ALGEBRAIC REFLEXIVITY OF SOME SUBSETS OF THE ISOMETRY GROUP

S. DUTTA AND T. S. S. R. K. RAO

Abstract. Let X be a compact first countable space. In this paper we show that the set of isometries of $C(X)$ that are involutions is algebraically reflexive. As a consequence of a recent work of Botelho and Jamison this leads to the conclusion that the set of generalized bi-circular projections on $C(X)$ is also algebraically reflexive. We also consider these questions for the space $C(X, E)$ where E is a uniformly convex Banach space.

1. Introduction

Let E be a complex Banach space. Let $\mathcal{G}(E)$ denote the group of isometries of E. For a non-empty, bounded set $L \subset \mathcal{L}(E)$ the algebraic closure $\overline{L}^a$ of L is defined as follows: $\Phi \in \overline{L}^a$ if for every $e \in E$ there exists $\Phi_e \in L$ such that $\Phi(e) = \Phi_e(e)$. L is said to be algebraically reflexive if $L = \overline{L}^a$. Algebraic reflexivity of the isometry group of function spaces and spaces of operators has received a lot of attention recently. See the Lecture Notes by Molnar [10] for a very comprehensive account of this theory.

In [10, Theorem 3.2.1] it was shown that if X is first countable compact Hausdorff space the $\mathcal{G}(C(X))$ is algebraically reflexive. In this paper we are interested in studying the algebraic reflexivity of some special subsets of $\mathcal{G}(C(X))$. This study is also related to a recently started or revived study of properties of projections on E that have some special geometric properties. Let $\mathbb{T}$ denote the unit circle. A linear projection $P : E \to E$ is said to be a generalized bi-circular projection (GBP for short) if for some $\lambda \in \mathbb{T}$, $\Phi = P + \lambda(I - P)$ is an isometry (see [5]). Clearly if P is a GBP then so is $I - P$. Recently in [8] it was shown that if a projection P on a complex Banach space X is such that $P + \lambda(I - P)$ is an isometry and λ is of infinite order in $\mathbb{T}$, then P is hermitian. Such GBP's are called trivial in [8]. Thus according to this terminology a GBP is non-trivial only if λ is a nth root of unity for some n. It now follows that (see [8, Corollary 2] that if P is a GBP then P is actually bi-contractive, that is, $\|P\| \leq 1$ and $\|I - P\| \leq 1$.

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It has been shown by Benau, Lacey [3] and Lima [7] that on certain function spaces for any bi-contractive projection P, $\Phi = 2P - I$ is an isometry. In this case the isometry is an involution i.e, $\Phi^2 = I$. These include the space $C(X)$ and the space of affine continuous functions on a Choquet simplex, $A(K)$ (see Section 2 for the definition). While it is known that in general bi-contractive projections need not be hermitian, in above cases they are and coincide with GBP's. In Section 2 we exhibit some more examples of spaces where these two notions coincide.

Since for a large class of Banach spaces one has precise description of surjective isometries, Botelho and Jamison [4] have initiated the study of structure of GBP based on the corresponding description of isometries. We thank Professor Botelho and Jamison for providing us with a copy of their work and for some clarifications.

Throughout this paper we assume that the isometry group under consideration is algebraically reflexive. Motivated by the above consideration in the first section of the paper we consider the algebraic reflexivity of the set of isometries that are involutions. It follows from our result that if X is a metrizable compact set then for the space $C(X)$, this set is algebraically reflexive. We extend (with a correct formulation) a result of Botelho and Jamison [4] that characterizes GBP's on the space of vector-valued continuous functions $C(X, E)$, to the space of affine continuous functions $A(K, E)$, where K is a compact Choquet simplex. If the set of extreme points $X = \partial_e K$ is a closed set, then the space $A(K, E)$ coincides with the continuous function space $C(X, E)$. As a corollary of our result we get that the set of bi-contractive projections on $C(X)$, for a metrizable X is algebraically reflexive.

2. ALGEBRAIC REFLEXIVITY OF INVOLUTIONS

Let $\mathcal{G}^2(C(X))$ denote the set of all involuting isometries of $C(X)$. In what follows we use the classical Banach-Stone theorem that describes the isometry group of $C(X)$.

