteq Y \supseteq X$ and $dim(Y/X) = 1$, there is a projection with norm* 1 *from Y onto X.*

By Proposition 2.2, (iii) implies that X is an AC-subspace of Z, while by (i), there is no norm 1 projection from Z onto X.

Definition 2.8. (a) [10] A Banach space X such that X^* is isometrically isomorphic to $L^1(\mu)$ for some positive measure μ is called an L^1-predual.

(b) A Banach space is a $\mathcal{P}_1$-space if it is constrained in every superspace.

Remark 2.9. (a) From the results of [10, Chapter 3], it follows that X is a real L^1-predual with $IP_{f,\infty}$ if and only if X is a real $\mathcal{P}_1$-space. In particular, X is constrained in X^{**}.

(b) It can be shown that the space X in Example 2.6 is not constrained in X^{**}. Therefore, it could have been a possible counterexample to the $IP_{f,\infty}$ question as well. But, from the construction in [11], it is clear that the space X is a real L^1-predual, but not a real $\mathcal{P}_1$-space. Thus it lacks the $IP_{f,\infty}$.

3. Some sufficient conditions

We now obtain sufficient conditions for an AC-subspace to be constrained. Some preliminaries first. As in [1], we introduce the following notation.

Definition 3.1. Let Y be a subspace of X. For $x \in X$ and $y^* \in Y^*$, put

$$\begin{aligned} U(x, y^*) &= \inf\{y^*(y) + \|x - y\| : y \in Y\}, \\ L(x, y^*) &= \sup\{y^*(y) - \|x - y\| : y \in Y\}. \end{aligned}$$

For $x^* \in X^*$, we will write $U(x, x^*)$ for $U(x, x^*|_Y)$. Let $C(x) = \{x^* \in B(X^*) : U(x, x^*) = L(x, x^*)\}$, for $x \in X$, and $C = \bigcap_{x \in X} C(x)$.

The following result is immediate from the proof of the Hahn-Banach Theorem (see, e.g., [14, Section 48]).

Lemma 3.2. *Let Y be a subspace of X, $x_0 \notin Y$ and $y^* \in S(Y^*)$. Then $L(x_0, y^*) \leq U(x_0, y^*)$ and α lies between these two numbers if and only if there exists $x^* \in HB(y^*)$ with $x^*(x_0) = \alpha$.*

Remark 3.3. It is clear that for any $x^* \in B(X^*)$ and $x \in X$, $L(x, x^*) \leq x^*(x) \leq U(x, x^*)$ and for any $y^* \in S(Y^*)$, HB(y^*) is singleton if and only if for all $x \in X$, $L(x, y^*) = U(x, y^*)$.

The next three results are from [1]. We include the proofs for the sake of completeness.

Lemma 3.4. *Let Y be a subspace of X. For $x_1, x_2 \in X$, the following are equivalent:*

(a) $x_2 \in \bigcap_{y \in Y} B_X[y, \|x_1 - y\|]$.

(b) *For all $x^* \in B(X^*)$, $U(x_2, x^*) \leq U(x_1, x^*)$.*

Proof. Clearly, $x_2 \in \bigcap_{y \in Y} B_X[y, \|x_1 - y\|]$ if and only if $\|x_2 - y\| \leq \|x_1 - y\|$, for all $y \in Y$.

(a) $\Rightarrow$ (b). If for all $y \in Y$, $\|x_2 - y\| \leq \|x_1 - y\|$, then for all $x^* \in B(X^*)$, $x^*(y) + \|x_2 - y\| \leq x^*(y) + \|x_1 - y\|$. Therefore, $U(x_2, x^*) \leq U(x_1, x^*)$.

(b) $\Rightarrow$ (a). Suppose $\|x_2 - y_0\| > \|x_1 - y_0\|$ for some $y_0 \in Y$. Then there exists $\varepsilon > 0$ such that $\|x_2 - y_0\| - \varepsilon \geq \|x_1 - y_0\|$. Choose $x^* \in B(X^*)$ such that $\|x_1 - y_0\| \leq \|x_2 - y_0\| - \varepsilon < x^*(x_2 - y_0) - \varepsilon/2$. Thus $U(x_1, x^*) \leq x^*(y_0) + \|x_1 - y_0\| < x^*(x_2) - \varepsilon/2 < U(x_2, x^*)$. $\square$

Proposition 3.5. *Let Y be a subspace of X, $x^* \in B(X^*)$ and $x_0 \in X \setminus Y$. The following are equivalent:*

(a) $x^* \in C(x_0)$.

