Weighted Chebyshev Centres and Intersection Properties of Balls in Banach Spaces

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Abstract. Veselý has studied Banach spaces that admit weighted Chebyshev centres for finite sets. Subsequently, Bandyopadhyay and Rao had shown, *inter alia*, that L_1-preduals have this property. In this work, we investigate why and to what extent are these results true and thereby explore when a more general family of sets admit weighted Chebyshev centres. We extend and improve upon some earlier results in this general set-up and relate them with a modified notion of minimal points. Special cases when we consider the family of all finite, or more interestingly, compact subsets lead to characterizations of L_1-preduals. We also consider some stability results.

1. Introduction

Let X be a Banach space. We will denote by $B_X[x,r]$ the closed ball of radius $r > 0$ around $x \in X$. We will identify any element $x \in X$ with its canonical image in X^{**}. Our notations are otherwise standard. Any unexplained terminology can be found in either [**6**] or [**10**].

In this paper we continue the study of Banach spaces that admit weighted Chebyshev centres that began with [**3**].

Definition 1.1. Let Y be a subspace of a Banach space X. For $A \subseteq Y$ and $\rho : A \longrightarrow \mathbb{R}_+$, define

$$\phi_{A,\rho}(x) = \sup\{\rho(a)\|x - a\| : a \in A\}$$

A point $x_0 \in X$ is called a weighted Chebyshev centre of A in X for the weight ρ if $\phi_{A,\rho}$ attains its minimum at x_0.

When A is finite, Veselý [**18**] has shown that if X is a dual space, A admits weighted Chebyshev centres in X for any weight ρ, that the infimum of $\phi_{A,\rho}$ over X and X^{**} are the same, and

Theorem 1.2. [**18**, Theorem 2.7] *For a Banach space X and $a_1, a_2, \ldots, a_n \in X$, the following are equivalent :*

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(*a*) *If* $r_1, r_2, \ldots, r_n > 0$ *and* $\cap_{i=1}^n B_{X^{**}}[a_i, r_i] \neq \emptyset$*, then* $\cap_{i=1}^n B_X[a_i, r_i] \neq \emptyset$.

(*b*) $\{a_1, a_2, \ldots, a_n\}$ *admits weighted Chebyshev centres for all weights* $r_1, r_2, \ldots, r_n > 0$.

(*c*) $\{a_1, a_2, \ldots, a_n\}$ *admits* f*-centres for every continuous monotone coercive* $f : \mathbb{R}_+^n \to \mathbb{R}$ *(see* [**18**] *for the definitions).*

In this work, we investigate why and to what extent are these results true and thereby explore when a more general family of sets admit weighted Chebyshev centres. Extending the notion of central subspaces introduced in [**3**], we define an $\mathcal{A}$-C-subspace Y of a Banach space X with the centres of the balls coming from a given family $\mathcal{A}$ of subsets of Y, the typical examples being those of finite, compact, bounded or arbitrary sets. The first gives us the central subspace *a la* [**3**] and the last one is related to the Finite Infinite Intersection Property ($IP_{f,\infty}$) [**8**]. We extend and improve upon some results of [**3, 18**] in this general set-up and relate them with a modified notion of minimal points. We also improve upon one of the main results of [**4**] on the structure of the set of minimal points of a compact set. As in [**3**], special cases when we consider the family of all finite, or more interestingly, compact subsets lead to characterizations of L_1-preduals. We also consider some stability results.

2. General Results

We first extend Veselý's result in [**18**] on dual spaces from finite sets to all the way upto bounded sets and also strengthens its conclusions. We need the following notions.

Definition 2.1. Let X be a Banach space and $A \subseteq X$.

(a) We define a partial ordering on X as follows : for $x_1, x_2 \in X$, we say that $x_1 \leq_A x_2$ if $\|x_1 - a\| \leq \|x_2 - a\|$ for all $a \in A$. We will denote by $m_X(A)$ the set of points of X that are minimal with respect to the ordering $\leq_A$ and often refer to them as $\leq_A$-minimal points of X.

Note that $\leq_A$ defines a partial order on any Banach space containing A and we will use the same notation in all such cases.

(b) A function $f : X \to \mathbb{R}^+$ is said to be A-monotone if $f(x_1) \leq f(x_2)$ whenever $x_1 \leq_A x_2$.

(c) Let Y be a subspace of X and $A \subseteq Y$. Following [**9**], we say $x \in X$ is a minimal point of A with respect to Y if for any $y \in Y$, $y \leq_A x$ implies $y = x$.

We denote the set of all minimal points of A with respect to Y in X by $A_{Y,X}$. Note $A_{Y,X} \supseteq A$. For $A \subseteq X$, the set $A_{X,X}$ will be called minimal points of A in X, and will be denoted simply by $\min A$.

(d) For $A \subseteq X$ bounded, the Chebyshev radius of A in X is defined by

$$r(A) = \inf_{x \in X} \sup_{a \in A} \|x - a\|.$$

Theorem 2.2. (*a*) *If* $A \subseteq X$ *is bounded and* $x \notin \overline{A + r(A)B(X)}$*, then there exists* $y \in X$ *such that* $y \leq_A x$.