Theorem 1. *Let X be compact Hausdorff space. If $\mathcal{G}(C(X))$ is algebraically reflexive then $\mathcal{G}^2(C(X))$ is also algebraically reflexive.*

Proof. We first recall from the Banach-Stone theorem that $U \in \mathcal{G}^2(C(X))$ if and only if there exists a homeomorphism ϕ of X such that $\phi^2(x) = x$ for all x , an unimodular function $\tau : X \to \mathbb{T}$ satisfying $\tau(x)\tau \circ \phi(x) = 1$ for all $x \in X$ and U is given by $Uf = \tau f \circ \phi$ for all $f \in C(X)$.

Let $T \in \overline{\mathcal{G}^2(C(X))}^a$. Thus for each f there exists a homeomorphism ϕ_f of X, $\phi_f^2(x) = x$ for all $x \in X$, and $\tau_f : X \to \mathbb{T}$ satisfying $\tau_f(x)\tau_f(\phi_f(x)) = 1$ such that $Tf = \tau_f f \circ \phi_f$. Since $\mathcal{G}(C(X))$ is algebraically reflexive it follows that T itself is an isometry. Thus there exists a homeomorphism ϕ of X and $\tau : X \to \mathbb{T}$ such that $Tf = \tau f \circ \phi$.

To show T is an involution we need to show that $\phi^2(x) = x$ for all x and $\tau(x)\tau(\phi(x)) = 1$.

First note that for any strictly positive function $f \in C(X)$, since $f(\phi(x)) = |\tau(x)f(\phi(x))| = |\tau_f(x)f(\phi_f(x))| = |f(\phi_f(x))| = f(\phi_f(x))$, we have $\tau = \tau_f$.

Now consider any point $x \in X$. We show that $\phi^2(x) = x$. If $x = \phi(x)$ then this indeed holds true. So let for some x, $x \neq \phi(x)$. Then for each neighborhood U of x and each neighborhood V of $\phi(x)$, disjoint from U, we choose $f_{UV} \in C(X)$ such that $1 \leq f_{UV}(x) \leq 2$ and $f_{UV}(x) = 1$ $f_{UV}(\phi(x)) = 2$, $f^{-1}\{1\} \subseteq U$, $f^{-1}\{2\} \subseteq V$. We take $y_{UV} = \phi_{f_{UV}}(x)$ and $x_{UV} = \phi(y_{UV})$. Since f is strictly positive we have $f(\phi_{f_{UV}}) = f(\phi)$ and hence $f_{UV}(y_{UV}) = f_{UV}(\phi_{f_{UV}}(x)) = f_{UV}(\phi(x)) = 2$. This implies $y_{UV} \in V$. Similarly $f_{UV}(x_{UV}) = f_{UV}(\phi(y_{UV})) = f_{UV}(\phi(\phi_{f_{UV}})(x)) = f_{UV}(\phi_{f_{UV}}(x)) = f(x) = 1$. Hence $x_{UV} \in U$. Since $\phi(x) = \lim_{UV} y_{UV}$ and $x = \lim_{UV} x_{UV}$, by continuity of ϕ it now follows that $\phi^2(x) = \phi(\lim_{UV} y_{UV}) = \lim_{UV} \phi(y_{UV}) = \lim_{UV} x_{UV} = x$. This proves $\phi^2(x) = x$.

It remains to show that $\tau(x)\tau(\phi(x)) = 1$. Note that for the constant function $\mathbf{1}$, $T\mathbf{1}(\phi(x)) = \tau(\phi(x))$. On the other hand $T\mathbf{1}(\phi(x)) = T_{\mathbf{1}}\mathbf{1}(\phi(x)) = \tau_{\mathbf{1}}(\phi_{\mathbf{1}}(\phi(x)))$. Hence $\tau(\phi(x)) = \tau_{\mathbf{1}}(\phi_{\mathbf{1}}(\phi(x)))$. Now we have $T^2\mathbf{1} = \tau(x)\tau(\phi(x)) = \tau_1(x)\tau_{\mathbf{1}}(\phi_{\mathbf{1}}(\phi(x))) = \tau_1(\phi^2(x))\tau_{\mathbf{1}}(\phi_{\mathbf{1}}(\phi(x))) = 1$. Thus $\tau(x)\tau(\phi(x)) = 1$ and the proof is complete. □

Combining Theorem 1 with the Molnar's result mentioned in the introduction we immediately have the following corollary.

