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STRONGLY PROXIMINAL SUBSPACES OF FINITE CODIMENSION IN $C(K)$

BY

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Abstract. We characterize strongly proximinal subspaces of finite codimension in $C(K)$ spaces. We give two applications of our results. First, we show that the metric projection on a strongly proximinal subspace of finite codimension in $C(K)$ is Hausdorff metric continuous. Second, strong proximinality is a transitive relation for finite-codimensional subspaces of $C(K)$.

1. Introduction. Let X be a Banach space and Y a closed subspace of X. The *metric projection* of X onto Y is the set-valued map defined by $P_Y(x) = \{y \in Y : \|x - y\| = \operatorname{dist}(x, Y)\}$ for $x \in X$. If for every $x \in X$, $P_Y(x) \neq \emptyset$, we say that Y is a *proximinal subspace* of X.

For a Banach space X, we denote the closed unit ball and the unit sphere by B_X and S_X respectively. We restrict ourselves to real scalars. All subspaces we consider are assumed to be closed.

In [7] and [8], G. Godefroy, V. Indumathi and F. Lust-Piquard studied the following stronger version of proximinality.

DEFINITION 1.1. Let Y be a closed subspace in a Banach space X and $x \in X$. For $\delta > 0$, consider the set

$$P_Y(x, \delta) = \{y \in Y : \|x - y\| < d(x, Y) + \delta\}.$$

A proximinal subspace Y is said to be *strongly proximinal* at $x \in X$ if given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$P_Y(x, \delta) \subseteq P_Y(x) + \varepsilon B_Y.$$

Necessary and sufficient conditions for strong proximinality of a finite-codimensional subspace Y in a Banach space X are given in [7]. To describe those results we need the notions of SSD-points and QP-points.

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Definition 1.2.

(a) Let X be a Banach space. The norm $\|\cdot\|$ is said to be *strongly subdifferentiable* (for short SSD) at $x \in X$ if the one-sided limit

$$\lim_{t \to 0^+} \frac{\|x + th\| - \|x\|}{t}$$

exists uniformly for $h \in S_X$.

We say that x is an *SSD-point* of X if the norm is SSD at x. Recall that the *duality map* J_{X^*} of X is defined as

$$J_{X^*}(x) = \{g \in B(X^*) : g(x) = \|x\|\} \quad \text{for } x \in X.$$

In [4], it was shown that x is an SSD-point if and only if the duality map J_{X^*} is (norm-norm) upper semicontinuous at x, that is, for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$J_{X^*}(z) \subseteq J_{X^*}(x) + \varepsilon B_{X^*} \quad \text{if } \|z - x\| < \delta,\ \|z\| = \|x\|.$$

(b) We say that x is a *QP-point* of X if there exists $\delta > 0$ such that

$$J_{X^*}(z) \subseteq J_{X^*}(x) \quad \text{if } \|z - x\| < \delta,\ \|z\| = \|x\|.$$

It was shown in [7, Lemma 3.3] that QP-points are SSD-points but the converse is not true.

The following two propositions describe the connections between strongly proximinal subspaces of finite codimension and QP- and SSD-points.

Proposition 1.3 ([7]). *Let Y be a finite-codimensional subspace of a Banach space X. If Y is strongly proximinal then $Y^\perp$ is contained in the set of SSD-points of X^*.*

It remains an open question if the converse of Proposition 1.3 is true. However, we have the following result. Recall that a finite-dimensional Banach space E is called *polyhedral* if B_E has finitely many extreme points.

Proposition 1.4 ([7]). *Let Y be a finite-codimensional subspace of a Banach space X such that $Y^\perp$ is contained in the set of QP-points of X^*. Then $Y^\perp$ is polyhedral and Y is strongly proximinal.*

In Theorem 2.1 of Section 2 we show that in $C(K)^*$, K a compact Hausdorff space, SSD-points and QP-points coincide and they are precisely the finitely supported measures on K. As a corollary, a finite-codimensional subspace Y of $C(K)$ is strongly proximinal if and only if $Y^\perp$ is contained in the set of SSD-points of $C(K)^*$.

Section 3 contains two applications of our results from Section 2. The first one is a continuity property of the metric projection P_Y, where Y is a strongly proximinal subspace of finite codimension in $C(K)$. We will need the following definitions.