(*b*) *If* $X = Z^*$ *is a dual space and* A *is bounded, then every* A*-monotone and* w^**-lower semicontinuous (henceforth, lsc)* $f : X \to \mathbb{R}^+$ *attains its minimum. In particular, for every* ρ, $\phi_{A,\rho}$ *attains its minimum.*

(c) *If $X = Z^*$ is a dual space, for every $x_0 \in X$, there is a $x_1 \in m_X(A)$ such that $x_1 \leq_A x_0$. In particular, the minimum in* (b) *is attained at a point of $m_X(A)$.*

Proof. (a). Let $x \notin \overline{A + r(A)B(X)}$. Then, there exists $\varepsilon > 0$ such that $\|x - a\| > r(A) + \varepsilon$ for all $a \in A$. By definition of $r(A)$, there exists $y \in X$ such that $\sup_{a \in A} \|y - a\| < r(A) + \varepsilon$. Clearly, $y \leq_A x$.

(b). By (a), if $x \notin \overline{A + r(A)B(X)}$, there exists $y \in X$ such that $y \leq_A x$, and hence, $f(y) \leq f(x)$. Thus, the infimum of f over X equals the infimum over $\overline{A + r(A)B(X)}$. Moreover, since X is a dual space and f is w*-lsc, it attains its minimum over any w*-compact set. Thus f actually attains its minimum over X as well.

Since the norm on X is w*-lsc, so is $\phi_{A,\rho}$ for every ρ.

(c). Consider $\{x \in X : x \leq_A x_0\}$. Let $\{x_i\}$ be a totally ordered subset. Let z be a w*-limit point of x_i. Since the norm is w*-lsc, we have

$$\|z - a\| \leq \liminf \|x_i - a\| = \inf \|x_i - a\| \text{ for all } a \in A.$$

Thus the family $\{x_i\}$ is $\leq_A$-bounded below by z.

By Zorn's lemma, there is a $x_1 \in m_X(A)$ such that $x_1 \leq_A x_0$.

Now let x_0 be a minimum for f. There is a $x_1 \in m_X(A)$ such that $x_1 \leq_A x_0$. Clearly, f attains its minimum also at x_1. □

Remark 2.3. (a) It follows that for any bounded set A, $\min A \subseteq \overline{A + r(A)B(X)}$. This improves the estimates in [**9**] or [**18**].

(b) Apart from $\phi_{A,\rho}$, there are many examples of A-monotone and w*-lsc $f : X = Z^* \to \mathbb{R}^+$. One particular example that has been treated extensively in [**4**] is the function ϕ_μ defined by $\phi_\mu(x) = \int_A \|x - a\|^2 d\mu(a)$, where μ is a probability measure on a compact set $A \subseteq X$.

(c) Observe that though minimal points of A are $\leq_A$-minimal, there is some distinction between the two notions. The two notions coincide if X is strictly convex. See Proposition 3.1 below.

Now, if A is a bounded subset of a Banach space X, then by Theorem 2.2, A has a weighted Chebyshev centre in X^{**}. But what about a weighted Chebyshev centre in X?

When A is finite, Veselý [**18**] has shown that the infimum of $\phi_{A,\rho}$ over X and X^{**} are the same, and A admits weighted Chebyshev centres in X for any weight ρ if and only if X satisfies Theorem 1.2(a). We now show that both of these are special cases of more general results. We need the following definition.

Definition 2.4. Let Y be a subspace of a Banach space X. Let $\mathcal{A}$ be a family of subsets of Y.

(a) We say that Y is an almost $\mathcal{A}$-C-subspace of X if for every $A \in \mathcal{A}$, $x \in X$ and $\varepsilon > 0$, there exists $y \in Y$ such that

$$\|y - a\| \leq \|x - a\| + \varepsilon \text{ forall } a \in A. \tag{1}$$

(b) We say that Y is an $\mathcal{A}$-C-subspace of X if we can take $\varepsilon = 0$ in (a).

(c) If $\mathcal{A}$ is a family of subsets of X, we say that X has the (almost) $\mathcal{A}$-IP if X is an (almost) $\mathcal{A}$-C-subspace of X^{**}.

Some of the special families that we would like to give names to are :
(i) $\mathcal{F}$ = the family of all finite sets,
(ii) $\mathcal{K}$ = the family of all compact sets,
(iii) $\mathcal{B}$ = the family of all bounded sets,
(iv) $\mathcal{P}$ = the power set.
Since these families depend on the space in which they are considered, we will use the notation $\mathcal{F}(X)$ etc. whenever there is a scope of confusion.

Remark 2.5. (a) Note that $\mathcal{F}$-C-subspaces were called central (C) subspaces in [**3**], $\mathcal{P}$-C-subspaces were called almost constrained (AC) subspaces in [**1, 2**]. Also if X has the $\mathcal{F}$-IP, it was said to belong to the class (GC) in [**18, 3**], and the $\mathcal{P}$-IP was called the Finite Infinite Intersection Property ($IP_{f,\infty}$) in [**7, 2**].
(b) The definition of almost $\mathcal{A}$-C-subspace is adapted from the definition of almost central subspace defined in [**17**]. The exact analogue of the definition in [**17**] would have, in place of condition (1),

$$\sup_{a\in A} \|y-a\| \le \sup_{a\in A} \|x-a\| + \varepsilon.$$

Clearly, our condition is stronger. We observe below (see Proposition 2.7) that this definition is more natural in our context.
(c) By the Principle of Local Reflexivity (henceforth, PLR), any Banach space has the almost $\mathcal{F}$-IP. More generally, if Y is an ideal in X (see definition below), then Y is an almost $\mathcal{F}$-C-subspace of X.

Definition 2.6. A subspace Y of a Banach space X is said to be an ideal in X if there is a norm 1 projection P on X^* with $\ker(P) = Y^\perp$.