Corollary 2. *Let X be a first countable compact space. Then $\mathcal{G}^2(C(X))$ is algebraically reflexive.*

Remark 3. It was shown in [9] that if X is not first countable, $C(X)$ need not be algebraically reflexive. Thus for such a space the proof of Theorem 1 does not work. However in [11] it was shown that for the space $\ell_\infty = C(\beta\mathbb{N})$, $\mathcal{G}(\ell_\infty)$ is algebraically reflexive. Hence form Theorem 1 it follows that $\mathcal{G}^2(\ell_\infty)$ is also algebraically reflexive though $\beta\mathbb{N}$ is not first countable.

We recall from [1] that a compact convex set K is said to be a Choquet simplex, if for every $k \in K$, there exists a unique regular Borel probability measure μ on K, that is maximal in the Choquet ordering, such that $a(k) = \int a \, d\mu$ for all $a \in A(K)$.

Before applying this result to the set of bi-contractive projections, we prove a theorem that extends the main result of [4] to the space $A(K, E)$.

We first recall the vector-valued Banach-Stone theorem for the space $A(K, E)$ (see [6]) . We note that analogous to the case of $C(X)$, isometries are determined by evaluation at the set of extreme points $\partial_e K$.

Let K be a Choquet simplex and let E^* be a strictly convex space. Let $\Phi : A(K,E) \to A(K,E)$ be a surjective isometry. Then there exists a weight function $\tau : \partial K \to \mathcal{G}(E)$ and an affine homeomorphism ϕ of K such that $\Phi(a)(k) = \tau(k)(a(\phi(k))$ for all $k \in \partial_e K$ and $a \in A(K,E)$.

In the proof of the following theorem we adapt the arguments given during the proof of [4, Theorem 2.1] to the simplex case.

Theorem 4. *Let K be a compact Choquet simplex and E be such that E^* is strictly convex. Let $P : A(K,E) \to A(K,E)$ be a GBP. Then the affine homeomorphism ϕ associated with Φ is an involution on $\partial_e K$. If further $\phi \neq I$ then $\lambda = -1$, $P(a)(k) = \frac{1}{2}\{\tau(k)(a(\phi(k)) + a(k)\}$ and the weight function τ satisfies $\tau(k) \circ \tau(\phi(k)) = I$ on $\partial_e K$. If $\phi = I$ then $P_k(e) = \frac{\tau(k)(e) - \lambda e}{(1-\lambda)}$, is a GBP in E for each $k \in \partial_e K$. Also, $P(a)(k) = P_k(a(k))$ for $k \in \partial_e K$ and $a \in A(K,E)$.*

Proof. Let $\Phi = P + \lambda(I - P)$. As Φ is a surjective isometry, from the result quoted above, we have an affine homeomorphism ϕ and a weight function $\tau : \partial_e K \to \mathcal{G}(E)$ such that $\Phi(a)(k) = \tau(k)(a(\phi(k)))$. Since P is a projection we further have $\lambda a(k) - (\lambda+1)\tau(k)((a(\phi(k))) + \tau(k)(\tau(a(\phi(\phi(k)))) = 0$, for all $a \in A(K,E)$ and $k \in \partial_e K$. Suppose $k \in \partial_e K$ is such that $\phi^2(k) \neq k$ and $\phi(k) \neq k$. As $\phi(k), \phi^2(k) \in \partial_e K$ and K is a simplex, convex hull of $\{\phi(k), \phi^2(k)\}$ is a face. Therefore by [1, Corllary II.5.20] there exist a $a_0 \in A(K)$ such that $a_0(k) = 1$ and $a_0(\phi(k)) = 0 = a_0(\phi^2(k))$. Evaluating the above formula at this k and a_0 we get that $\lambda = 0$, a contradiction. Thus $\phi(k) = k$ or $\phi^2(k) = k$. In either case $\phi^2(k) = k$ on $\partial_e K$. If $\phi \neq I$, then for some $k_o \in \partial_e K$, using a separation arguments once again in the above equation, we get that $\lambda = -1$ and $\tau(k) \circ \tau(\phi(k)) = I$ for $k \in \partial_e K$. Thus $P(a)(k) = \frac{1}{2}\{\tau(k)(a(\phi(k)) + a(k)\}$.

