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Norm-To-Weak Upper Semi-continuity of the Pre-duality Map

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Abstract. For a Banach space X we study the norm-weak upper semi-continuity of the pre-duality map. We give an example to show that these points in general do not continue to be points of norm-weak upper semi-continuity for the pre-duality map on the third dual. We give several geometric conditions on X that ensures the existence of these points independent of the predual and also that they continue to be points of norm-weak upper semi-continuity in all the duals of odd order. We also consider these questions in the space of operators.

Keywords. Pre-duality map, Unitaries, Semi L-summand, Spaces of operators.

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1. Introduction

Motivated by the recent work of Godefroy and Indumathi [4] on the points of norm-weak upper semi-continuity of the pre-duality map of a Banach space X, in this paper we under take a detailed study of these points. Throughout this paper we only work with unit vectors of X^* that attain their norm (denoted by $NA(X)$). Given a unit vector $x_0^* \in NA(X)$ we denote its pre-state space and state space by $S_{x_0^*} = \{x \in X_1 : x_0^*(x) = 1\}$ and $S^{x_0^*} = \{\tau \in X_1^{**} : \tau(x_0^*) = 1\}$ where for a Banach space X we denote its closed unit ball by X_1. We recall that the pre-duality map $\rho : X^* \to 2^X$ is defined by $\rho(x^*) = S_{x^*}$. Note that this set can be empty.

Definition 1. [[4]] We say the pre-duality map ρ is norm-weak upper semi-continuous ($(n-w)usc$ for short) at $x_0^* \in NA(X)$ if given any weak neighborhood V of 0 in X there exists a $\delta > 0$ such that for all $y^* \in X^*$ with $\|y^* - x_0^*\| < \delta$ we have $\rho(y^*) \subset \rho(x_0^*) + V$.

It follows from Lemma 2.2 of [4] that $x^* \in NA(X)$ is a point of $(n-w)usc$ for ρ if and only if under the canonical embedding the pre-state space is weak*-dense in the state space. For a von Neumann algebra A it is a folk lore that for any unitary of A the pre-state space is weak*-dense in the state space (see Proposition 16). Thus any unitary is a point of $(n-w)usc$ for ρ. This is one of

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our prime motivations for considering these points in general spaces of operators and also the reason for the deviation from the notation of [4] and considering the pre-state space and the state space.

Unlike the case of a von Neumann algebra where the predual is unique up to isometry, the existence of points of $(n-w)usc$ in general depends on the predual X. Also since the bidual of a von Neumann algebra A is again a von Neumann algebra, the unitaries, under the appropriate canonical embedding continue to be points of $(n-w)usc$ for the pre-duality map in any dual of even order of A.

We start our main results with a simple example that shows that this phenomenon is not true in general. Our example also illustrates how the points of $(n-w)usc$ depend on the pre-dual. Thus an interesting question here is to consider geometric conditions on X^* or a local condition on x_0^* to ensure the upper semi-continuity independent of the pre-dual as well as conditions that ensure that these points are preserved in higher duals.

In this paper we give several such geometric conditions. We show that if X is a M-embedded space then any point of $(n-w)usc$ for the pre-duality map on X^* continues to be such a point for the pre-duality map on all the duals of odd order of X. Another condition involves the notion of a semi-L-summand considered by Lima in [6]. We show that any x^* for which span$\{x^*\}$ is a semi-L-summand is a point of $(n-w)usc$ for the pre-duality map.

We show that for $1 < q < \infty$ a norm attaining functional in the ℓ_q-direct sum of a family of dual spaces is a point of $(n-w)usc$ for ρ if and only if it is a point of $(n-w)usc$ in each component space where it is non-zero. We also obtain a similar result in the case of c_0 direct sums.

In the concluding result of this paper we consider this question for the space of bounded linear operators on a dual space. Here we show that for a dual space the identity operator is a point of $(n-w)usc$ for ρ.