Proposition 2.7. *Let Y be a subspace of a Banach space X. Let $\mathcal{A}$ be a family of bounded subsets of Y. Then the following are equivalent :*
(a) *Y is an almost $\mathcal{A}$-C-subspace of X*
(b) *for all $A \in \mathcal{A}$ and $\rho : A \to \mathbb{R}^+$, if $\cap_{a\in A} B_X[a, \rho(a)] \neq \emptyset$, then for every $\varepsilon > 0$, $\cap_{a\in A} B_Y[a, \rho(a) + \varepsilon] \neq \emptyset$.*
(c) *for every bounded ρ, the infimum of $\phi_{A,\rho}$ over X and Y are equal.*

Proof. Equivalence of (a) and (b) is immediate and does not need A to be bounded.

(a) ⇒ (c). Let Y be an almost $\mathcal{A}$-C-subspace of X, $A \in \mathcal{A}$ and $\rho : A \to \mathbb{R}^+$ be bounded. Let $M = \sup \rho(A)$. Let $\varepsilon > 0$. By definition, for $x \in X$, there exists $y \in Y$ such that

$$\|y-a\| \le \|x-a\| + \varepsilon \text{ for all } a \in A.$$

It follows that

$$\rho(a)\|y-a\| \le \rho(a)\|x-a\| + \rho(a)\varepsilon \le \rho(a)\|x-a\| + M\varepsilon \text{ for all } a \in A.$$

and hence,

$$\phi_{A,\rho}(y) \le \phi_{A,\rho}(x) + M\varepsilon.$$

Therefore,

$$\inf \phi_{A,\rho}(Y) \le \inf \phi_{A,\rho}(X) + M\varepsilon.$$

As ε is arbitrary, the infimum of $\phi_{A,\rho}$ over X and Y are equal.

$(c) \Rightarrow (a)$. Let $A \in \mathcal{A}$, $x \in X$ and $\varepsilon > 0$. We need to show that there exists $y \in Y$ such that

$$\|y - a\| \leq \|x - a\| + \varepsilon \text{ for all } a \in A.$$

If $x \in Y$, nothing to prove. Let $x \in X \setminus Y$. Let $N = \sup_{a \in A} \|x - a\|$. Let $\rho(a) = 1/\|x - a\|$. Since $x \notin Y$ and $A \subseteq Y$, ρ is bounded. Then $\phi_{A,\rho}(x) = 1$, and therefore, $\inf \phi_{A,\rho}(X) \leq 1$. By assumption, $\inf \phi_{A,\rho}(Y) = \inf \phi_{A,\rho}(X) \leq 1$, and so, there exists $y \in Y$, such that $\phi_{A,\rho}(y) \leq 1 + \varepsilon/N$. This implies $\|y - a\| \leq \|x - a\| + \varepsilon\|x - a\|/N \leq \|x - a\| + \varepsilon$ for all $a \in A$. □

As noted before, by PLR, any Banach space has the almost $\mathcal{F}$-IP. And therefore, the result of [**18**] follows.

Proposition 2.8. *Let $\mathcal{A}$ and $\mathcal{A}_1$ be two families of subsets of Y such that for every $A \in \mathcal{A}$ and $\varepsilon > 0$, there exists $A_1 \in \mathcal{A}_1$ such that $A \subseteq A_1 + \varepsilon B(Y)$. If Y is an almost $\mathcal{A}_1$-C-subspace of X, then Y is an almost $\mathcal{A}$-C-subspace of X as well.*

Consequently, any ideal is an almost $\mathcal{K}$-C-subspace and any Banach space has the almost $\mathcal{K}$-IP. In particular, if A is a compact subset of X and $\rho : A \longrightarrow \mathbb{R}_+$ is bounded, then the infimum of $\phi_{A,\rho}$ over X and X^{**} are the same.

Proof. Let $A \in \mathcal{A}$ and $\varepsilon > 0$. By hypothesis, there exist $A_1 \in \mathcal{A}_1$ such that $A \subseteq A_1 + \varepsilon B(Y)$. Let $x \in X$. Since Y is an almost $\mathcal{A}_1$-C-subspace of X, there exists $y \in Y$ such that

$$\|y - a_1\| \leq \|x - a_1\| + \varepsilon/3 \text{ for all } a_1 \in A_1.$$

Now fix $a \in A$. Then there exists $a_1 \in A_1$ such that $\|a - a_1\| < \varepsilon/3$. Then

$$\begin{aligned} \|y - a\| &\leq \|y - a_1\| + \|a - a_1\| \leq \|x - a_1\| + 2\varepsilon/3 \\ &\leq \|x - a\| + \|a - a_1\| + 2\varepsilon/3 \leq \|x - a\| + \varepsilon. \end{aligned}$$

Therefore, Y is an almost $\mathcal{A}$-C-subspace of X as well.

Since any Banach space has the almost $\mathcal{F}$-IP, by the above, it has the almost $\mathcal{K}$-IP too. The rest of the result follows from Proposition 2.7. □

Example 2.9. Veselý [**18**] has shown that if A is infinite, the infimum of $\phi_{A,\rho}$ over X and X^{**} may not be the same. His example is $X = c_0$, $A = \{e_n : n \geq 1\}$ is the canonical unit vector basis of c_0 and $\rho \equiv 1$. Then $\inf \phi_{A,\rho}(X) = 1$ and $\inf \phi_{A,\rho}(X^{**}) = 1/2$. The example clearly also excludes countable, bounded, or, taking $A \cup \{0\}$, even weakly compact sets. Thus c_0 fails the almost $\mathcal{B}$-IP, almost $\mathcal{P}$-IP and if $\mathcal{A}$ is the family of countable or weakly compact sets, then c_0 fails the almost $\mathcal{A}$-IP too.

