ALMOST CONSTRAINED SUBSPACES OF BANACH SPACES - II

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Abstract. A subspace Y of a Banach space X is an almost constrained (AC) subspace of X if any family of closed balls centred at points of Y that intersects in X also intersects in Y. In this paper, we show that a subspace H of finite codimension in the space $C(K)$ of continuous functions on a compact Hausdorff space K is an AC-subspace if and only if H is the range of a norm one projection in $C(K)$. We also give a simple proof that the implication "$AC \Rightarrow$ 1-complemented" holds for any subspace of the spaces $c_0(\Gamma)$ and c.

1. Introduction

Let X be a Banach space over real or complex scalars. A closed subspace Y of X is called 1-complemented or constrained if it is the range of a norm 1 projection on X.

Definition 1.1. [1, 2] A subspace Y of X is an almost constrained (AC) subspace of X if any family of closed balls centred at points of Y that intersects in X also intersects in Y.

Clearly, any 1-complemented subspace is an AC-subspace. In this paper, we continue our study [2] of the converse. As observed in [2, Example 2.6], the converse is not true in general, even for finite codimensional subspaces. In [2], working with real scalars, we obtained sufficient conditions for the converse to hold. But it remains an open question for X in its bidual X^{**} (see [7]).

In two recent preprints [10, 11], using different terminology, it has been shown that the converse holds for any subspace of the *real* sequence spaces

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c_0, c, ℓ_1, the Lorentz sequence space $d(\omega, 1)$ and some subspaces of Musielak-Orlicz sequence spaces equipped with the Luxembourg norm.

Let $C(K)$ denote the Banach space of all scalar-valued continuous functions on a compact Hausdorff space K with the supremum norm. And let $C_0(S)$ denote the Banach space of all scalar-valued continuous functions "vanishing at infinity" on a locally compact Hausdorff space S with the supremum norm. In this paper, we show in particular that, irrespective of the scalar field, an AC-subspace of finite codimension in $C(K)$ (or $C_0(S)$) is 1-complemented. Our proof also leads to an explicit description of such a subspace in terms of the measures defining it. In particular we show

Theorem 1.2. *Let H be a subspace of codimension n of $C(K)$. The following are equivalent :*

- (a) *H is an AC-subspace.*
- (b) *H is 1-complemented in $C(K)$.*
- (c) *There exist measures $\mu_1, \mu_2, \ldots, \mu_n$ and distinct isolated points $\{k_1, k_2, \ldots, k_n\}$ of K such that*
  - (i) $H = \bigcap_{i=1}^{n} \ker \mu_i$.
  - (ii) $2|\mu_i(\{k_i\})| \geq \|\mu_i\|, \ i = 1, 2, \cdots, n$.

In [12, 13], 1-complemented subspaces of real $C(K)$ spaces have been characterized as being isometric to some $C(S)$ space. In [5, Theorem 1 and Proposition 1.11], for real or complex scalars, the general form of norm 1 projections onto subspaces of $C_0(S)$ is obtained in terms of a *simultaneous extension operator* E and some restriction operator Q, where S is locally compact Hausdorff space. Thus, our result is in a different direction, more in the line of [4]. Moreover, these results do not help in proving $(a) \Rightarrow (b)$ above. It would be interesting to see if one can characterize general AC-subspaces of $C_0(S)$ in the framework of [5].

Our technique also yields a simple proof that the converse holds for any subspace of the spaces $c_0(\Gamma)$ and c.

As in [2], an important tool in our study is the ortho-complement of a subspace Y in X.

Definition 1.3. (a) [9] Let X be a Banach space and $x, y \in X$. We say y is orthogonal to x (written $y \perp x$) in the sense of Birkhoff, if $\|y\| \leq \|\alpha x + y\|$ for every scalar α.

(b) [1] Let Y be a subspace of X. The ortho-complement of Y in X is defined as

$$O(Y, X) = \{x \in X : y \perp x \text{ for all } y \in Y\}$$

or, equivalently $O(Y, X) = \{x \in X : \|x + y\| \geq \|y\| \text{ for all } y \in Y\}$.