Our notation and terminology is standard and can be found in [2] and [3]. For a Banach space X by $\partial_e X_1$ we denote the set of extreme points of the closed unit ball. For $n > 3$, $X^{(n)}$ denotes the dual of n-th order of X. We always consider a Banach space as canonically embedded in its bidual. By Γ we denote the unit circle.

2. Main Results

The following simple example shows that the points of $(n-w)usc$ of the pre-duality map do not in general remain as points of $(n-w)usc$ for the pre-duality map of the higher duals .

Example 1. *Consider $c^* = \ell^1$. Let $f = (\frac{1}{2^n}) \in \ell^1$. Then f attains its norm on c and $S_f = \{e\}$ where $e \in c$ is the vector with 1 in all the coordinates. Similarly $S^f \in \ell_\infty$ consists of e only. Thus in particular f is a smooth point of ℓ^1 and also* *the pre-duality map on ℓ^1 is $(n-w)$usc at f. But considering $f \in \ell^{1^{**}}$ the pre duality map on $\ell^{1^{**}}$ is not $(n-w)$usc at f. For then f will be a very smooth point of ℓ^1 and ℓ^1 being L-embedded space we know (see Proposition 23 in* [10]*) that there is no very smooth point.*

This also shows that points of $(n-w)$usc depend on the predual. For example considering $c_0^* = \ell_1$, we see that f taken as above does not even attain its norm.

We recall from [5] that a closed subspace $M \subset X$ is said to be an M-ideal if there exists a linear projection P on X^* such that $ker(P) = M^\perp$ and $\|x^*\| = \|P(x^*)\| + \|x^* - P(x^*)\|$ for all $x^* \in X^*$. It is well known that the range of P is isometric to M^* and functionals in M^* have unique norm-preserving extension to X. In the following proposition we study stability of points of $(n-w)usc$ for the pre-duality map on subspaces.

Proposition 1. Let $M \subset X$ be an M-ideal. If a unit vector $m^* \in M^*$ is a point of $(n-w)usc$ for the pre-duality map on M^* then its unique norm preserving extension, still denoted by m^*, is a point of $(n-w)usc$ for the pre-duality map on X^*.

Proof. We continue to denote by m^* the unique norm preserving extension in X^*. We note that $X^{**} = M^{\perp\perp} \oplus_\infty (M^*)^\perp$. We also recall that M^{**} is canonically identified with $M^{\perp\perp}$. Thus if $(\alpha, \beta) \in X_1^{**}$ is such that $(\alpha, \beta)(m^*, 0)$ $= \alpha(m^*) = 1$. Then since $\alpha \in S^{m^*}$ we have by hypothesis a net $\{\alpha_t\} \subset S_{m^*}$ converging to it in weak*-topology.

Now for any

$$\begin{aligned}
(g,h) \in X^* &= M^* \oplus_1 M^\perp, \\
\lim\ (\alpha_t, \beta)(g,h) &= \lim\ \alpha_t(g) + \beta(h) \\
&= \alpha(g) + \beta(h) \\
&= (\alpha, \beta)(g,h).
\end{aligned}$$

Also $(\alpha_t, \beta)(m^*, 0) = m^*(\alpha_t) = 1$ for all t. This completes the proof. ∎

We use this proposition to give a geometric condition that ensures that points of $(n-w)usc$ gets preserved in higher duals. We recall from [5] that X is said to be an M-embedded space if it is an M-ideal in its bidual under the canonical embedding. See [5] Chapters III and VI for a large collection of examples of Banach spaces that are M-embedded.

Theorem 2. *Let X be an M-embedded space. Let $x^* \in X^*$ be a unit vector that is a point of $(n-w)usc$ for the pre-duality map. Then it continues to be a point of $(n-w)usc$ for the pre-duality map in all the duals of odd order of X.*

Proof. Rao has proved in [9] that the hypothesis implies that X continues to be an M-ideal under appropriate canonical embeddings in all the duals of even order of X. Thus the conclusion follows from the above proposition. ∎

We next consider other geometric conditions on a Banach space X that allow us to exhibit points that remain $(n-w)usc$ in all the duals of odd order of X. For a compact set K when $C(K)$ is the dual of a space X, the constant function 1 satisfies the hypothesis of the following theorem.