Stronger conclusions are possible for $\mathcal{A}$-IP.

Lemma 2.10. *Let Y be a subspace of a Banach space X. For $A \subseteq Y$, the following are equivalent :*

- (a) *For every A-monotone $f : A \longrightarrow \mathbb{R}_+$ and $x \in X$, there exists $y \in Y$ such that $f(y) \leq f(x)$.*
- (b) *For every $\rho : A \longrightarrow \mathbb{R}_+$ and $x \in X$, there exists $y \in Y$ such that $\phi_{A,\rho}(y) \leq \phi_{A,\rho}(x)$.*
- (c) *For every continuous $\rho : A \longrightarrow \mathbb{R}_+$ and $x \in X$, there exists $y \in Y$ such that $\phi_{A,\rho}(y) \leq \phi_{A,\rho}(x)$.*
- (d) *For every bounded $\rho : A \longrightarrow \mathbb{R}_+$ and $x \in X$, there exists $y \in Y$ such that $\phi_{A,\rho}(y) \leq \phi_{A,\rho}(x)$.*

(*e*) *Any family of closed balls centred at points of A that intersects in X also intersects in Y.*

(*f*) *for any $x \in X$, there exists $y \in Y$ such that $y \leq_A x$.*

It follows that whenever any of the above conditions is satisfied, for every A-monotone $f : A \longrightarrow \mathbb{R}_+$, the infimum of f over X and Y are equal and if A has a weighted Chebyshev centre in X, it has a weighted Chebyshev centre in Y.

Proof. $(a) \Rightarrow (b) \Rightarrow (c)$, $(b) \Rightarrow (d)$ and $(e) \Leftrightarrow (f) \Rightarrow (a)$ are obvious.

(c) or $(d) \Rightarrow (f)$. As in the proof of Proposition 2.7, let $\rho(a) = 1/\|x-a\|$. Then ρ is continuous and bounded and $\phi_{A,\rho}(x) = 1$. Thus, there exists $y \in Y$ such that $\phi_{A,\rho}(y) \leq 1$. This implies $\|y - a\| \leq \|x - a\|$ for all $a \in A$. □

We now conclude the discussion so far by obtaining the extension of Theorem 1.2.

Theorem 2.11. *For a Banach space X and a family $\mathcal{A}$ of bounded subsets of X, the following are equivalent :*

(*a*) *X has the $\mathcal{A}$-IP.*

(*b*) *For every $A \in \mathcal{A}$ and every $f : X^{**} \longrightarrow \mathbb{R}_+$ that is A-monotone and w^*-lsc, the infimum of f over X^{**} and X are equal and is attained at a point of X.*

(*c*) *For every $A \in \mathcal{A}$ and every ρ, the infimum of $\phi_{A,\rho}$ over X^{**} and X are equal and is attained at a point of X.*

Moreover, the point in (b) or (c) can be chosen to be $\leq_A$-minimal.

We now study different aspects of $\mathcal{A}$-C-subspaces.

Definition 2.12. Let Y be a subspace of a Banach space X. Let $A \subseteq Y$. For $x \in X$ and $x^* \in B(X^*)$, define

$$\begin{aligned} U(x, A, x^*) &= \inf\{x^*(y) + \|x - y\| : y \in A\} \\ L(x, A, x^*) &= \sup\{x^*(y) - \|x - y\| : y \in A\} \end{aligned}$$

The following lemma is in [**1**]. We include the proof for completeness.

Lemma 2.13. *Let Y be a subspace of a Banach space X and $A \subseteq Y$. For $x_1, x_2 \in X$, $x_2 \leq_A x_1$ if and only if for all $x^* \in B(X^*)$, $U(x_2, A, x^*) \leq U(x_1, A, x^*)$.*

Proof. If $x_2 \leq_A x_1$, then for all $x^* \in B(X^*)$, $x^*(y) + \|x_2 - y\| \leq x^*(y) + \|x_1 - y\|$. And therefore, $U(x_2, A, x^*) \leq U(x_1, A, x^*)$.

Conversely, suppose $\|x_2 - y_0\| > \|x_1 - y_0\|$ for some $y_0 \in A$. Then there exists $\varepsilon > 0$ such that $\|x_2 - y_0\| - \varepsilon \geq \|x_1 - y_0\|$. Choose $x^* \in B(X^*)$ such that $\|x_1 - y_0\| \leq \|x_2 - y_0\| - \varepsilon < x^*(x_2 - y_0) - \varepsilon/2$. Thus $U(x_1, A, x^*) \leq x^*(y_0) + \|x_1 - y_0\| < x^*(x_2) - \varepsilon/2 < U(x_2, A, x^*)$. □

Remark 2.14. Instead of $B(X^*)$, it suffices to consider the unit ball of any norming subspace of X^*.

We compile in the following propositions several interesting facts about $\mathcal{A}$-C-subspaces and the $\mathcal{A}$-IP.

Proposition 2.15. *Let Y be a subspace of a Banach space X. For a family $\mathcal{A}$ of subsets of Y, the following are equivalent :*

(*a*) *Y is an $\mathcal{A}$-C-subspace of X*

(*b*) *for every* $x \in X$ *and* $A \in \mathcal{A}$, *there exists* $y \in Y$ *such that* $U(y, A, x^*) \leq U(x, A, x^*)$ *for every* $x^* \in B(X^*)$.
(*c*) *for any* $A \in \mathcal{A}$, $A_{Y,X} \subseteq Y$.