DEFINITION 1.5. Suppose Y is a proximinal subspace of a Banach space X.

(a) P_Y is called *lower semicontinuous* at $x \in X$ if given $\varepsilon > 0$ and $y_0 \in P_Y(x)$, there exists $\delta > 0$ such that for z satisfying $\|z - x\| < \delta$, one can find $y \in P_Y(z)$ such that $\|y - y_0\| < \varepsilon$.
(b) P_Y is called *lower Hausdorff semicontinuous* (henceforth lHsc) at x if it is lower semicontinuous at x and the δ in (a) above can be chosen independent of $y_0 \in P_Y(x)$. Equivalently, P_Y is lHsc at x if for any $x_n \to x$,
$$\sup\{d(y, P_Y(x_n)) : y \in P_Y(x)\} \to 0.$$
(c) P_Y is called *upper Hausdorff semicontinuous* (henceforth uHsc) at x if given $\varepsilon > 0$, there exists $\delta > 0$ such that for every z satisfying $\|z - x\| < \delta$ we have $P_Y(z) \subseteq P_Y(x) + \varepsilon B_Y$.
(d) P_Y is called *Hausdorff metric continuous* at x if it is continuous as a single-valued map from X to 2^Y with respect to the Hausdorff metric defined by
$$h(A, B) = \max\{\sup_{x \in A} \operatorname{dist}(x, B), \sup_{y \in B} \operatorname{dist}(y, A)\}, \qquad A, B \in 2^Y.$$

REMARK 1.6.

(a) It is a simple consequence of the definition that if Y is a strongly proximinal subspace then P_Y is uHsc.
(b) If Y is proximinal in X, then P_Y is Hausdorff metric continuous if and only if P_Y is both lHsc and uHsc.
(c) Sometimes in the literature (see [2, 11]) lower Hausdorff semicontinuity is referred to as strong lower semicontinuity.

In [11] it was shown that if $X \subseteq c_0$ and $Y \subseteq X$ is a strongly proximinal subspace of finite codimension in X, then P_Y is Hausdorff metric continuous. A more general result was obtained in [2], where the authors showed that if X is a Banach space with property $(*)$ (see [2, 3] for the definition) and $Y \subseteq X$ is a proximinal subspace of finite codimension, then P_Y is lHsc. By [2], every separable polyhedral space has a renorming with property $(*)$. In particular, if $1 \leq \alpha < \omega_1$ is a countable ordinal then the space $C(\omega^\alpha)$ is an ℓ_1-predual and hence isomorphically polyhedral space. Thus $C(\omega^\alpha)$ has a renorming with property $(*)$ (see [5]).

As a first application of our results, we show in Section 3 that if Y is a strongly proximinal subspace of finite codimension in $C(K)$, then P_Y is Hausdorff metric continuous.

A second application is to show that the relation of being a strongly proximinal subspace is transitive for finite-codimensional subspaces in $C(K)$. In particular, we show that if Y and M are finite-codimensional subspaces in $C(K)$ such that $Y \subseteq M \subseteq X$, Y is strongly proximinal in M and M is strongly proximinal in $C(K)$, then Y is strongly proximinal in $C(K)$. A similar result for proximinal subspaces of finite codimension in subspaces of c_0 was established in [10]. However, in [1], the authors constructed an example to show that the transitivity of proximinal subspaces of finite codimension in $C(K)$ fails in general.

2. Strongly proximinal subspaces of $C(K)$. The following theorem is our main result in this section.

THEOREM 2.1. *Let $\mu \in C(K)^*$ with $\|\mu\| = 1$. Then the following assertions are equivalent.*

(a) *μ is finitely supported.*
(b) *μ is a QP-point.*
(c) *μ is an SSD-point.*

Proof. (a)$\Rightarrow$(b). We write $\mu = \sum_{i=1}^n \alpha_i \delta_{k_i}$ where $k_i \in K$ and $\sum_{i=1}^n |\alpha_i| = 1$. If $F \in J_{C(K)^{**}}(\mu)$ then $F(\mu) = \sum_{i=1}^n \alpha_i \delta_{k_i}(F) = 1 = \sum_{i=1}^n |\alpha_i|$. It follows that $F \in J_{C(K)^{**}}(\mu)$ if and only if $\delta_{k_i}(F) = \operatorname{sign}(\alpha_i)$, $i = 1, \dots, n$.