Theorem 3. *Let x_0^* be such that $X_1 = \overline{CO}(\Gamma S_{x_0^*})$. Then the pre-duality map is $(n-w)usc$ at x_0^* . More over under the appropriate canonical embedding x_0^* continues to be a point of $(n-w)usc$ for the pre-duality map on $X^{(2n+1)}$ for $n \geq 0$.*

Proof. Since X_1 is weak*-dense in X_1^{**}, our assumption implies that $X_1^{**} = \overline{CO}\left(\Gamma S_{x_0^*}\right)$ (closure taken in the weak*-topology). We shall show that $\partial_e S^{x_0^*} \subset (S_{x_0^*})^-$ then since $S^{x_0^*}$ is a weak*-compact convex set the conclusion follows.

Let $\Lambda \in \partial_e S^{x_0^*}$. Since $S^{x_0^*}$ is an extreme subset of X_1^{**}, $\Lambda \in \partial_e X_1^{**}$. Thus by Milman's theorem [see [2], page 151] $\Lambda \in (\Gamma S_{x_o^*})^-$, where the closure is taken in the weak*-topology. Thus there exists a net $\{t_\alpha x_\alpha\}_{\{\alpha \in \Delta\}} \subset \Gamma S_{x_0^*}$ such that $t_\alpha x_\alpha \to \Lambda$ in the weak*-topology. Evaluating at x_0^* we get that $t_\alpha \to 1$. Therefore $x_\alpha \to \Lambda$ in the weak *-topology and hence $\Lambda \in (S_{x_0^*})^-$.

We next show that the pre-duality map of X^{***} is $(n-w)usc$ at x_0^*. The proof that this is the case for any dual of odd order of X follows by induction.

By an application of the Milman's theorem again we have $\mid \Lambda(x_0^*) \mid = 1$ for all $\Lambda \in \partial_e X_1^{**}$. Note that the canonical embedding Φ of X^* into $C(S^{x_0^*})$ is an isometry and x_0^* gets mapped to the constant function 1. It is easy to see that $\Phi^*(C(S^{x_0^*})_1^*) = X_1^{**}$. Let $\mu \in C(S^{x_0^*})_1^*$, $\mu(1) = 1$ be a probability measure. Clearly $\Phi^*(\mu)(x_0^*) = 1$. Since $C(S^{x_0^*})_1^*$ is the absolute norm closed convex hull of probability measures, we get that X_1^{**} is the absolute norm closed convex hull of $S^{x_0^*}$. Therefore by repeating the arguments given during the first part of the proof we get that $S^{x_0^*}$ is weak*-dense in $S = \{\tau \in X_1^{(IV)} : \tau(x_0^*) = 1\}$. Thus the pre-duality map of X^{***} is $(n-w)usc$ at x_0^*. ∎

Remark 1. It follows from our remarks in the introduction that $I \in \mathcal{L}(\ell^2)$ is a point of $(n-w)usc$ for ρ. For any orthogonal unit vectors $x, y \in \ell^2$, for the operator $x \otimes y \in \partial_e \mathcal{L}(\ell^2)_1^*$, we have $(x \otimes y)(I) = 0$. So in this set up the hypothesis of Theorem 5 does not hold. Thus we have different behavior for commutative and non-commutative von Neumann algebras.

To see an application of this result we now recall an equivalent definition of a semi-L-summand from [6] (Theorem 5.6). A closed subspace $J \subset X$ is a semi-L-summand if and only if for every $x \in X$ there exists a unique $y \in J$ such that $\|x - y\| = d(x, J)$ and this unique y satisfies $\|x\| = \|y\| + \|x - y\|$. Also $J \subset X$ is said to be a semi-M-ideal if $J^\perp$ is a semi-L-summand in X^*.

These concepts played an important role in the study of geometry of Banach spaces. Any semi-L-summand is a Chebyshev subspace (i.e., every element of

X has unique best approximation in J) and any semi-M-ideal is a proximinal subspace (i.e., every element of X has a best approximation in J) .