PROOF. This follows from Lemma 2.13 and the definition of $A_{Y,X}$. □

COROLLARY 2.16. *X has the $\mathcal{P}$-IP if and only if for every $x^{**} \in X^{**}$, there exists $x \in X$ such that x is dominated on $B(X^*)$ by the upper envelop of x^{**} considered as a function on $B(X^*)$ equipped with the w*-topology.*

PROOF. Observe that for any $x \in X$, $U(x, X, \cdot) \equiv x$ on $B(X^*)$ and for $x^{**} \in X^{**}$, $U(x^{**}, X, x^*)$ is the upper envelop of x^{**} considered as a function on $B(X^*)$ equipped with the w*-topology (see [**8**]). □

PROPOSITION 2.17. (*a*) *Let X be a Banach space and let Y be a subspace of X. Let $\mathcal{A}$ be a family of subsets of Y and let $\mathcal{A}_1$ be a subfamily of $\mathcal{A}$. If Y is a $\mathcal{A}$-C-subspace of X, then Y is a $\mathcal{A}_1$-C-subspace of X as well. In particular, $\mathcal{P}$-IP implies $\mathcal{B}$-IP implies $\mathcal{K}$-IP implies $\mathcal{F}$-IP.*
(*b*) *1-complemented subspaces are $\mathcal{A}$-C-subspaces for any $\mathcal{A}$.*
(*c*) *Let $Z \subseteq Y \subseteq X$ and let $\mathcal{A}$ be a family of subsets of Z. If Z is an $\mathcal{A}$-C-subspace of X, then Z is an $\mathcal{A}$-C-subspace of Y. And, if Y is an $\mathcal{A}$-C-subspace of X, then the converse also holds.*

PROOF. The proof follows the same line of argument as in [**3**, Proposition 2.2]. We omit the details. □

PROPOSITION 2.18. *For a family $\mathcal{A}$ of subsets of a Banach space X, the following are equivalent :*
(*a*) *X has the $\mathcal{A}$-IP*
(*b*) *X is a $\mathcal{A}$-C-subspace of some dual space.*
(*c*) *for all $A \in \mathcal{A}$ and $\rho : A \to \mathbb{R}^+$, $\cap_{i=1}^n B_X[a_i, \rho(a_i) + \varepsilon] \neq \emptyset$ for all finite subset $\{a_1, a_2, \ldots, a_n\} \subseteq A$ and for all $\varepsilon > 0$ implies $\cap_{a \in A} B_X[a, \rho(a)] \neq \emptyset$.*
In particular, any dual space has the $\mathcal{A}$-IP for any $\mathcal{A}$. Let $\mathcal{S}$ be any of the families $\mathcal{F}$, $\mathcal{K}$, $\mathcal{B}$ or $\mathcal{P}$. The $\mathcal{S}$-IP is inherited by $\mathcal{S}$-C-subspaces, in particular, by 1-complemented subspaces.

PROOF. Clearly, $(a) \Rightarrow (b)$, while $(c) \Rightarrow (a)$ follows from the PLR.
$(b) \Rightarrow (c)$. Let X be an $\mathcal{A}$-C-subspace of Z^*. Consider the family $\{B_{Z^*}[a, \rho(a) + \varepsilon] : a \in A, \varepsilon > 0\}$ in Z^*. Then, by the hypothesis, any finite subfamily intersects. Hence, by w*-compactness, $\cap_{a \in A} B_{Z^*}[a, \rho(a)] \neq \emptyset$. Since X is an $\mathcal{A}$-C-subspace of Z^*, we have $\cap_{a \in A} B_X[a, \rho(a)] \neq \emptyset$. □

The following result significantly improves [**3**, Proposition 2.8] and provides yet another characterization of the $\mathcal{A}$-IP.

PROPOSITION 2.19. *Let Y be an almost $\mathcal{F}$-C subspace of a Banach space X. Let $\mathcal{A}$ be a family of subsets of Y. If Y has the $\mathcal{A}$-IP, then Y is an $\mathcal{A}$-C-subspace of X. In particular, the conclusion holds when Y is an ideal in X.*

PROOF. Let $x \in X$, $A \in \mathcal{A}$. Since Y be an almost $\mathcal{F}$-C subspace of X, for all finite subset $\{a_1, a_2, \ldots, a_n\} \subseteq A$ and for all $\varepsilon > 0$, $\cap_{i=1}^n B_Y[a_i, \|x - a_i\| + \varepsilon] \neq \emptyset$. Since Y has the $\mathcal{A}$-IP, by Proposition 2.18(c), $\cap_{a \in A} B_Y[a, \|x - a\|] \neq \emptyset$. □

Since X is always an ideal in X^{**}, the following corollary is immediate.

COROLLARY 2.20. *For a Banach space X and a family $\mathcal{A}$ of subsets of X, the following are equivalent :*

(a) *X has $\mathcal{A}$-IP.*

(b) *X is an $\mathcal{A}$-C-subspace of every superspace Z in which X embeds as an almost $\mathcal{F}$-C subspace.*

(c) *X is an $\mathcal{A}$-C-subspace of every superspace Z in which X embeds as an ideal.*

3. Strict convexity and minimal points

PROPOSITION 3.1. *If a Banach space X is strictly convex, then for every $A \subseteq X$, $\min A = m_X(A)$.*

PROOF. As we have already observed, $\min A \subseteq m_X(A)$.