Let $\nu \in S_{C(K)^*}$ be such that $\|\mu - \nu\| < \varepsilon$ where $\varepsilon = \min\{|\alpha_i| : 1 \le i \le n\}$. Then $k_1, \dots, k_n$ are atoms of ν and $\operatorname{sign}(\alpha_i) = \operatorname{sign}(\nu(k_i))$.

Now let $G \in J_{C(K)^{**}}(\nu)$. We claim $\delta_{k_i}(G) = \operatorname{sign}(\nu(k_i)) = \operatorname{sign}(\alpha_i)$, $i = 1, \dots, n$. Indeed, assume $\delta_{k_i}(G) \neq \operatorname{sign}(\nu(k_i))$ for some i. We can write $\nu = \sum_{i=1}^n \nu(k_i)\delta_{k_i} + \nu_1$ where $\nu_1 = \nu|_{K \setminus \{k_1, \dots, k_n\}}$. But then $G(\nu) = \sum_{i=1}^n \nu(k_i)\delta_{k_i}(G) + G(\nu_1) < \sum_{i=1}^n |\nu(k_i)|\,|\delta_{k_i}(G)| + |G(\nu_1)| = 1$. This contradicts $G \in J_{C(K)^{**}}(\nu)$. Thus $\delta_{k_i}(G) = \operatorname{sign}(\alpha_i)$, $i = 1, \dots, n$, and hence $G \in J_{C(K)^{**}}(\mu)$. This proves μ is a QP-point.

(b)$\Rightarrow$(c). Follows from [7, Lemma 3.3].

(c)$\Rightarrow$(a). Let $\mu = \mu^+ - \mu^-$ be the Jordan decomposition of μ. Then $|\mu| = \mu^+ + \mu^-$ is a probability measure and $\operatorname{supp}(|\mu|) = \operatorname{supp}(\mu)$. It is now straightforward to verify that $|\mu|$ is an SSD-point if and only if μ is. Thus without loss of generality, we may assume that μ is a probability measure.

We first note that if $f \in J_{C(K)}(\mu)$ then $f = 1$ on $\operatorname{supp}(\mu)$.

Suppose $\operatorname{supp}(\mu)$ is not finite. By regularity of μ we can find a decreasing sequence $\{V_n\}_{n \ge 1}$ of open sets in K such that $\mu(V_n) > 0$ and $\lim_n \mu(V_n) = 0$. Let $\mu_n = \mu_{K \setminus V_n}/\|\mu_{K \setminus V_n}\|$. Then $\|\mu_n\| = 1$ and $\|\mu - \mu_n\| = \mu(V_n)/\|\mu_{K \setminus V_n}\| \to 0$.

Fix $x_n \in V_n \cap \operatorname{supp}(\mu)$. Let $f_n : K \to [0,1]$ be such that $f_n(x_n) = 0$ and $f_n = 1$ on $K \setminus V_n$. Then $f_n \in J_{C(K)}(\mu_n)$ but $\|f_n - f\| = 1$. This contradicts μ being an SSD-point. ∎

REMARK 2.2.

(a) Recall that $x^* \in S_{X^*}$ is called a *point of (norm-weak) upper semicontinuity of the preduality map* of X^* if given any weak neighborhood V of the origin in B_X, there exists $\delta > 0$ such that if $y^* \in S_{X^*}$ satisfies $\|x^* - y^*\| < \delta$, then $J_X(y^*) \subseteq J_X(x^*) + V$. It was proved in [6] that an SSD-point of a dual space attains its norm and is a point of (norm-weak) upper semicontinuity of the preduality map. Now if $\mu \in C(K)^*$ with $\operatorname{supp}(\mu)$ uncountable then we can actually show that μ is not even point of (norm-weak) upper semicontinuity of the preduality map of $C(K)^*$. To see this, first note that μ is not a purely atomic measure. Thus there exists $k \in \operatorname{supp}(\mu)$ such that $\mu(k) = 0$. Let $f_n \in S_{C(K)}$ be such that $f_n(k) = 0$ for all n and $\mu(f_n) \to 1$. Set $V = \{g \in B_{C(K)} : g(k) < 1/2\}$. We have $f_n \notin J_{C(K)}(\mu) + V$ for all n. By [6, Lemma 2.1], the preduality map is not (norm-weak) upper semicontinuous at μ.