(b) $\|x^*|_Y\| = 1$ *and every $x_1^* \in HB(x^*|_Y)$ takes the same value at x_0.*

(c) $\|x^*|_Y\| = 1$ *and if* $\{x^*_\alpha\} \subseteq S(X^*)$ *is a net such that* $x^*_\alpha|_Y \to x^*|_Y$ *in the* w^**-topology of* Y^*, *then* $\lim_\alpha x^*_\alpha(x_0) = x^*(x_0)$.
(d) $\|x^*|_Y\| = 1$ *and if* $\{x^*_n\} \subseteq S(X^*)$ *is a sequence such that* $x^*_n|_Y \to x^*|_Y$ *in the* w^**-topology of* Y^*, *then* $\lim x^*_n(x_0) = x^*(x_0)$.

Proof. (a) $\Leftrightarrow$ (b). Let $\|x^*|_Y\| = \alpha$. Then $\alpha \leq \|x^*\| \leq 1$ and it suffices to show that $\alpha = 1$. Let $x^*_1 \in \mathrm{HB}(x^*|_Y)$. Then $\|x^*_1\| = \alpha$ and therefore, for any $y \in Y$, $|x^*_1(x_0 - y)| \leq \alpha\|x_0 - y\| \leq \|x_0 - y\|$. It follows that

$$\begin{aligned} L(x_0, x^*) &\leq \sup\{x^*(y) - \alpha\|x_0 - y\| : y \in Y\} \leq x^*_1(x_0) \\ &\leq \inf\{x^*(y) + \alpha\|x_0 - y\| : y \in Y\} \leq U(x_0, x^*). \end{aligned}$$

Since $x^* \in C(x_0)$, equality holds everywhere.

Now if $\alpha < 1$, let $0 < \delta < d(x_0, Y)$ and let $0 < \varepsilon < (1-\alpha)\delta$. Then for all $y \in Y$, $(1-\alpha)\|x_0 - y\| > \varepsilon$. Therefore, for all $y \in Y$,

$$y^*(y) - \|x_0 - y\| + \varepsilon < y^*(y) - \alpha\|x_0 - y\|.$$

Thus, the first inequality must be strict. Contradiction!

The result now follows from Lemma 3.2.

(b) $\Rightarrow$ (c). Let $\{x^*_\alpha\} \subseteq S(X^*)$ be a net such that $\lim_\alpha x^*_\alpha(y) = x^*(y)$ for all $y \in Y$. It follows that any w*-cluster point of $\{x^*_\alpha\}$ is in $\mathrm{HB}(x^*|_Y)$. By (b), therefore, $\lim x^*_\alpha(x_0) = x^*(x_0)$.

(c) $\Rightarrow$ (d) is clear.

(d) $\Rightarrow$ (b). If $x^*_1 \in \mathrm{HB}(x^*|_Y)$ with $x^*(x_0) \neq x^*_1(x_0)$, then the constant sequence $x^*_n = x^*_1$ clearly satisfies $\lim_n x^*_n(y) = x^*(y)$ for all $y \in Y$, but $\{x^*_n(x_0)\}$ cannot converge to $x^*(x_0)$. □

Proposition 3.6. *Let* Y *be a subspace of* X. *For* $x^* \in B(X^*)$, *the following are equivalent:*

(a) $x^* \in C$.
(b) $\|x^*|_Y\| = 1$ *and* $HB(x^*|_Y) = \{x^*\}$.
(c) $\|x^*|_Y\| = 1$ *and if* $\{x^*_\alpha\} \subseteq S(X^*)$ *is a net such that* $x^*_\alpha|_Y \to x^*|_Y$ *in the* w^**-topology of* Y^*, *then* $x^*_\alpha \to x^*$ *in the* w^**-topology of* X^*.
(d) $\|x^*|_Y\| = 1$ *and if* $\{x^*_n\} \subseteq S(X^*)$ *is such that* $x^*_n|_Y \to x^*|_Y$ *in the* w^**-topology of* Y^*, *then* $x^*_n \to x^*$ *in the* w^**-topology of* X^*.

Here is our first sufficient condition for an AC-subspace to be constrained.