If $\phi = I$, then as in the proof of [4, Theorem 2.1], using [4, Lemma 2.1] we get that $P_k(e) = \frac{\tau(k)(e) - \lambda e}{(1-\lambda)}$, is a GBP in E for each $k \in \partial_e K$. Hence the conclusion follows. □

Remark 5. *A remarkable feature of the above description is the additional requirement on the function τ, $\tau(k)\tau(\phi(k)) = k$ on $\partial_e K$. Since by the Banach-Stone theorem, $\tau : \partial_e K \to \mathcal{G}(E)$ is continuous when the range space is equipped with the strong operator topology, $\tau' : \partial_e K \to \mathcal{G}(E)$ defined by $\tau'(k) = \tau(k)^{-1} = \tau(\phi(k))$ is also continuous. While considering the algebraic reflexivity question for GBP's, we have been able to overcome the difficulty induced by this requirement on τ only in the scalar-valued case.*

Corollary 6. *Let X be a compact Hausdorff space such that $\mathcal{G}(C(X))$ is algebraically reflexive. Then the set of GBP's on $C(X)$ is algebraically reflexive.*

Proof. Let the set of all GBP's on $C(X)$ be denoted by $\mathcal{P}$. Let $T \in \overline{\mathcal{P}}^a$. Then for each $f \in C(X)$, there exists $P_f \in \mathcal{P}$ such that $Tf = P_f f$. Thus by [4], for each f there exists a homeomorphism ϕ_f of X, $\phi_f^2(x) = x$ for all $x \in X$ and $\tau_f : X \to \mathbb{T}$ satisfying $\tau_f(x)\tau_f(\phi_f(x)) = 1$ such that $Tf = \frac{1}{2}(f + \tau_f f(\phi_f))$.

Thus for each f, $(2T - I)f = \tau_f f \circ \phi_f$ is given by an involuting isometry. The conclusion follows from Theorem 1. □

We recall [7] that a closed subspace $J \subseteq E$ is called a semi L-summand if for every $e \in E$ there exists a unique $j \in J$ such that $\|e - j\| = \mathrm{dist(e, J)}$ and moreover j satisfies $\|e\| = \|j\| + \|e - j\|$. In [7, Theorem 4.4] it was proved that if E is such that for every $e^* \in \partial_e E_1^*$ $\mathrm{span\{e^*\}}$ is a semi L-summand then $P : E \to E$ is a bi contractive projections if and only if $U = 2P - I$ is an isometry and in this case $U^2 = I$. It is known that the spaces $C(X)$ and for a Choquet simplex K, $A(K)$, satisfy the hypothesis of [7, Theorem 4.4].

Let E_1^* denote the dual unit ball and $\partial_e E_1^*$, its set of extreme points.

Lemma 7. *Let E be such that for every $e^* \in \partial_e E_1^*$ $\mathrm{span\{e^*\}}$ is a semi L-summand. Then the same is true for $C(X, E)$.*

Proof. Let $\tau \in \partial_e C(X, E)_1^*$. Then $\tau = \delta(x) \otimes e^*$ for some $x \in X$ and $e^* \in \partial_e E_1^*$. By hypotheses $\mathrm{span\{e^*\}}$ is a semi L-summand. Since $C(X, E)^* = (\mathrm{span\{\delta(x)\} \otimes E^*}) \oplus_1 \mathrm{N}$, for a closed subspace N, it follows that $\mathrm{span}\{\tau\}$ is a semi L-summand in $C(X, E)^*$. □