(b) If K is countably compact, it is not true in general that if μ is a point of (norm-weak) upper semicontinuity of the preduality map of $C(K)^*$ then μ is a finitely supported measure. For example, consider the space of convergent sequences c and $\mu = (1/2^n)_{n=1}^{\infty} \in \ell_1$. Then $J_c(\mu) = J_{\ell_\infty}(\mu) = (1, 1, \dots)$ and by [6, Theorem 2.3], μ is a point of (norm-weak) upper semicontinuity of the preduality map of ℓ_1.

We can now describe strongly proximinal subspaces of finite codimension in $C(K)$.

COROLLARY 2.3. *Let Y be a finite-codimensional subspace in $C(K)$. Then the following assertions are equivalent.*

(a) *Y is strongly proximinal.*
(b) *Every closed subspace Z of finite codimension with $Y \subseteq Z \subseteq X$ is strongly proximinal.*
(c) *Every hyperplane containing Y is strongly proximinal.*
(d) *$Y^\perp \subseteq \{f \in X^* : f$ is an SSD-point of $X^*\} = \{f \in X^* : f$ is an QP-point of $X^*\}$.*

Proof. (a)$\Rightarrow$(b). Since Y is strongly proximinal, $Y^\perp \subseteq$ SSD-points of X^* by Proposition 1.3. By the equivalence of (b) and (c) in Theorem 2.1, $Y^\perp \subseteq$ QP-points of X^*. Since $Z^\perp \subseteq Y^\perp$ the result follows from Proposition 1.4.

(b)$\Rightarrow$(c). Follows trivially.

(c)$\Rightarrow$(d). Follows from Proposition 1.3 and the equivalence of (b) and (c) in Theorem 2.1.

(d)$\Rightarrow$(a). Follows from Proposition 1.4. ∎

3. Applications. Our first application is the following result on continuity of the metric projection.

THEOREM 3.1. *Let Y be a strongly proximinal subspace of finite codimension in $C(K)$. Then P_Y is Hausdorff metric continuous.*

We need to fix some notation for which we closely follow [11].

Let Y be a proximinal subspace of codimension n in a Banach space X. For $x \in X$, set $Q_Y(x) = x - P_Y(x)$. For $\{f_1, \ldots, f_k\} \subseteq Y^\perp$ with $1 \leq k \leq n$ we define

$$Q_{f_1,\ldots,f_k}(x) = \bigcap_{i=1}^{k} \{z \in B_X : f_i(z) = f_i(x)\}.$$

Note that $Q_Y(x) \subseteq Q_{f_1,\ldots,f_k}(x)$, and if $\{f_1, \ldots, f_n\}$ is a basis of $Y^\perp$ then $Q_Y(x) = Q_{f_1,\ldots,f_n}(x)$.

Let E be an n-dimensional polyhedral space and $\Phi \in S_E$. Consider the sets

$$A_\Phi = \{f \in B_{E^*} : f(\Phi) = 1\}, \qquad C_\Phi = \{f \in \operatorname{ext} B_{E^*} : f(\Phi) = 1\}.$$

Then C_Φ is a finite set and $\bigcap_{f \in A_\Phi} J_E(f) = \bigcap_{f \in C_\Phi} J_E(f)$. Let $\{f_1, \ldots, f_k\}$, $1 \leq k \leq n$, be a maximal linearly independent subset of C_Φ. Then $\bigcap_{i=1}^{k} J_E(f_i)$ is a minimal face of B_E containing x.

Let $D(Y) = \{x \in X : \operatorname{dist}(x, Y) = 1\}$.

DEFINITION 3.2. Suppose $x \in D(Y)$ and there exists a maximal independent set $\{f_1, \ldots, f_k\} \subseteq S_{Y^\perp}$, $1 \leq k \leq n$, such that

$$Q_{f_1,\ldots,f_k}(x) = \bigcap_{i=1}^{k} J_X(f_i).$$

Then we say x is a *k-corner point* with respect to $\{f_1, \ldots, f_k\}$.