Proposition 3.7. *For a subspace* Y *of* X, *the following are equivalent:*

(a) Y *is an* AC*-subspace of* X *and* $O(Y, X)$ *is a closed subspace of* X.
(c) Y *is an* AC*-subspace of* X *and* $O(Y, X)$ *is a linear subspace of* X.
(c) Y *is constrained in* X *and for all* $x \in X$, $\mathfrak{P}(x)$ *is a singleton.*

Moreover, in this case, Y *is constrained by a* unique *norm* 1 *projection.*

Proof. (a) $\Rightarrow$ (b) is trivial.

(b) $\Rightarrow$ (c). Since Y is an AC-subspace of X, by Proposition 2.2, any $x \in X$ can be written as $x = y + z$, where $y \in Y$ and $z \in O(Y, X)$. Since both Y and $O(Y, X)$ are linear subspaces and $Y \cap O(Y, X) = \{0\}$, this representation is unique and $x \mapsto y$ is a well-defined linear map. Since $z \in O(Y, X)$, this map is of norm 1. Hence Y is constrained in X. Moreover, since $y \in \mathfrak{P}(x)$, $\mathfrak{P}(x)$ is single-valued.

(c) $\Rightarrow$ (a). Let Y be constrained in X by a norm 1 projection P and for all $x \in X$, let $\mathfrak{P}(x)$ be a singleton. Clearly, Y is an AC-subspace of X and for all

$x \in X$, $\mathfrak{P}(x) = \{P(x)\}$. It is easy to see that $\ker(P) \subseteq O(Y, X)$ and since for all $x \in X$, $\mathfrak{P}(x) = \{P(x)\}$, $\ker(P) \supseteq O(Y, X)$. Thus, $O(Y, X) = \ker(P)$ is a closed subspace of X. □

Remark 3.8. (a) Even in the case of $IP_{f,\infty}$, this observation is new. References [6] and [7] discuss more complicated situations when $O(X)$, being a linear subspace of X^{**}, automatically implies that it is a w*-closed subspace of X^{**}.

(b) We do not know if (c) can be replaced by "Y is constrained by a unique norm 1 projection".

(c) It follows from the proof that

$$\bigcup\{\ker(P) : P \text{ is a norm 1 projection onto } Y\} \subseteq O(Y, X).$$

Are these two sets equal?

The following result significantly improves [3, Lemma 2], which was also the key tool in [8].

Lemma 3.9. *Let Y be a subspace of X. Let $x_1, x_2 \in X$ be such that $x_1 \in \bigcap_{y \in Y} B_X[y, \|x_2 - y\|]$. Then for any $x^* \in C(x_2)$, $x^*(x_1 - x_2) = 0$.*

Proof. Let $x_1, x_2 \in X$ be such that $x_1 \in \bigcap_{y \in Y} B_X[y, \|x_2 - y\|]$. Then, by Lemma 3.4, for all $x^* \in B(X^*)$,

$$L(x_2, x^*) \le L(x_1, x^*) \le U(x_1, x^*) \le U(x_2, x^*).$$

Thus for $x^* \in C(x_2)$, equality holds. By Lemma 3.2, the result follows. □

Here is our main theorem.

Theorem 3.10. *Let Y be a subspace of X. Suppose*

$$\text{(1)} \qquad \textit{for every } x_1, x_2 \in X, \quad C(x_1) \cap C(x_2) \textit{ separates points of } Y.$$

If Y is an AC-subspace of X, then Y is constrained in X. Moreover, the projection is unique and $O(Y, X)$ is a closed subspace of X.

Proof. Since Y is an AC-subspace of X, $\mathfrak{P}(x) \neq \emptyset$ for all $x \in X$. By Lemma 3.9, for all $x \in X$,

$$\text{(2)} \qquad x^*(x - y) = 0 \quad \text{for any } x^* \in C(x),\ y \in \mathfrak{P}(x).$$

Now if $y_1, y_2 \in \mathfrak{P}(x)$, then for any $x^* \in C(x)$, $x^*(x - y_1) = x^*(x - y_2) = 0$. Therefore, $x^*(y_1 - y_2) = 0$. By (1), $y_1 = y_2$. That is, $\mathfrak{P}(x)$ is single-valued. Let $\mathfrak{P}(x) = \{P(x)\}$. Then, P satisfies all the properties listed in Corollary 2.4. So, it only remains to show that P is additive.