Proposition 2. Let $x_0^* \in X_1^*$ be such that span$\{x_0^*\}$ is a semi-L-summand. Then the pre-duality map is $(n-w)usc$ at x_0^* and x_0^* continues to be a point of $(n-w)usc$ under the appropriate canonical embedding, for the pre-duality map in all the duals of X of odd order.

Proof. Let x_0^* be such that span$\{x_0^*\}$ is a semi-L-summand. It is easy to see that $x_0^* \in \partial_e X_1^*$. As ker x_0^* is a semi-M -ideal we have that x_0^* attains its norm. (Note that this is irrespective of the scalar field). It now follows from the arguments given in Theorem 3.5 of [7] that in either scalar field, $X_1 = CO\left(\Gamma S_{x_0^*}\right)^-$. Thus the conclusions follow from the above theorem.

We also note here that by using Theorem 6.14 of [6] repeatedly we see that span$\{x_0^*\}$ is a semi-L-summand under the appropriate canonical embedding, in all the duals of odd order of X. ∎

Remark 2. See [6] for examples of Banach spaces in which the space spanned by any extreme point of the unit ball is a semi-L-summand. It follows from the results of [6] that for any Banach space X whose dual is isometric to a $L^1(\mu)$ has this property. This latter class for example includes spaces of continuous functions on a compact set.
That span$\{x_0\}$ is a semi-L-summand depends on the scalar field. As remarked in [7] in the real scalar field any $f \in \partial_e C(K)_1$ spans a semi-L-summand, where as this is not true in the complex case. Thus by taking the complex function space $X = L^1(\mu)$ and x_0^* as the identically one function, we see that the hypothesis of Theorem 5 is satisfied but span$\{x_0^*\}$ is not a semi-L-summand.

The following proposition shows that points of $(n-w)usc$ are preserved in weak*-closed subspaces.

Proposition 3. Let $Y \subset X$ be a closed subspace and suppose a unit vector $x_0^* \in Y^\perp \subset X^*$ is a point of $(n-w)usc$ for the pre-duality map of X^*. Then it is also a point of $(n-w)usc$ for the pre-duality map of $Y^\perp$.

Proof. Let $\pi : X \to X/Y$ denote the quotient map. Since $\|x_0^*\| = 1 = x_0^*(x) = \|x\|$ for some $x \in X$ clearly $x_0^*(\pi(x)) = 1 = \|\pi(x)\|$. Let S' denote the state space in X/Y and S'' the corresponding object in the bidual. Thus $\pi(S_{x_0^*}) \subset S'$. We note for later use that the sets coincide when Y is a proximinal subspace of X.

Since we are considering canonical embedding of a space in its bidual we have that $\pi^{**} : X^{**} \to X^{**}/Y^{\perp\perp}$ is the natural quotient map. Since $Y^{\perp\perp}$ being weak*-closed is a proximinal subspace, we have, $\pi^{**}(S^{x_0^*}) = S''$. Now since $S_{x_0^*}$ is weak*-dense in $S^{x_0^*}$ by the weak*-continuity of π we get the required conclusion.

∎

We next consider the stability of points of $(n-w)usc$ for ρ in l_q-sums of dual spaces.

Theorem 4. *Suppose (X_n) be a sequence of Banach spaces and let $X = \bigoplus_p X_n$ where $1 < p < \infty$. Suppose $f = (x_1^*, x_2^*, \ldots) \in X^*$ be a norm one functional which attains its norm on X_1. Then f is a point of $(n-w)usc$ of the pre-duality map if and only if for each n, $\frac{x_n^*}{\|x_n^*\|}$ is either 0 or a point of $(n-w)usc$ of the pre-duality map on X_n^*.*

Proof. We have $X^* = \bigoplus_q X_n^*$ where q is the conjugate exponent of p. We first show that a norm one vector $f = (x_1^*, x_2^*, \ldots) \in X^*$ attains its norm at some $g = (x_1, x_2, \ldots) \in X$ with $\|g\|_p = 1$ if and only if $x_n = \|x_n^*\|^{q-1} y_n$ for some $y_n \in S_{\frac{x_n^*}{\|x_n^*\|}}$.