We summarize the above discussion in the following lemma. Note that if $Y^\perp$ is polyhedral, then so is X/Y.

LEMMA 3.3. *Let Y be a proximinal subspace of codimension n in X. Suppose $Y^\perp$ is polyhedral and $x \in D(Y)$. Then there exists a maximal independent set $\{f_1, \ldots, f_k\} \subseteq S_{Y^\perp}$, $1 \leq k \leq n$, such that x is a k-corner point with respect to $\{f_1, \ldots, f_k\}$.*

The following result was proved in [11]. Though it is stated for Hausdorff metric continuity of P_Y, it is evident from the proof given there that it is valid for both lHsc and uHsc.

THEOREM 3.4 ([11, Theorem 3.10]). *Let X be a Banach space and Y a proximinal subspace of codimension n in X with $Y^\perp$ polyhedral. Assume that whenever $x \in D(Y)$ is a k-corner point with respect to a set of linearly* *independent functionals* $\{f_1, \ldots, f_k\}$ *in* $Y^\perp$ *for some positive integer* $1 \le k \le n$, *then the map* $Q_{f_1,\ldots,f_k}$ *is Hausdorff metric continuous at* x. *Then the metric projection* P_Y *is Hausdorff metric continuous on* X.

We are now ready to prove Theorem 3.1. We separate out the following simple lemma from the proof of Theorem 2.1.

LEMMA 3.5. *Suppose* $\mu \in S_{C(K)^*}$ *is given by* $\mu = \sum_{i=1}^n \alpha_i \delta_{k_i}$.

(a) *If* $f \in J_{C(K)}(\mu)$ *then* $f(k_i) = \mathrm{sign}(\alpha_i)$, $i = 1, \ldots, n$.
(b) *If* $f_n \in B_{C(K)}$ *are such that* $\mu(f_n) \to 1$ *then* $f_n(k_i) \to \mathrm{sign}(\alpha_i)$, $i = 1, \ldots, n$.

Proof of Theorem 3.1. By Remark 1.6(a), P_Y is uHsc. Thus by Remark 1.6(b) we only need to show that P_Y is lHsc. Also, scaling by the norm of $z \in X$, it is evident that P_Y is lHsc at z if and only if P_Y is lHsc at every $x \in D(Y)$.

Let $x \in D(Y)$. Since Y is strongly proximinal it follows from Corollary 2.3 that $Y^\perp \subseteq$ QP-points of $C(K)^*$. By Proposition 1.4, $Y^\perp$ is polyhedral. Thus Lemma 3.3 implies there exists a maximal linearly independent set $\{\mu_1, \ldots, \mu_m\} \subseteq S_{Y^\perp}$, $1 \le m \le n$, such that x is an m-corner point with respect to $\{\mu_1, \ldots, \mu_m\}$. By Theorem 3.4, it is enough to prove that $Q_{\mu_1,\ldots,\mu_m}$ is lHsc at x.

By Theorem 2.1, for each $j = 1, \ldots, m$, $\mathrm{supp}(\mu_j)$ is a finite set, say $\bigcup_{j=1}^m \mathrm{supp}(\mu_j) = \{k_1, \ldots, k_l\}$. For each $i = 1, \ldots, l$, we choose a neighborhood θ_i of k_i such that $\theta_i \cap \theta_j = \emptyset$, $i \ne j$.

Let $\varepsilon > 0$ be given and $x_n \in D(Y)$ with $x_n \to x$. Suppose $y \in Q_{\mu_1,\ldots,\mu_m}(x)$. We need to produce an n_0 such that for $n \ge n_0$, there exists $v_n \in Q_{\mu_1,\ldots,\mu_m}(x_n)$ such that $\|v_n - y\| < \varepsilon$.

Since $Q_{\mu_1,\ldots,\mu_m}(x) = \bigcap_{i=1}^m J_{C(K)}(x)$, it follows from Lemma 3.5 that if $k_i \in \mathrm{supp}(\mu_j)$ for some $j = 1, \ldots, m$ then $y(k_i) = \mathrm{sign}(\mu_j(k_i))$.