As noted in [2], Y is an AC-subspace of X if and only if $X = Y + O(X, Y)$. On the other end of the spectrum are what we called very nonconstrained (VN) subspaces in [1], where other equivalent formulations can be found.

Definition 1.4. [1] Y is said to be a very non-constrained (VN-) subspace of X if $O(Y, X) = \{0\}$.

Thus a proper subspace cannot be simultaneously VN- as well as AC-subspace.

The results of this paper hold for both real and complex scalars. For this purpose, we first show that the results from [1, 2] that we need here are scalar independent. In particular, in Section 2, we begin by characterizing $O(Y, X)$. We give a necessary condition for a subspace H of $C(K)$ to be a VN-subspace. If H is weakly separating in $C(K)$ (see Definition 2.4), this condition is also sufficient. However, it is not sufficient in general.

In Section 3, we prove our main result, Theorem 1.2. As a corollary, we have that if K has at most n isolated points, $n = 0, 1, \ldots$, then there is no AC-subspace of codimension $n + 1$ in $C(K)$.

For a Banach space X, we will denote by B_X and S_X respectively the closed unit ball and the unit sphere of X. All subspaces we consider are norm closed. For a closed bounded convex set C, $\operatorname{ext} C$ denotes the set of extreme points of C. For $y^* \in Y^*$, the set of all Hahn-Banach (*i.e.*, norm-preserving) extension of y^* to X is denoted by $\mathrm{HB}_X(y^*)$. We will omit the subscript when the space is understood. We will denote by T the set of scalars of modulus 1, i.e., $T = \{-1, 1\}$ in the real case and $T =$ the unit circle in the complex case. Our notations are otherwise standard and can be found in [8].

2. VN-subspace of $C(K)$

We begin by characterizing elements of $O(Y, X)$. This is a variant of [1, Lemma 2.10] and [2, Lemma 3.14] with a simpler proof that works for both real and complex scalars.

Definition 2.1. (*a*) A set $B \subseteq S_{X^*}$ is a norming set for X if for every $x \in X$, $\sup_{x^* \in B} |x^*(x)| = \|x\|$.

(*b*) A set $B \subseteq S_{X^*}$ is a boundary for X if for every $x \in X$, there exists $x^* \in B$ such that $|x^*(x)| = \|x\|$.

Lemma 2.2. *Let Y be a subspace of a Banach space X. For $x \in X$, the following are equivalent :*

(*a*) $x \in O(Y, X)$

(*b*) *For every $y^* \in S_{Y^*}$, there exists $x^* \in HB(y^*)$ such that $x^*(x) = 0$.*

(*c*) *$S_{\ker x|_Y}$ is a boundary for Y.*

(*d*) *$S_{\ker x|_Y}$ is a norming set for Y.*

Proof. (*a*) $\Rightarrow$ (*b*). Given $y^* \in S_{Y^*}$, define z^* on $Z = \text{span}(Y \cup \{x\})$ as

$$z^*(y + \alpha x) = y^*(y), \quad y \in Y, \alpha \text{ scalar}$$

Clearly, $z^*|_Y = y^*$ and $z^*(x) = 0$. Moreover, since $x \in O(Y, X)$, $\|z^*\| = 1$. Thus, any $x^* \in \text{HB}(z^*)$ works.

(*b*) $\Rightarrow$ (*c*) $\Rightarrow$ (*d*) is clear.

(*d*) $\Rightarrow$ (*a*). Suppose $S_{\ker x|_Y}$ is a norming set for Y. Then for any $y \in Y$,

$$\|y\| = \sup_{x^* \in S_{\ker x}} |x^*(y)| = \sup_{x^* \in S_{\ker x}} |x^*(x + y)| \leq \|x + y\|.$$

Thus $x \in O(Y, X)$. ☐

The following lemma is again adapted from [2, Proposition 3.15]. Let Y be a subspace of a Banach space X. Define,

$$C = \{x^* \in S_{X^*} : \text{HB}_X(x^*|_Y) = \{x^*\}\}.$$

Lemma 2.3. *If $C|_Y$ is a norming set for Y, then $O(Y, X)$ is a closed subspace of X. Hence if Y is in addition assumed to be an AC-subspace of X, then Y is complemented by a unique norm one projection in X.*

Proof. We claim $C_\perp := \{x \in X : x^*(x) = 0 \text{ for all } x^* \in C\} = O(Y, X)$.