Suppose g is as above. Then $f(g) = \sum_n \|x_n^*\|^{q-1} x_n^*(x_n) = \sum_n \|x_n^*\|^{q-1}\|x_n^*\| = 1$. Conversely, let $g = (x_1, x_2, \ldots) \in X$ with $\|g\|_p = 1$ be such that $f(g) = 1$. Then using Holder's inequality

$$1 = \sum_n x_n^*(x_n) \leq \sum_n |x_n^*(x_n)| \leq \sum_n \|x_n^*\|\|x_n\| \leq \sum_n \|x_n^*\|^{q^{\frac{1}{q}}} \sum_n \|x_n\|^{p^{\frac{1}{p}}} \leq 1.$$

Thus equality holds throughout and hence $x_n^*(x_n) = \|x_n^*\|\|x_n\|$ and $\|x_n\| = \alpha\|x_n^*\|^{q-1}$. By summing up $\|x_n\|^p$ and noting that p and q are conjugate exponent we have $\alpha = 1$. Taking $y_n = \|x_n^*\|^{q-1} x_n$ we have the desired conclusion.

Suppose that the pre-duality map is $(n-w)usc$ at $\frac{x_n^*}{\|x_n^*\|}$ whenever $\|x_n^*\| \neq 0$. Let $\Lambda \in X^{**} = \bigoplus_p X_n^{**}$ be such that $\Lambda(f) = 1$ and $\|\Lambda\| = 1$. By the same argument as above we have $\Lambda = (\|x_n^*\|^{q-1}\tau_n)$ where $\tau_n \in S^{\frac{x_n^*}{\|x_n^*\|}}$. Thus for each n we have a net $y_n^\beta \to \tau_n$ in the w^* topology of X_n^{**} and $y^\beta \in S_{\frac{x_n^*}{\|x_n^*\|}}$. Put $x_n^\beta = \|x_n^*\|^{q-1} y_n^\beta$ and $x_n^\beta = 0$ if $x_n^* = 0$. Then for $g^\beta = (x_n^\beta)$ we have $g^\beta \in S_f$ and $g^\beta \to \Lambda$ in w^* topology of X^{**}. Thus S_f is w^* dense in S^f and the pre-duality map on X^* is $(n-w)usc$ at f.

Conversely, let the pre-duality map on X^* be $(n-w)usc$ at f. Then S_f is w^* dense in S^f. Let $x_n^* \neq 0$. We will show that the pre-duality map on X_n^* is $(n-w)usc$ at $\frac{x_n^*}{\|x_n^*\|}$.

Let $\tau \in S^{\frac{x_n^*}{\|x_n^*\|}}$. Take any $\Lambda \in S^f$. Define $\tilde{\Lambda} \in S^f$ as same as Λ except the nth coordinate replaced by $\|x_n^*\|^{q-1}\tau$. Get $g^\beta \in S_f$ which converges to $\tilde{\Lambda}$ in the w^* topology of X^{**}. For any $z^* \in X_n^*$, evaluating g^β at the functional in X^* which is z^* at the nth coordinate and 0 elsewhere we get a net in $S_{\frac{x_n^*}{\|x_n^*\|}}$ which converges to τ in the w^* topology of X_n^{**}. ∎

We do not know if the above Theorem holds true for the Bochner $\mathcal{L}_p$ spaces. However we can give a description of norm attaining functionals on the dual.

Lemma 5. *Let $(\Omega, \mathcal{A}, \mu)$ be a positive measure space. For a Banach space X and $1 < p < \infty$ consider the space $\mathcal{L}_p(\mu, X)$. Let $f \in \mathcal{L}_q(\mu, X^*)$ be of norm one and attains its norm at some $g \in \mathcal{L}_p(\mu, X)_1$, where q is the conjugate exponent of p. Then for almost all $t \in \Omega$, either $f(t) = 0$ or $\frac{f(t)}{\|f(t)\|}$ attains its norm on X_1. Conversely if X is separable and $f \in \mathcal{L}_q(\mu, X^*)$ be such that for almost all $t \in \Omega$ either $f(t) = 0$ or $\frac{f(t)}{\|f(t)\|}$ attains its norm on X_1, then f attains its norm at some $g \in \mathcal{L}_p(\mu, X)_1$.*

Proof. The proof of the only if first part follows exactly with the same line of argument as in the Proposition 4. We show the other part.