Let $x_0 \in m_X(A)$ and $x_0 \notin \min A$. Then there is an $x \in X$ such that $x \neq x_0$ and $x \leq_A x_0$. Since $x_0 \in m_X(A)$, we must have $\|x - a\| = \|x_0 - a\|$ for all $a \in A$. Since X is strictly convex, $\|(x + x_0)/2 - a\| < \|x_0 - a\|$ for all a. This contradicts that $x_0 \in m_X(A)$. Hence $x_0 \in \min A$. □

REMARK 3.2. If X is strictly convex, by a similar argument, for every $x_0 \in X$, there is at most one $x_1 \in m_X(A)$ such that $x_1 \leq_A x_0$. Thus for a strictly convex dual space, for every $x_0^* \in X^*$, there is a unique $x_1^* \in m_{X^*}(A)$ such that $x_1^* \leq_A x_0^*$.

PROPOSITION 3.3. *Let X be strictly convex. Let A be a compact subset of X. For each continuous ρ, A admits at most one weighted Chebyshev centre.*

PROOF. Suppose A admits two distinct weighted Chebyshev centres $x_0, x_1 \in X$. Then $\phi_{A,\rho}(x_0) = \phi_{A,\rho}(x_1) = r$ (say). Then for all $a \in A$, we have x_1, $x_0 \in B_X[a, r/\rho(a)]$. By rotundity $z = (x_1 + x_0)/2$ is in the interior of $B_X[a, r/\rho(a)]$ for all a. Thus, $\rho(a)\|z - a\| < r$, for all a. Since ρ is continuous, $\phi_{A,\rho}(z) < r$, which contradicts that minimum value is r. □

THEOREM 3.4. *Let X be a Banach space such that*

(i) *X has the $\mathcal{F}$-IP; and*

(ii) *for every compact set $A \subseteq X$, $m_X(A)$ is weakly compact.*

Then X has the $\mathcal{K}$-IP. Moreover, if X^{**} is strictly convex, then the converse also holds.

PROOF. Let X have the $\mathcal{F}$-IP and for every compact set $A \subseteq X$, let $m_X(A)$ be weakly compact. Observe that for any $B \subseteq A$, we have $m_X(B) \subseteq m_X(A)$.

Let $A \subseteq X$ be compact and let $x^{**} \in X^{**}$. By Lemma 2.10, it suffices to show that there is a $z_0 \in X$ such that $\|z_0 - a\| \leq \|x^{**} - a\|$ for all $a \in A$.

Let $\{a_n\}$ be a norm dense sequence in A. Take a sequence $\varepsilon_k \to 0$. By compactness of A, for each k, there is a n_k such that $A \subseteq \bigcup_1^{n_k} B_X[a_n, \varepsilon_k]$. Since X has the $\mathcal{F}$-IP, there exists $z_k \in \cap_1^{n_k} B_X[a_n, \|x^{**} - a_n\|]$ and $z_k \in m_X(\{a_1, a_2 \cdots a_{n_k}\}) \subseteq m_X(A)$. Then $\|z_k - a\| \leq \|x^{**} - a\| + 2\varepsilon_k$ for all $a \in A$. Now, by weak compactness of $m_X(A)$, we have, by passing to a subsequence if necessary, $z_k \to z_0$ weakly for some $z_0 \in X$. Since the norm is weakly lsc, we have $\|z_0 - a\| \leq \liminf \|z_k - a\| \leq \|x^{**} - a\|$ for all $a \in A$.

Conversely, let X have the $\mathcal{K}$-IP and X^{**} be strictly convex. Let $A \subseteq X$ be compact. It is enough to show that any sequence $\{x_n\} \subseteq m_X(A)$ has a weakly convergent subsequence. Without loss of generality, we may assume that $\{x_n\}$ are all distinct. By Remark 2.3 (a), $m_X(A) \subseteq \overline{A + r(A)B(X)}$ is bounded. Let x^{**} be a w*-cluster point of $\{x_n\}$ in X^{**}. It suffices to show that $x^{**} \in X$.

Suppose $x^{**} \in X^{**} \setminus X$. Since X has the $\mathcal{K}$-IP, there exists $x_0 \in m_X(A)$ such that $\|x_0 - a\| \leq \|x^{**} - a\|$ for all $a \in A$. Since X^{**} is strictly convex, $\|(x^{**} + x_0)/2 - a\| < \|x^{**} - a\|$ for all $a \in A$. Since $(x^{**} + x_0)/2 \in X^{**} \setminus X$, by $\mathcal{K}$-IP again, there exists $z_0 \in m_X(A)$ such that $\|z_0 - a\| \leq \|(x^{**} + x_0)/2 - a\| < \|x^{**} - a\|$ for all $a \in A$.

Since A is compact, there exists $\varepsilon > 0$ such that $\|z_0 - a\| < \|x^{**} - a\| - \varepsilon$ for all $a \in A$. Observe that

$$\|z_0 - a\| < \|x^{**} - a\| - \varepsilon \leq \liminf_n \|x_n - a\| - \varepsilon \text{ for all } a \in A.$$

Therefore, for every $a \in A$, there exists $N(a) \in \mathbb{N}$ such that for all $n \geq N(a)$, $\|z_0 - a\| < \|x_n - a\| - \varepsilon$. By compactness, there exists $N \in \mathbb{N}$ such that $\|z_0 - a\| < \|x_n - a\| - \varepsilon/4$ for all $n \geq N$ and $a \in A$. Thus, $z_0 \leq_A x_n$ for all $n \geq N$. Since $x_n \in m_X(A)$ and X is strictly convex, $z_0 = x_n$ for all $n \geq N$. This contradiction completes the proof. □

Remark 3.5. In proving sufficiency, one only needs that $\{z_k\}$ has a subsequence convergent in a topology in which the norm is lsc. The weakest such topology is the ball topology, b_X. So it follows that if X has the $\mathcal{F}$-IP and for every compact set $A \subseteq X$, $m_X(A)$ is b_X-compact, then X has the $\mathcal{K}$-IP. Is the converse true?