Let $x_1, x_2 \in X$. If $x^* \in C(x_1) \cap C(x_2)$, then by Proposition 3.5, $x^* \in C(x_1 + x_2)$ and by (2), $x^*(x_1 - P(x_1)) = x^*(x_2 - P(x_2)) = x^*((x_1 + x_2) - P(x_1 + x_2)) = 0$. Therefore, $x^*(P(x_1 + x_2) - P(x_1) - P(x_2)) = 0$. By (1), $P(x_1 + x_2) = P(x_1) + P(x_2)$.

The rest of the result follows from Proposition 3.7. □

By Theorem 3.10, the condition "C separates points of Y" is sufficient for an AC-subspace to be constrained by a unique norm 1 projection. This condition is clearly satisfied if Y is a U-subspace, or even a weakly U subspace of X.

It is shown in [8, Theorem 2] that an AC-subspace Y is constrained in X by a unique norm 1 projection if every point of $S(Y)$ is a smooth point of $B(X)$. By the following result, our condition is much weaker.

Proposition 3.11. *Every point of $S(Y)$ is a smooth point of $B(X)$ if and only if every subspace Z of Y is a weakly U-subspace of X. In particular, X is smooth if and only if every subspace of X is a weakly U-subspace of X.*

Proof. Suppose every point of $S(Y)$ is a smooth point of $B(X)$. Let Z be any subspace of Y. Suppose $z^* \in S(Z^*)$ attains its norm at $z_0 \in S(Z)$. By assumption, z_0 is a smooth point of $B(X)$. Now, $z^* \in D_Z(z_0)$ and $\mathrm{HB}(z^*) \in D_X(z_0)$. Since $D_X(z_0)$ is a singleton, so is $\mathrm{HB}(z^*)$. Thus, Z is a weakly U-subspace of X.

Conversely, suppose there exists $y_0 \in S(Y)$ such that $D_X(y_0)$ is not a singleton. Suppose $\{x_1^*, x_2^*\} \subseteq D_X(y_0)$ and $x_1^* \neq x_2^*$. Let $Z = \{x \in Y : x_1^*(x) = x_2^*(x)\}$. Then $y_0 \in S(Z)$ and therefore, $\|x_1^*|_Z\| = \|x_2^*|_Z\| = 1$. Thus, $z^* = x_1^*|_Z \in S(Z^*)$ attains its norm at $y_0 \in S(Z)$, but $\{x_1^*, x_2^*\} \subseteq \mathrm{HB}(z^*)$. □

Example 3.12. As noted in [6], the space $X = L^\infty$ gives an example of a dual space such that there are infinitely many norm 1 projections from X^{**} onto X. This produces an example of a space with $IP_{f,\infty}$ that is constrained in X^{**}, but $O(X)$ is not a closed subspace of X^{**}. This also shows that our sufficient condition, although weaker than the known ones, is still not necessary for an AC-subspace to be constrained.

We conclude the paper with some necessary and/or sufficient conditions for $O(Y, X)$ to be a closed subspace of X. First we need a characterization of $O(Y, X)$. This is a slight improvement over that in [1].

Definition 3.13. We say $A \subseteq B(X^*)$ is a norming set for X if $\|x\| = \sup\{x^*(x) : x^* \in A\}$ for all $x \in X$.

A subspace F of X^* is called a norming subspace if $B(F)$ is a norming set for X.

Lemma 3.14. *Let Y be a subspace of X. For $x \in X$, the following are equivalent:*

(a) $x \in O(Y, X)$.
(b) $\ker(x)|_Y \subseteq Y^*$ *is a norming subspace for* Y.
(c) $0 \in \bigcap_{y \in Y} B_Y[y, \|x - y\|]$.
(d) *For every* $x^* \in B(X^*)$, $L(x, x^*) \leq 0 \leq U(x, x^*)$.
(e) *For every* $y^* \in B(Y^*)$, $L(x, y^*) \leq 0 \leq U(x, y^*)$.

Further, for a w-closed subspace $F \subseteq X^*$, $F|_Y$ is a norming subspace for Y if and only if $F_\perp \subseteq O(Y, X)$, where $F_\perp = \{x \in X : f(x) = 0$ for all $f \in F\}$.*

Proof. Let $F \subseteq X^*$ be a w*-closed subspace such that $F_\perp \subseteq O(Y, X)$. Then $F = (X/F_\perp)^*$ and therefore, it suffices to show that $\|y\| = \|y + F_\perp\| = d(y, F_\perp)$.