Let $A \subseteq \Omega$ be such that $t \in A \Rightarrow f(t) = 0$ or $\frac{f(t)}{\|f(t)\|}$ attains its norm over X_1. Then A^c is a μ-null set. Let $B = \{t \in A : \ f(t) = 0\}$. For $t \in B^c$ consider the set valued map from Ω to X given by $F(t) = S^{\frac{f(t)}{\|f(t)\|}}$. Then $F(t)$ is non-empty,closed set-valued and we claim that the graph $G(F)$ of F is measurable. To see this note that

$$G(F) = \{(t, x) \in \Omega \times X : \ x \in F(t)\} = \{(t, x) \in \Omega \times X : \ f(t)(x) = \|f(t)\|\}.$$

Since f is measurable, $G(F)$ is measurable. Thus by von Neumann's selection Theorem [see [8]] we have $g_1 : \ B^c \to X$ with g_1 measurable such that $g_1(t) \in F(t)$ for almost all $t \in B^c$. Define $g : \ \Omega \to X$ as follows:

$$\begin{aligned} g(t) &= \|f(t)\|^{q-1} g_1(t) \text{ for } t \in B^c \\ &= 0 \text{ otherwise.} \end{aligned}$$

Since X is separable and

$$\int \|g(t)\|^p d\mu = \int \|f(t)\|^{(p(q-1)} d\mu = \int \|f(t)\|^q d\mu = 1$$

we have $g \in \mathcal{L}_p(\mu, X)$ and $\|g\| = 1$. It is now easy to check that f attains its norm at g. ∎

Question 1. Suppose X^* is separable. Let $f \in \mathcal{L}_q(\mu, X^*) = \mathcal{L}_p(\mu, X)^*$ be such that for almost all $t \in \Omega$ either $f(t) = 0$ or the pre-duality map on X^* is $(n-w)usc$ at $\frac{f(t)}{\|f(t)\|}$. Is the pre-duality map $(n-w)usc$ at f?

We next consider the stability of the points of $(n-w)usc$ in c_0 direct sums of Banach spaces. In the following proposition we only consider countable sums, however it follows from the arguments given during the proof that the same result holds in general.

Proposition 4. Let (X_n) be sequence of Banach spaces. Let $X = \bigoplus_0 X_n$. Let $f = (x_1^*, x_2^*, \ldots) \in X^*$ be a norm one element which attains its norm. Then the pre-duality map of X^* is $(n-w)usc$ at f if and only if $\frac{x_n^*}{\|x_n^*\|}$ is either 0 or a point of $(n-w)usc$ of the pre-duality map on X_n^*.

Proof. Since $X^* = \bigoplus_1 X_n^*$ it is easy to see if $f = (x_n^*)$ is of norm one and attains its norm then only finitely many x_n^* are nonzero. Without loss of generality we assume $f = (x_1^*, x_2^*, \ldots, x_k^*, 0, 0, \ldots)$. Let $(x_n) \in X$ be of norm one and f attains its norm at (x_n). Then,

$$1 = \sum_1^k x_n^*(x_n) \leq \sum_1^k \|x_n^*\| \|(x_n)\| \leq \sup \|x_n\| \sum_1^k \|x_n^*\| \leq \sum_1^k \|x_n^*\|$$

.