Corollary 3.6. [**4**, Corollary 1] *Let X be a reflexive and strictly convex Banach space. Let $A \subseteq X$ be a compact set. Then* $\min(A)$ *is weakly compact.*

Remark 3.7. Clearly, our proof is simpler than the original proof of [**4**].

If Z is a non-reflexive Banach space with Z^{***} strictly convex, then $X = Z^*$ is a non-reflexive Banach space with $\mathcal{K}$-IP such that X^{**} is strictly convex. Thus, our result is also stronger than [**4**, Corollary 1].

4. L^1-preduals and $\mathcal{P}_1$-spaces

Our next theorem extends [**3**, Theorem 7], exhibits a large class of Banach spaces with the $\mathcal{K}$-IP and produces a family of examples where the notions of $\mathcal{F}$-C-subspaces and $\mathcal{K}$-C-subspaces are equivalent.

Definition 4.1. (a) [**12**] A Banach space X is called an L^1-predual if X^* is isometrically isomorphic to $L^1(\mu)$ for some positive measure μ.
(b) [**11**] A family $\{B_X[x_i, r_i]\}$ of closed balls is said to have the weak intersection property if for all $x^* \in B(X^*)$ the family $\{B_{\mathbb{R}}[x^*(x_i), r_i]\}$ has nonempty intersection in $\mathbb{R}$.

Theorem 4.2. *For a Banach space X, the following are equivalent :*
(a) X is a $\mathcal{K}$-C-subspace of every superspace
(b) X is a $\mathcal{K}$-C-subspace of every dual superspace
(c) X is a $\mathcal{F}$-C-subspace of every superspace
(d) X is an almost $\mathcal{F}$-C-subspace of every superspace
(e) X is a $\mathcal{F}$-C-subspace of every dual superspace
(f) X is an almost $\mathcal{F}$-C-subspace of every dual superspace
(g) X is an L^1-predual.

Proof. Observe that if $X \subseteq Y \subseteq Y^{**}$ and X is a $\mathcal{A}$-C-subspace of Y^{**}, then X is a $\mathcal{A}$-C-subspace of Y. Thus $(a) \Leftrightarrow (b)$ and $(c) \Leftrightarrow (e)$. And clearly, $(a) \Rightarrow (c) \Rightarrow (d) \Rightarrow (f)$.

$(f) \Rightarrow (g)$. Since the definition of almost central subspaces in [**17**] is weaker than our definition of almost $\mathcal{F}$-C-subspaces, this follows from [**17**, Theorem 1, $2 \Rightarrow 3$]

$(g) \Rightarrow (a)$. Suppose X is an L^1-predual, and let $X \subseteq Y$. Let $A \subseteq X$ be compact with at least three points. Let $y_0 \in Y$. Then the family of balls $\{B_X[a, \|y_0 - a\|] : a \in A\}$ have the weak intersection property. Since X is an L^1-predual and since the centres of the balls are in a compact set, by [**14**, Proposition 4.4], $\cap_{a \in A} B_X[a, \|y_0 - a\|] \neq \emptyset$.

If A has two points, observe that two balls intersect if and only if the distance between the centres is less than or equal to the sum of the radii, it is independent of the ambient space. □

Corollary 4.3. *Every L^1-predual has the $\mathcal{K}$-IP and hence also the $\mathcal{F}$-IP.*

Proposition 4.4. *Suppose X is an L_1-predual space. Then for a subspace $Y \subseteq X$, the following are equivalent :*

- (a) *Y is an ideal in X*
- (b) *Y is a $\mathcal{K}$-C-subspace of X*
- (c) *Y is a $\mathcal{F}$-C-subspace of X*
- (d) *Y is an almost $\mathcal{F}$-C-subspace of X*
- (e) *Y itself is an L_1-predual*

Proof. $(e) \Rightarrow (b)$ follows from Theorem 4.2 and $(e) \Rightarrow (a)$ follows from [**16**, Proposition 1]. And clearly, $(b) \Rightarrow (c) \Rightarrow (d)$ and $(a) \Rightarrow (d)$.

$(d) \Rightarrow (e)$. This again is an easy adaptation of the proof of [**17**, Theorem 1, $2 \Rightarrow 3$]. We omit the details. □

The analog of Theorem 4.2 for $\mathcal{P}$-C-subspaces involves $\mathcal{P}_1$-spaces.

Definition 4.5. Recall that a Banach space is a $\mathcal{P}_1$-space if it is 1-complemented in every superspace.

Theorem 4.6. *For a Banach space X, the following are equivalent :*

- (a) *X is a $\mathcal{P}_1$-space*
- (b) *X is 1-complemented in every dual space that contains it*
- (c) *X is a $\mathcal{P}$-C-subspace of every superspace*
- (d) *X is a $\mathcal{P}$-C-subspace of every dual space that contains it*
- (e) *X is isometric to $C(K)$ for some extremally disconnected compact Hausdorff space K.*

Proof. $(a) \Leftrightarrow (b)$ and $(c) \Leftrightarrow (d)$ follow as in the first paragraph of Theorem 4.2. And clearly, $(a) \Rightarrow (c)$.