Clearly, $\|y\| \geq d(y, F_\perp)$. Also, since $F_\perp \subseteq O(Y, X)$, for any $y \in Y$ and $z \in F_\perp$, $\|y + z\| \geq \|y\|$. Thus, $d(y, F_\perp) \geq \|y\|$.

Specializing to $F = \ker(x)$, we get (a) $\Rightarrow$ (b).

(b) $\Rightarrow$ (a). Since $\ker(x)|_Y$ norms Y, $\|y\| = \|y|_{\ker(x)}\| = d(y, \mathbb{R}x)$ for all $y \in Y$. Hence $\|x - y\| \geq \inf_{\lambda \in \mathbb{R}} \|y - \lambda x\| = \|y\|$ for all $y \in Y$. Thus, $x \in O(Y, X)$.

Now suppose $F \subseteq X^*$ is a w*-closed subspace such that $F|_Y$ is a norming subspace for Y. If $x \in F_\perp$, then $F \subseteq \ker(x)$ and therefore, $x \in O(Y, X)$. That is, $F_\perp \subseteq O(Y, X)$.

(a) $\Leftrightarrow$ (c) and (d) $\Rightarrow$ (e) are immediate from definition, while (c) $\Rightarrow$ (d) follows from Lemma 3.4.

(e) $\Rightarrow$ (a). For every $y^* \in B(Y^*)$, $0 \le U(x, y^*)$ implies for all $y^* \in B(Y^*)$ and $y \in Y$,

$$0 \le y^*(y) + \|x - y\| \implies y^*(-y) \le \|x - y\|.$$

Since this is true for all $y^* \in B(Y^*)$, $\|y\| \le \|x - y\|$ for all $y \in Y$. That is, $x \in O(Y, X)$. ☐

Let $\mathcal{N} = \{F : F$ is a w*-closed subspace of X^* and $F|_Y$ is a norming subspace for $Y\}$ and $N = \bigcap \mathcal{N}$. Similar to [6], we observe

Proposition 3.15. *Let Y be a subspace of X. $O(Y, X)$ is a closed subspace of X if and only if $N|_Y$ is a norming subspace for Y. In particular, this happens if $C|_Y$ is a norming set for Y.*

Proof. By Lemma 3.14, $F \in \mathcal{N}$ if and only if $F_\perp \subseteq O(Y, X)$. Thus if $N|_Y$ norms Y, then $N \in \mathcal{N}$ and hence, $N_\perp \subseteq O(Y, X)$. On the other hand, if $x \in O(Y, X)$, then $\ker(x) \in \mathcal{N}$, and hence, $N \subseteq \ker(x)$. That is, $x \in N_\perp$. Therefore, $O(Y, X) = N_\perp$, and $O(Y, X)$ is a closed subspace of X.

Conversely, if $O(Y, X)$ is a closed subspace of X and $M = O(Y, X)^\perp$, then $M_\perp = O(Y, X)$ and therefore, $M \in \mathcal{N}$. Moreover, for every $F \in \mathcal{N}$, $F_\perp \subseteq O(Y, X) = M_\perp$, and hence, $M \subseteq F$. This shows $N = M$ and $N \in \mathcal{N}$.

Now, if $C|_Y$ is a norming set for Y, then as above, $C_\perp \subseteq O(Y, X)$.

Conversely let $x \in O(Y, X)$. Let $x^* \in C$. By Lemmas 3.2 and 3.14, there exists $z^* \in \mathrm{HB}(x^*|_Y)$ such that $z^*(x) = 0$. Since $x^* \in C$, $\mathrm{HB}(x^*|_Y) = \{x^*\}$, and we have $x^*(x) = 0$. Thus, $C_\perp = O(Y, X)$. ☐

Definition 3.16. (a) [16] Let Y be a subspace of X. Let

$$A(Y) = \{x^* \in B(X^*) : x^*|_Y \text{ is an extreme point of } B(Y^*)\}.$$

Y is a weakly separating subspace of X if Y separates points of $A(Y)$.

(b) [9] A subspace $Y \subseteq X$ is said to be an M-ideal if there exists a subspace $N \subseteq X^*$ such that $X^* = Y^\perp \oplus_1 N$.