Suppose the isometries of $C(X, E)$ are described by generalized Banach-Stone Theorem, that is, if $\Phi \in \mathcal{G}(C(X, E))$ then there exists a map $\tau_\Phi : X \to \mathcal{G}(E)$ continuous in s.o.t. and a homeomorphism ϕ_Φ of X such that $\Phi f(x) = \tau_\Phi(x) f(\phi_\Phi(x))$, for all $f \in C(X, E)$. Let $\mathcal{G}(C(X, E))$ be algebraically reflexive. Take $\mathcal{I}$ the subset of $\mathcal{G}(C(X, E))$ which are described by involuting homeomorphism of X, that is,

$$\mathcal{I} = \{\Phi \in \mathcal{G}(C(X, E)) : \ \phi_\Phi^2(x) = x, \ \forall x \in X\}.$$

Now if $\Phi \in \overline{\mathcal{I}}^a$ then by algebraic reflexivity of $\mathcal{G}(C(X, E))$, Φ is an isometry and hence $\Phi f(x) = \tau(x) f(\phi(x))$ where $\tau(x) \in \mathcal{G}(E)$ for each x and ϕ is a homeomorphism of X. Now if $\phi(x) \neq x$ for some x, choose $h \in C(X)$ such that $0 \leq h(x) \leq 1, \ x \in X$ and $h^{-1}(1) = \{x\}, \ h^{-1}(0) = \{\phi(x)\}$. Fix a norm one vector $e \in E$ and take $f = h \otimes e$. Evaluating $\tau f \circ \phi = \tau_f f \circ \phi_f$ at x and $\phi(x)$ one can show as in the proof of Theorem 1 that $\phi^2(x) = x$.

The description of isometries of $C(X, E)$ given by the generalized Banach-Stone Theorem, holds when E has trivial centralizer. This is the case, in particular, when E *or* E^* is strictly convex. It was shown in [6] that if E is uniformly convex and $\mathcal{G}(E)$ is algebraically reflexive then $\mathcal{G}(C(X, E))$ is algebraically reflexive. Combining this with our preceding discussion we obtain the following.

Proposition 8. *Let E be a uniformly convex Banach space such that $\mathcal{G}(E)$ is algebraically reflexive. Let*

$$\mathcal{I} = \{\Phi \in \mathcal{G}(C(X,E)) : \ \phi_\Phi^2(x) = x, \ \forall x \in X\}.$$

Then $\mathcal{I}$ is algebraically reflexive.

Using Proposition 8 we can describe the algebraic closure of GBP's in $C(X,E)$ for some class of spaces E. Note that for finite dimensional spaces E, GBP's were studied in [5]. We also recall [6, Theorem 6] that for a finite dimensional space E with trivial centralizer and X metrizable isometry group of $C(X,E)$ is algebraically reflexive.

We omit the proof of the following Proposition.

Proposition 9. *Let E be a uniformly convex Banach space such that $\mathcal{G}(E)$ is algebraically reflexive. Assume further that E does not have non-trivial GBP. Let $\mathcal{P}$ denote the GBP's in $C(X,E)$. If $T \in \overline{\mathcal{P}}^a$ then there exists a $\tau : X \to \mathcal{G}(E)$ continuous in s.o.t. and a homeomorphism ϕ of X satisfying $\phi^2(x) = x$ for all $x \in X$ such that for all $f \in C(X,E)$, $Tf(x) = \tau(x)f(\phi(x)$.*

Remark 10. *Even under some very generous assumptions on E, when the isometry group is non-trivial , we have not been able to establish the algebraic reflexivity of GBP's for $C(X,E)$. An independent question of interest here is to examine the algebraic reflexivity of the set of isometries which are given by a weight function τ such that $\tau' : X \to \mathcal{G}(E)$ defined by $\tau'(x) = \tau(x)^{-1}$ is continuous.*

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(S. Dutta) Department of Mathematics and Statistics, Indian Institute of Technology, Kanpur, India, *E-mail : sudipta@iitk.ac.in*

(T. S. S. R. K. Rao) Stat–Math Unit, Indian Statistical Institute, R. V. College P.O., Bangalore 560059, India, *E-mail : tss@isibang.ac.in*