To see this, let $x \in C_\perp$. Then $\ker x \supseteq C$ and hence, $S_{\ker x|_Y}$ is a norming set for Y. By Lemma 2.2, it follows that $x \in O(Y, X)$. Conversely, if $x \in O(Y, X)$, by Lemma 2.2, it follows that $x^*(x) = 0$ for every $x^* \in C$. Thus $x \in C_\perp$.

The rest follows from [2, Proposition 2.2 and 3.7] and these results are easily seen to hold for both real and complex scalars. ☐

For a subspace $H \subseteq C(K)$ which separates points in K, the Choquet boundary of H is defined in [14] as

$$\partial H = \{k \in K : \phi k \in \text{ext} B_{H^*}\},$$

where for $k \in K$, $\phi k \in H^*$ is the evaluation functional. This definition coincides with the classical definition of the Choquet boundary when H also contains the constants. In this paper, we will use the same notation even when H does not necessarily separate points of K.

Definition 2.4. [16] A subspace Y of X is said to be weakly separating if Y separates points of the set

$$D(Y) = \{x^* \in B_{X^*} : x^*|_Y \in \text{ext} B_{Y^*}\}.$$

As noted in [16], if $H \subseteq C(K)$ separates points of K and contains the constants, or, if H is a closed ideal in $C(K)$, then H is weakly separating.

We now obtain a necessary condition for a subspace H of $C(K)$ to be a VN-subspace.

Proposition 2.5. *Let H be a subspace of $C(K)$. If H is VN-subspace of $C(K)$ then $\overline{\partial H} = K$. Moreover, if H is weakly separating, the converse is also true.*

Proof. Suppose, $\overline{\partial H} \neq K$. We can get an nonzero $f \in C(K)$ such that $f|_{\overline{\partial H}} = 0$. Since ∂H is a boundary for H, it follows from Lemma 2.2 that $f \in O(H, C(K))$.

For the converse, suppose H is weakly separating.

Claim. If $k \in \partial H$, $\text{HB}_{C(K)}(\phi k) = \{\delta_k\}$.

Indeed, since $\phi k \in \text{ext} B_{H^*}$, $\text{HB}(\phi k)$ is a face of $B_{C(K)^*}$ containing δ_k. So if $\text{HB}(\phi k)$ is not a singleton, it contains extreme points of $B_{C(K)^*}$ other than δ_k. But any such point is of the form $\alpha\delta_{k'}$ for some $k' \in K$ and $\alpha \in T$. Thus, $\delta_k|_H = \alpha\delta_{k'}|_H$. This contradicts that H is weakly separating.

Now let $f \in O(H, C(K))$. By Lemma 2.2 and the above claim, we have $f(k) = 0$ for any $k \in \partial H$. Thus if $\overline{\partial H} = K$, then $f \equiv 0$ and hence, H is a VN-subspace of $C(K)$. □

Remark 2.6. The proof of the above claim essentially shows that if Y is a weakly separating subspace of X, then $D(Y) \subseteq C$. This is also implicit in the proof of [16, Lemma 1]. Thus if Y is also an AC-subspace, then by Lemma 2.3, it is complemented by a *unique* norm 1 projection.

As a corollary, we can characterize M-ideals in $C(K)$ which are VN-subspaces. Recall that any M-ideal in $C(K)$ is of the form $M = \{f \in C(K) : f|_D = 0\}$ for some closed set $D \subseteq K$ (see [8, Example 1.4 (a)]) and that such subspaces are weakly separating.