Thus f attains its norm at (x_n) if and only if $x_n^*(x_n) = \|x_n^*\|$ for $n = 1, 2, \ldots, k$. Now let $\Lambda \in S^f \subseteq \bigoplus_\infty X_n^{**}$. Again we have $\Lambda = (\tau_n)$ with $\tau_n \in X_n^{**}$ and $\tau_n(x_n^*) = \|x_n^*\|$ for $n = 1, 2, \ldots, k$. Since the pre-duality map on X_n^* is $(n-w)usc$ at $\frac{x_n^*}{\|x_n^*\|}$ we have $y_n^\alpha \in S_{\frac{x_n^*}{\|x_n^*\|}}$ such that $y_n^\alpha \to \tau_n$ in w^* topology of X_n^{**}. Get a net $g^\alpha = (x_n^\alpha) \in X$ such that $g^\alpha \to \Lambda$ in w^* topology of $X^*{*}$. Define

$$\begin{aligned} \tilde{g^\alpha}(n) &= y_n^\alpha \text{ for } n = 1, 2, \ldots, k \\ &= x_n^\alpha \text{ otherwise.} \end{aligned}$$

Then we still have $\tilde{g^\alpha} \to \Lambda$ in w^* and by the observation above $\tilde{g^\alpha} \in S_f$. Thus the pre-duality map on X^* is $(n-w)usc$ at f.

The converse is easy to see. ■

Corollary 6. *Suppose $f \in c_0^* = \ell^1$ attains its norm. Then the pre-duality map on ℓ^1 is $(n-w)usc$ at f.*

Remark 3. In the Corollary 6 one can also check that $f \in \ell^1$ continues to be a point of $(n-w)usc$ of the pre-duality map on $\ell^{1^{**}}$. Thus Example 2 also illustrates that it is important which pre-dual of ℓ^1 we consider.

For a Banach space X we recall that the space of bounded linear operators $\mathcal{L}(X^*)$ is a dual space. For example the projective tensor product of $X \otimes_\pi X^*$ is an isometric predual of this space (see [3], page 230). Our last result shows that $I \in \mathcal{L}(X^*)$ a point of $(n-w)usc$ for the pre-duality map w.r.t this predual.

Proposition 5. The pre-duality map on $\mathcal{L}(X^*)$ is $(n-w)usc$ at I.

Proof. Note that

$$S_I \supset \{x \otimes x^* : \ x^*(x) = 1 = \|x\| = \|x^*\|\}.$$

We shall show that for any $T \in \mathcal{L}(X^*)$, *sup* $|T|$ is the same over either state space. A simple separation argument then shows that $\overline{CO}\{\Gamma S_I\} = \overline{CO}\{\Gamma S^I\}$ (closure w.r.t weak*-topology). It is then easy to see that S_I is weak*-dense in S^I and hence I is a point of $(n-w)usc$ for the pre-duality map.

Let $T \in \mathcal{L}(X^*)$. It follows from Theorem 4 on page 84 in [1] that

$$sup \ \{|(x \otimes x^*)(T) : x \otimes x^* \in S_I\} = sup \ \{|\tau(T)| : \tau \in S^I\}.$$

Thus the conclusion follows. ∎

In the following example we show that $I \in \mathcal{L}(\ell^1)$ is a point of $(n-w)usc$ for ρ independent of the predual. It is easy to see that this space can be identified with ℓ^∞-direct sum of countably many copies of ℓ^1. Note that the predual of this space is not uniquely determined. We consider the real scalar field and obtain an answer which is independent of the predual.

Example 7. *Let $I \in \mathcal{L}(\ell^1)$. Then span$\{I\}$ is a semi-L-summand and hence a point of $(n-w)usc$ for ρ. To see this note that $\mathcal{L}(\ell^1)^*_1 = \overline{CO}\{u \otimes \delta(t) : u \in \ell^\infty, |u| = 1$ and $t \in \beta(\boldsymbol{N})\}$. Since $|(u \otimes \delta(t))(I)| = 1$ an application of Milman's theorem again shows that $|\tau(I)| = 1$ for any $\tau \in \partial_e \mathcal{L}(\ell^1)^*_1$. Thus it again follows from the results in* [7] *that span$\{I\}$ is a semi-L-summand. Similar arguments show that for any $T \in \partial_e \mathcal{L}(\ell^1)_1$, span$\{T\}$ is a semi-L-summand.*

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