Proposition 3.17. *In each of the following cases, $O(Y, X)$ is a closed subspace of X,* a fortiori*, if Y is an AC-subspace, then Y is constrained by a unique norm 1 projection.*

(a) *Y is a weakly separating subspace of X.*
(b) *Y is an M-ideal in X.*
(c) *Y is a subspace of $X = C(K)$ containing the constants and separating points of K.*

Proof. (a) A careful examination of the proof of [16, Lemma 1] actually shows that $A(Y) \subseteq C$. It is easy to see that $A(Y)$ is a norming set for Y. The result follows from Proposition 3.15.

(b) [9, Theorem I.1.12] observes that an M-ideal is a U-subspace.

(c) As observed in [16], such a Y is weakly separating. ☐

Remark 3.18. (a) In [16], it is shown that for a weakly separating subspace in $C(K)$, if there is a norm 1 projection, it must be unique. Clearly, our conclusion is stronger.

(b) In [13], it is shown that an M-ideal with the $IP_{f,\infty}$ is an M-summand. An argument similar to [2, Proposition 2.8] shows that an M-ideal Y in X with the

$IP_{f,\infty}$ is an AC-subspace of X. Thus, Proposition 3.17(b) improves the result in [13].

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References

[1] P. Bandyopadhayay, S. Basu, S. Dutta and B. L. Lin, *Very nonconstrained subspaces of Banach spaces*, Preprint 2002 (submitted).

[2] Pradipta Bandyopadhyay and T. S. S. R. K. Rao, *Central subspaces of Banach spaces*, J. Approx. Theory, **103** (2000), 206–222. MR **2001b:**46022

[3] B. Beuzamy and B. Maurey, *Points minimaux et ensembles optimaux dans les espaces de Banach*, J. Functional Analysis, **24** (1977), 107–139. MR **55:**1044

[4] J. Diestel and J. J. Uhl, Jr., *Vector measures*, Mathematical Surveys, No. 15, Amer. Math. Soc., Providence, R. I. (1977). MR **56:**12216

[5] S. R. Foguel, *On a Theorem by A. E. Taylor*, Proc. Amer. Math. Soc., **9** (1958), 325. MR **20:**219

[6] G. Godefroy, *Existence and uniqueness of isometric preduals: a survey*, Banach space theory (Iowa City, IA, 1987), 131–193, Contemp. Math., 85, Amer. Math. Soc., Providence, RI, 1989. MR **90b:**46035

[7] G. Godefroy and N. J. Kalton, *The ball topology and its applications*, Banach space theory (Iowa City, IA, 1987), 195–237, Contemp. Math., 85, Amer. Math. Soc., Providence, RI, 1989. MR **90c:**46022

[8] G. Godini, *On minimal points*, Comment. Math. Univ. Carolina, **21** (1980), 407–419. MR **81j:**46023

[9] P. Harmand, D. Werner and W. Werner, *M-ideals in Banach spaces and Banach algebras*, Lecture Notes in Mathematics, 1547, Springer-Verlag, Berlin, 1993. MR **94k:**46022

[10] H. E. Lacey, *Isometric theory of classical Banach spaces*, Die Grundlehren der mathematischen Wissenschaften, Band **208**, Springer-Verlag, New York-Heidelberg, 1974. MR **58:**12308

[11] J. Lindenstrauss, *On projections with norm 1—an example*, Proc. Amer. Math. Soc., **15** (1964), 403–406. MR **28:**4335

[12] J. Lindenstrauss, *Extension of compact operators*, Mem. Amer. Math. Soc., No. **48**, 1964. MR **31:**3828

[13] T. S. S. R. K. Rao, *Intersection properties of balls in tensor products of some Banach spaces, II*, Indian J. Pure Appl. Math., **21** (1990), 275–284. MR **91g:**46020

[14] G. F. Simmons, *Introduction to topology and modern analysis*, McGraw-Hill Book Co., Inc., New York-San Francisco, Calif.-Toronto-London (1963). MR **26:**4145

[15] A. E. Taylor, *The extension of linear functionals*, Duke Math. J., **5** (1939), 538-547. MR **1:**58b

[16] D. E. Wulbert, *Some complemented function spaces in $C(X)$*, Pacific J. Math., **24** (1968), 589–602. MR **36:**6915

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