Corollary 2.7. *Let $D \subseteq K$ be a closed set. Let $M = \{f \in C(K) : f|_D = 0\}$. Then M is a VN-subspace of $C(K)$ if and only if $K \setminus D$ is dense in K.*

We now give an example to show that in general the above condition does not ensure that H is a VN-subspace of $C(K)$.

Example 2.8. Let X be any Banach space. Let $K = \overline{\text{ext}B_{X^*}}^{w^*}$. Then X naturally embeds as a point separating subspace of $C(K)$. Clearly we have $\partial X = \text{ext}B_{X^*}$ and $\mathbf{1} \notin X$ where $\mathbf{1}$ is the constant function 1 in $C(K)$. Now for $x \in X$, get $x^* \in \text{ext}B_{X^*}$ such that $x^*(x) = -\|x\|$. Then $\|\mathbf{1} - x\|_\infty \geq |(\mathbf{1} - x)(x^*)| = 1 + \|x\| > \|x\|$. Thus $\mathbf{1} \in O(X, C(K))$ and X is not an VN-subspace of $C(K)$.

3. Proof of the Theorem 1.2

Let H be a finite codimensional subspace of $C(K)$. We will need the following result on the size of the set $K \setminus \partial H$. If H separates points, this follows directly from [6, Lemma 5.6, Theorem 7.3], and in the general case, we indicate how to modify the proof of [6].

Proposition 3.1. *Let H be a subspace of codimension n in $C(K)$. Then the set $K \setminus \partial H$ contains at most n points.*

Proof. (Sketch): We adapt the argument in [6]. First we need a little modification of the proof of [6, Lemma 7.2].

Consider the map $p : T \times K \to T\phi(K)$ given by $p(\alpha, k) = \alpha\phi k$. We claim p^{-1} admits a Borel measurable selection $s : T\phi(K) \to T \times K$, i.e., for each $L \in T\phi(K)$, if $s(L) = (\alpha, k)$ then $L = \alpha\phi k$ on H.

To see this, we first define a Borel measurable map $s_1 : T\phi(K) \to T$. Let $L \in T\phi(K)$. In the *real* case, just define $s_1(L) = 1$ if $L = \phi k$ and $s_1(L) = -1$ if $L = -\phi k$. Then s_1 is continuous.

In the *complex* case, define,

$$\theta(L) = \inf\{\theta \in [0, 2\pi) : e^{-i\theta} L \in \phi(K)\}$$

Then θ is lower semicontinuous, and hence, the map $s_1(L) = e^{i\theta(L)}$ is Borel.

We now define a measurable selection π for ϕ^{-1} as follows: First define an equivalence relation on K by letting $k \approx k'$ if $h(k) = h(k')$ for all $h \in H$, or, equivalently, $\phi k = \phi k'$. Since $\operatorname{codim}(H) = n$, all equivalence classes are finite and only finitely many are not singletons. For each $k \in K$, choose and fix one element from the equivalence class of k and call it $\pi(k)$.

Then the final map s defined in [6, Lemma 7.2], namely,

$$s(L) = (s_1(L), \pi(s_1(L)^{-1} L))$$

has the desired properties.

Now following [6, Theorem 7.3], for each $L \in S_{H^*}$, we can get a regular Borel measure ν on K as follows: By Choquet's Theorem [15], there exists a maximal probability measure λ on B_{H^*} whose resultant is L. Since λ is maximal, its support is contained in $T\phi(K)$. Let μ be the probability measure on $T \times K$ induced by s, *i.e.*, $\mu(f) = \lambda(f \circ s)$ for $f \in C(T \times K)$. Now take $\nu = \mathcal{H}\mu$, where $\mathcal{H}$ is the Hustad map of μ defined by

$$(\mathcal{H}\mu)(g) = \int_{T\times K} \alpha g(k) d\mu(\alpha, k), \quad g \in C(K).$$

As in the proof of [6, Theorem 7.3], it is easily verified that ν satisfies,

(i) $\nu = L$ on H.
(ii) $\|\nu\| = \|L\| = 1$.
(iii) ν is a boundary measure.