Fix $z_n \in Q_{\mu_1,\ldots,\mu_m}(x_n)$. Since $x_n \to x$, we have $\mu_j(z_n) \to 1$, $j = 1, \ldots, m$. Thus by Lemma 3.5, there exists n_0 such that $|z_n(k_i) - \mathrm{sign}(\mu_j)(k_i)| < \varepsilon/2$ whenever $k_i \in \mathrm{supp}(\mu_j)$.

We define further neighborhoods B_i of k_i as follows:

$$B_i = \begin{cases} \theta_i \cap \{s \in K : y(s) > 1 - \varepsilon\} & \text{if } y(k_i) = 1, \\ \theta_i \cap \{s \in K : y(s) < -1 + \varepsilon\} & \text{if } y(k_i) = -1. \end{cases}$$

Let $v_n' \in B_{C(K)}$ be such that

$$v_n'(k) = \begin{cases} z_n(k) & \text{if } k \in \{k_1, \ldots, k_l\}, \\ y(k) & \text{if } k \in K \setminus \bigcup_{i=1}^n B_i. \end{cases}$$

Then set

$$v_n'' = v_n' \wedge (y + \varepsilon) \quad \text{and} \quad v_n = v_n'' \vee (y - \varepsilon).$$

Then $v_n(k_i) = x_n(k_i)$ and by construction $v_n \in Q_{\mu_1,\ldots,\mu_m}(x_n)$. Also, it is straightforward to verify that $\|v_n - y\| < \varepsilon$ for $n > n_0$. This completes the proof.

Remark 3.6. Recall that a proximinal subspace Y of X is called a *Chebyshev subspace* if $P_Y(x)$ is single-valued for each $x \in X$. Clearly, if Y is a Chebyshev subspace of X then the Hausdorff metric continuity of P_Y amounts to the continuity of P_Y in the usual sense. R. R. Phelps in [13] constructed a Chebyshev subspace Y in $C(K)$ with $\operatorname{codim} Y = 2$ for some extremally disconnected compact Hausdorff space K. However, a result of P. D. Morris (see [12, Theorem 4]) states that if Y is a Chebyshev subspace of finite codimension greater than one in $C(K)$, then P_Y cannot be continuous. Thus the strong proximinaliy condition in Theorem 3.1 cannot be replaced by Y being proximinal (even Chebyshev) in $C(K)$.

Our next application of Theorem 2.1 is to show that strong proximinality is transitive for finite-codimensional subspaces of $C(K)$. We begin by the following lemma which is a simple consequence of the definition of an SSD-point in terms of one-sided differentiability mentioned in the introduction.

Lemma 3.7. *Let Y and Z be two closed subspaces of a Banach space X such that $X = Y \oplus_{\ell_1} Z$ and $x \in X$. Let $x = y + z$ where $y \in Y$, $z \in Z$. Then x is an SSD-point in X if and only if both y and z are SSD-points in the respective subspaces.*

Proposition 3.8. *Let Y be a strongly proximinal subspace of finite codimension in $C(K)$. Suppose $\mu \in C(K)^*$ attains its norm on Y and $\mu|_Y = F$ is an SSD-point of Y^*. Then μ is an SSD-point of $C(K)^*$.*

Proof. Suppose the codimension of Y is n and $\mu_1, \ldots, \mu_n$ $\operatorname{span} Y^\perp$. By Theorem 2.1, $\operatorname{supp}(\mu_i)$ is a finite set for every i, $1 \le i \le n$. So $D = \bigcup_{i=1}^n \operatorname{supp}(\mu_i)$ is finite.

Let $J = \{h \in C(K) : h|_D = 0\}$. Then J is an M-ideal of finite codimension in $C(K)$. Observe that $J \subseteq Y \subseteq C(K)$ and by [9, Corollary I.1.19], J is an M-ideal in Y as well.

By [9, Example I.1.4(a)], there exists a subspace $N \subseteq C(K)^*$ isometric to J^* such that $C(K)^* = J^\perp \oplus_{\ell_1} N$. Similarly, we can write $Y^* = J_1^\perp \oplus_{\ell_1} M$ where $J_1^\perp$ is $J^\perp$ for J considered as a subspace of Y, and $M \subseteq Y^*$ is isometric to J^*.