To conclude the proof, if there are $(n+1)$ distinct points $k_1, k_2, \ldots, k_{n+1} \in K \setminus \partial H$, by the argument above, there exist boundary measures $\nu_1, \nu_2, \ldots, \nu_{n+1}$ such that the measures $\mu_i = \delta_{k_i} - \nu_i \in H^\perp$. Since ν_i's are boundary measures, $\mu_i(k_j) = \delta_{ij}$ and hence $\mu_1, \mu_2, \ldots, \mu_{n+1}$ are linearly independent. This contradicts that the dimension of $H^\perp$ is n. □

Now we prove our main theorem.

Proof of Theorem 1.2. $(a) \Rightarrow (b)$. Suppose H is of codimension n and is an AC-subspace of $C(K)$.

Modifying the definition of $\approx$ used above, let $k \sim k'$ if there exists $\alpha \in T$ such that $h(k) = \alpha h(k')$ for all $h \in H$. Again, since this implies $\delta_k - \alpha\delta_{k'} \in H^\perp$ and $\operatorname{codim}(H) = n$, all equivalence classes are finite and only finitely many are not singletons.

Let $K_0 = \{k \in K : \mathrm{HB}_{C(K)}(\phi k) = \{\delta_k\}\}$.

Claim 1. The set $K \setminus K_0$ is finite.

By Proposition 3.1, $(K \setminus K_0) \cap (K \setminus \partial H)$ contains at most n points.

Let $k \in \partial H \setminus K_0$. Then, $\mathrm{HB}(\phi k)$ is not a singleton and as in the proof of Proposition 2.5, there exists $k' \in K$ such that $k \neq k'$ and $k \sim k'$. Thus k belongs to an equivalence class that is not singleton. By the observation above, $\partial H \setminus K_0$ is finite. This proves the claim.

Now let $I = \{k \in K : f(k) \neq 0 \text{ for some } f \in O(H, C(K))\}$.

If $f \in O(H, C(K))$ and $k \in K_0$, then since $\mathrm{HB}(\phi k) = \{\delta_k\}$, by Lemma 2.2, $f(k) = 0$. Thus $I \subseteq K \setminus K_0$.

Therefore, by the claim above, I is finite and since there are nonzero $f \in O(H, C(K))$, each point of I is an isolated point of K.

Claim 2. If K_1 is a non-singleton equivalence class, then $K_1 \setminus I$ is at most singleton.

Let $k_1, k_2 \in K_1$. Let $f \in C(K)$ be such that $f(k_i) = i$. Since H is an AC-subspace, there is $h \in H$ such that $f - h \in O(H, C(K))$. By definition of $\sim$, $|h|$ is constant on K_1. Thus, $f - h$ cannot be zero at both k_1 and k_2. That is, at least one of them must be in I. Since this is true for any pair of points $k_1, k_2 \in K_1$, the claim is proved.

Now, if $K_1 \setminus I$ is a singleton, call that element k_0. Otherwise, choose and fix $k_0 \in K_1$ arbitrarily. By definition, for any $k \in K_1$, there exists $\alpha(k) \in T$ such that $\delta_{k_0} - \alpha(k)\delta_k \in H^\perp$. That is, $H \subseteq \cap_{k \in K_1} \ker[\delta_{k_0} - \alpha(k)\delta_k]$. Let

$$H_1 = \bigcap_{K_1} \bigcap_{k \in K_1} \ker[\delta_{k_0} - \alpha(k)\delta_k]$$

where the intersection is taken over all non-singleton equivalence class K_1. In other words, H_1 is the space of all $g \in C(K)$ such that $g(k) = \alpha(k)g(k_0)$ if $k \in K_1$, which is a non-singleton equivalence class, with the above choice of k_0 and $\alpha(k)$.

Then, $H \subseteq H_1 \subseteq C(K)$. Therefore, H is an AC-subspace of H_1.

Claim 3. For any $z^* \in \text{ext}B_{H^*}$, $\text{HB}_{H_1}(z^*)$ is a singleton. And hence, by Lemma 2.3, there exists a unique norm 1 projection P_1 from H_1 onto H.

If $\text{HB}_{C(K)}(z^*)$ is a singleton, nothing to prove.

Suppose $\text{HB}_{C(K)}(z^*)$ is not a singleton. As before, any two extreme points of $\text{HB}_{C(K)}(z^*)$ are $\sim$-equivalent, and, by definition of H_1, they coincide on H_1. This proves the claim.

Claim 4. There exists a norm 1 projection P_2 from $C(K)$ to H_1.

Let $f \in C(K)$. Define $P_2 f$ as follows : If $\{k\}$ is an equivalence class, let $P_2 f(k) = f(k)$. If $k \in K_1$, which is a non-singleton equivalence class, then with the choice of k_0 and $\alpha(k)$ as above, let $P_2 f(k) = \alpha(k) f(k_0)$. Since $K_1 \setminus \{k_0\} \subseteq I$ and each point of I is an isolated point of K, $P_2 f \in H_1$ and the claim is proved.

The composition $P = P_1 P_2$ is a norm 1 projection from $C(K)$ to H.

Remark 3.2. In this entire argument, the finite codimensionality of H is used only to prove the continuity of $\alpha(k)$ on K_1. Thus, the same proof goes through for any AC-subspace if any point of K_1 is an isolated point. This is used in Proposition 3.4 below.

$(b) \Rightarrow (a)$ is immediate.

$(b) \Rightarrow (c)$. Now, let P be a norm 1 projection on $C(K)$ with range H. Then $\ker P \subseteq O(H, C(K))$ and is of dimension n. Thus we can choose n distinct points $k_1, k_2, \ldots, k_n \in I$ and n linearly independent functions $f_1, f_2, \ldots, f_n \in \ker P$ such that $f_i(k_j) = \delta_{ij}$.

Get n measures $\mu_1, \mu_2, \ldots, \mu_n$ such that $H = \bigcap_{i=1}^n \ker \mu_i$ and $\mu_i(f_j) = \delta_{ij}$. Then for any $f \in C(K)$,

$$Pf = f - \sum_{i=1}^n \mu_i(f) f_i.$$

Fix $1 \leq i \leq n$. Let $\mu_i(\{k_i\}) = \beta$, that is, $\mu_i = \beta\delta_{k_i} + \sigma$.

Now given $\varepsilon > 0$, choose $g \in S_{C(K)}$ such that $\|\sigma\| - \varepsilon \leq -\sigma(g)$. Define $g_1 \in C(K)$ by

$$g_1(k) = \begin{cases} 1 & \text{if } k = k_i \\ g(k) & \text{otherwise} \end{cases}$$

Note that $\|g_1\| = 1$, $\mu_i(g_1) = \beta + \sigma(g)$, and

$$|Pg_1(k_i)| = |g_1(k_i) - \mu_i(g_1)| = |1 - (\beta + \sigma(g))| \leq 1.$$

Thus, $\mathrm{Re}(\beta) + \sigma(g) \geq 0$ and hence, $\mathrm{Re}(\beta) \geq -\sigma(g) \geq \|\sigma\| - \varepsilon$, or,

$$2|\beta| \geq |\beta| + \|\sigma\| - \varepsilon = \|\mu_i\| - \varepsilon.$$

Since ε was arbitrary, we have $2|\beta| \geq \|\mu_i\|$.

$(c) \Rightarrow (b)$. Suppose there exist norm 1 measures $\mu_1, \mu_2, \ldots, \mu_n$ and distinct isolated points $\{k_1, k_2, \ldots, k_n\}$ satisfying (i) and (ii). Then $1_{k_1}, 1_{k_2}, \ldots, 1_{k_n} \in C(K)$ and it suffices to show that $\mathrm{span}\{1_{k_i} : 1 \leq i \leq n\} \subseteq O(H, C(K))$. To see this let $f = \sum_{i=1}^n a_i 1_{k_i}$ where a_i's are scalars. We show that $\ker f$ in $C(K)^*$ is a boundary for H.