We write $F = F_1 + F_2$, where $F_1 \in J_1^\perp$, $F_2 \in M$. Since F is an SSD-point of Y^*, by Lemma 3.7, F_1, F_2 are SSD-points in the respective summands. Since SSD-points of a dual Banach space are norm attaining (see [6]), it follows that F_2 attains its norm on J.

Now write $\mu = \mu_1 + \mu_2$ where $\mu_1 \in J^\perp,\ \mu_2 \in N$. The support of μ_1 is contained in D, and thus μ_1 is finitely supported. Hence by Theorem 2.1, μ_1 is an SSD-point of $J^\perp$. It remains to show that μ_2 is an SSD-point in N. Without loss of generality we assume $\|\mu_2\| = 1$. Since $\mu_2|_J = F_2|_J$, $\mu_2|_J$ attains its norm on J.

If μ_2 is not an SSD-point of N, then there exist $\varepsilon > 0$ and $\nu_n \in C(K)^*$ with $\|\nu_n\| = \|\nu_n|_J\| = 1$ and $h_n \in J$ such that $\|\nu_n - \mu_2\| \to 0$, $\nu_n(h_n) = 1$ but $\operatorname{dist}(h_n, \{x \in J : \ \mu_2(x) = 1\}) \geq \varepsilon$ for all n.

But $\mu_2(x) = F_2(x)$ for all $x \in J$ and $\|\nu_n|_Y - F_2\| \to 0$. Thus $\operatorname{dist}(h_n, \{x \in J : F_2(x) = 1\}) \geq \varepsilon$ for all n. This contradicts F_2 being an SSD-point in M. ∎

Corollary 3.9. *Let $Y \subseteq C(K)$ be a subspace of finite codimension and M a subspace of $C(K)$ such that $Y \subseteq M \subseteq C(K)$. If Y is strongly proximinal in M and M is strongly proximinal in $C(K)$, then Y is strongly proximinal in $C(K)$. In other words, strong proximinality is transitive for finite-codimensional subspaces of $C(K)$.*

Proof. Considering Y as a subspace of M, by Proposition 1.3, we know that $Y^\perp$ is contained in the set of SSD-points of M^*. Since M is strongly proximinal, by Proposition 3.8, $Y^\perp$ is contained in the set of SSD-points of $C(K)^*$. The conclusion follows from Corollary 2.3. ∎

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REFERENCES

[1] S. Dutta and D. Narayana, *Strongly proximinal subspaces in Banach spaces*, to appear.

[2] V. Fonf and J. Lindenstrauss, *On the metric projection in a polyhedral space*, preprint.

[3] —, —, *On quotients of polyhedral spaces*, preprint.

[4] C. Franchetti and R. Payá, *Banach spaces with strongly subdifferentiable norm*, Boll. Un. Mat. Ital. B (7) 7 (1993), 45–70.

[5] A. Gleit and R. McGuigan, *A note on polyhedral Banach spaces*, Proc. Amer. Math. Soc. 33 (1972), 398–404.

[6] G. Godefroy and V. Indumathi, *Norm-to-weak upper semi-continuity of the duality and preduality mappings*, Set-Valued Anal. 10 (2002), 317–330.

[7] —, —, *Strong proximinality and polyhedral spaces*, Rev. Mat. Complut. 14 (2001), 105–125.

[8] G. Godefroy, V. Indumathi and F. Lust-Piquard, *Strong subdifferentiability of convex functionals and proximinality*, J. Approx. Theory 116 (2002), 397–415.

[9] P. Harmand, D. Werner and W. Werner, *M-Ideals in Banach Spaces and Banach Algebras*, Lecture Notes in Math. 1547, Springer, Berlin, 1993.

[10] V. Indumathi, *On transitivity of proximinality*, J. Approx. Theory 49 (1987), 130–143.

[11] —, *Metric projections of closed subspaces of c_0 onto subspaces of finite codimension*, Colloq. Math. 99 (2004), 231–252.

[12] P. D. Morris, *Metric projections onto subspaces of finite codimension*, Duke Math. J. 35 (1968), 799–808.

[13] R. R. Phelps, *Chebyshev subspaces of finite codimension in $C(X)$*, Pacific J. Math. 13 (1963), 647–655.

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