Let $h \in H$, $h \neq 0$. If there exists $k \in K$, $k \notin \{k_1, k_2, \ldots, k_n\}$ such that $|h(k)| = \|h\|$, then δ_k, which is in $\ker f$, norms h.

So suppose $\{k \in K : |h(k)| = \|h\|\} \subseteq \{k_1, k_2, \ldots, k_n\}$. Without loss of generality, we may assume $\{k \in K : |h(k)| = \|h\|\} = \{k_1, k_2, \ldots, k_m\}$ for some $m \leq n$.

Let $1 \leq i \leq m$. We decompose μ_i in its atomic and non-atomic parts as $\mu_i = \lambda_i + \nu_i$. Let $\lambda_i = \sum_j \alpha_{ij} \delta_{k_{ij}}$ with $k_{i1} = k_i$. Thus, by (ii), $|\alpha_{i1}| \geq 1/2$ and $\sum_j |\alpha_{ij}| + \|\nu_i\| = \|\mu_i\| = 1$. Note that since $0 = \mu_i(h) = \lambda_i(h) + \nu_i(h)$, we have,

$$\begin{aligned} |\lambda_i(h)| &= \left|\sum \alpha_{ij} h(k_{ij})\right| \geq |\alpha_{i1}| \cdot |h(k_{i1})| - \sum_{j\geq 2} |\alpha_{ij}| \cdot |h(k_{ij})| \\ &\geq \|h\| \left(|\alpha_{i1}| - \sum_{j \geq 2} |\alpha_{ij}| \right) = \|h\|(2|\alpha_{i1}| - \sum |\alpha_{ij}|) \\ &\geq \|h\| \|\nu_i\| \geq |\nu_i(h)| = |\lambda_i(h)| \end{aligned}$$

Thus equality holds throughout. It follows that if $\lambda_i(h) = 0$ then $\nu_i = 0$. Moreover, for all j, $|h(k_{ij})| = \|h\|$. Thus, $\{k_{ij}\} \subseteq \{k_1, k_2, \ldots, k_m\}$.

Claim 5. $\lambda_i(h) \neq 0$ for some $1 \leq i \leq m$.

If not, let $\lambda_i(h) = 0$ for every $1 \leq i \leq m$.

Then, for every $1 \leq i \leq m$, μ_i is of the form $\mu_i = \sum_{j=1}^m \alpha_{ij} \delta_{k_j}$. But $\mu_1, \mu_2, \ldots, \mu_m$ are linearly independent, and $0 = \mu_i(h) = \sum_{j=1}^m \alpha_{ij} h(k_j)$, for every $1 \leq i \leq m$. This implies $h(k_j) = 0$, for all $1 \leq j \leq m$. But $\|h\| = |h(k_j)| \neq 0$. A contradiction that proves the claim.

So let $\lambda_{i_0}(h) \neq 0$ for some $1 \leq i_0 \leq m$. Now, define

$$\nu_0 = -\frac{\|h\|}{\lambda_{i_0}(h)}\nu_{i_0}.$$

Then $\nu_0(h) = \|h\|$ and since equality holds above,

$$\|\nu_0\| = \frac{\|h\|}{|\lambda_{i_0}(h)|}\|\nu_{i_0}\| = 1.$$

Further since ν_0 is a non-atomic measure, $\nu_0 \in \ker f$, showing that $\ker f$ is a boundary for H. □

Remark 3.3. Natural modifications of the proof of $(a) \Rightarrow (b)$ above show that the implication "$AC \Rightarrow$ 1-complemented" also holds for finite codimensional subspaces of $C_0(S)$ for a locally compact Hausdorff space S.

Proposition 3.4. *In the space c of all convergent sequence of scalars, any AC-subspace is 1-complemented.*

For any set Γ, in the space $c_0(\Gamma)$, any AC-subspace is 1-complemented.

Proof. Let H be an AC-subspace of c. We define the equivalence relation $\sim$ on $\mathbb{N}$ as in the proof of $(a) \Rightarrow (b)$ in Theorem 1.2. If each non-singleton equivalence class K_1 is finite, we can proceed exactly as before to define H_1 and the projections P_1 and P_2. The finiteness of K_1 ensures that P_2 takes values in c, and hence, in H_1.

If some K_1 is infinite, note that for any $h \in H$, $|h_n|$ is constant on K_1. Since $h \in c$, this constant is $|\lim_n h_n|$. Thus, there is at most one infinite equivalence class. If we now further partition K_1 with the equivalence relation $m \approx n$ if $h_m = h_n$ for all $h \in H$, then again since $h \in c$, only one subclass—the one on which $h_n = \lim_n h_n$ for all $h \in H$—will be infinite and we can proceed as before, defining $(P_2 f)_n = \lim_n f_n$ on that subclass.

If H is an AC-subspace of $c_0(\Gamma)$, we can proceed as above to define an equivalence relation $\sim$ on Γ. Again, at most one equivalence class K_1 is infinite, and $h \equiv 0$ on K_1. Thus defining $P_2 f(\gamma) = 0$ for $\gamma \in K_1$ works. □

Remark 3.5. Notice that the proof for c and c_0 in [11] is essentially similar, but our argument is simpler and straightforward and works also for complex scalars and uncountable Γ.

From the proof of $(c) \Rightarrow (b)$ in Theorem 1.2 it follows that if H is complemented by a unique norm one projection, then for each $i = 1, 2, \cdots, n$, the condition $2|\mu_i(k_i)| \geq \|\mu_i\|$ holds for exactly one isolated atom k_i of μ_i. If H is a hyperplane in $C(K)$, this condition is also sufficient. That is,

Proposition 3.6. *Let $\mu \in S_{C(K)^*}$ and $H = \ker \mu$. Then H is complemented by a* unique *norm one projection if and only if $|\mu(\{k\})| \geq 1/2$ holds for exactly one isolated atom of μ.*

Proof. Let P be projection of norm one on $C(K)$ with range H. Then there exists $f_0 \in O(H, C(K))$ such that $\mu(f_0) = 1$ and $Pf = f - \mu(f)f_0$ for all $f \in C(K)$.

As before, let $K_0 = \{k \in K : \mathrm{HB}_{C(K)}(\phi k) = \{\delta_k\}\}$. Let $k \in K \setminus K_0$. Then there exists a measure $\nu \in B_{C(K)^*}$ such that $\nu \neq \delta_k$ and $\nu|_H = \phi k$. It follows that $\nu - \delta_k = \alpha\mu$ for some scalar $\alpha \neq 0$. Let $\mu(\{k\}) = \beta$, that is, $\mu = \beta\delta_k + \lambda$. Then $\|\lambda\| = 1 - |\beta|$ and

$$1 \geq \|\nu\| = \|(1 + \alpha\beta)\delta_k + \alpha\lambda\| = |1 + \alpha\beta| + |\alpha|(1 - |\beta|) \geq 1 + |\alpha|(1 - 2|\beta|).$$

Since $\alpha \neq 0$, we get $|\beta| \geq 1/2$. Thus $\{k\}$ is an atom of μ with $|\mu(\{k\})| \geq 1/2$. Now, if $|\mu(\{k\})| \geq 1/2$ holds only for $k = k_0$, it follows from the above argument that $K \setminus K_0 = \{k_0\}$. Since $f_0|_{K_0} = 0$ we conclude f_0 must be a scalar multiple of 1_{k_0}. This shows P is unique. □

It was shown in [1] that a hyperplane H in any Banach space is 1-complemented if and only if it is an AC-subspace and if and only if it is not an VN-subspace. The following corollary is immediate from Theorem 1.2.

Corollary 3.7. *For $n = 0, 1, 2, \ldots$, if K is a compact Hausdorff space with at most n isolated points, then there is no AC-subspace in $C(K)$ of codimension $n + 1$. In particular, if K has no isolated points, there is no 1-complemented hyperplane in $C(K)$. Thus every hyperplane is a VN-subspace.*

Remark 3.8. It is easy to check that the norm of a projection onto a hyperplane of $C(K)$ is at least 2 if K does not have isolated points.

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