$(d) \Rightarrow (a)$. By Proposition 2.18 and Theorem 4.2, (d) implies X is an L^1-predual with $\mathcal{P}$-IP. Recall that [**12**, Theorem 3.8] a Banach space X is a $\mathcal{P}_1$-space if and only if every pairwise intersecting family of closed balls in X intersects. And that X is a L^1-predual if and only if X^{**} is a $\mathcal{P}_1$-space.

Now given a pairwise intersecting family of closed balls in X, since X^{**} is a $\mathcal{P}_1$-space, they intersect in X^{**}. And since X has $\mathcal{P}$-IP, they intersect in X too.

$(a) \Leftrightarrow (e)$ is also observed in [**12**, Section 11]. □

PROPOSITION 4.7. *Let $\mathcal{A}$ be a family of subsets of X such that $\mathcal{F} \subseteq \mathcal{A}$. Then, the following are equivalent :*

(a) *X is an L_1-predual with $\mathcal{A}$-IP*
(b) *X is an $\mathcal{A}$-C-subspace of every superspace*
(c) *for every $A \in \mathcal{A}$, every pairwise intersecting family of closed balls in X with centres in A intersects.*

PROOF. (a) $\Rightarrow$ (b). Since X has the $\mathcal{A}$-IP, it is an $\mathcal{A}$-C-subspace of every superspace in which it is an ideal (Proposition 2.20) and since X is an L_1-predual, it is an ideal in every superspace [**16**, Proposition 1]. Thus (b) follows.

(b) $\Rightarrow$ (a). Since $\mathcal{F} \subseteq \mathcal{A}$, this is immediate.

(a) $\Rightarrow$ (c). This is similar too the proof of Theorem 4.6 (d) $\Rightarrow$ (a).

(c) $\Rightarrow$ (a). If every finite family of pairwise intersecting closed balls in X intersects, then X is an L_1-predual. And that X has the $\mathcal{A}$-IP follows from Proposition 2.18 (c). $\square$

Let $C(T, X)$ be the space of all X-valued bounded continuous functions on a topological space T equipped with the sup norm. We now characterize when $C(T, X)$ is a real L_1-predual. First we need the following lemma.

LEMMA 4.8. *Suppose Y is a subspace of a Banach space X and Y is a real L_1-predual. Let $A \subseteq Y$ be a compact set and $r : A \to \mathbb{R}^+$ be such that $\cap_{a\in A} B_X[a, r(a)] \neq \emptyset$. Let $y \in \cap_{a\in A} B_Y[a, r(a) + \varepsilon]$ for some $\varepsilon > 0$. Then there exists $z \in \cap_{a\in A} B_Y[a, r(a)]$ such that $\|y - z\| \leq \varepsilon$.*

PROOF. Since $\cap_{a\in A} B_X[a, r(a)] \neq \emptyset$, and intersection of intervals is an interval, for any $y^* \in B(Y^*)$, $\cap_{a\in A} B_{\mathbb{R}}[y^*(a), r(a)] \neq \emptyset$ and is a closed interval. As $y^*(y) \in \cap_{a\in A} B_{\mathbb{R}}[y^*(a), r(a) + \varepsilon]$ for any $y^* \in B(Y^*)$, the family $\{B_Y[y, \varepsilon], B_Y[a, r(a)] : a \in A\}$ is a weakly intersecting family of balls in Y. Since Y is a L_1-predual, $B_Y[y, \varepsilon] \cap \cap_{a\in A} B_Y[a, r(a)] \neq \emptyset$. $\square$

PROPOSITION 4.9. *A Banach space X is a real L_1-predual if and only if for each paracompact space T, $C(T, X)$ is a real L_1-predual.*

PROOF. Since X is 1-complemented in $C(T, X)$, hence a $\mathcal{K}$-C-subspace, by Proposition 4.4, if $C(T, X)$ is an L_1-predual, then so is X.

Conversely, suppose X is a real L_1-predual. Let $Z = C(T, X)$, $\{f_1, f_2, \ldots, f_n\} \subseteq Z$ and $r_1, r_2, \ldots, r_n > 0$ be such that the family $\{B_Z[f_i, r_i] : i = 1, \ldots, n\}$ intersects weakly. Then for each $t \in T$, the family $\{B_X[f_i(t), r_i] : i = 1, \ldots, n\}$ intersects weakly, and since X is a real L_1-predual, they intersect in X. Consider the multi-valued map $F : T \to X$ given by $F(t) = \cap_{i=1}^n B_X[f_i(t), r_i]$. Note for each t, $F(t)$ is a nonempty closed convex subset of X.

CLAIM : F is lower semicontinuous, that is, for each U open in X, the set $V = \{t \in T : F(t) \cap U \neq \emptyset\}$ is open in T.

Let $t_0 \in V$. Let $x_0 \in F(t_0) \cap U$. Let $\varepsilon > 0$ be such that $\|x - x_0\| < \varepsilon$ implies $x \in U$. Let W be an open subset of t_0 such that $t \in W$ implies $\|f_i(t) - f_i(t_0)\| < \varepsilon/2$ for all $i = 1, \ldots, n$. We will show that $W \subseteq V$.

Let $t \in W$. Then for any $i = 1, \ldots, n$, $\|x_0 - f_i(t)\| \leq \|x_0 - f_i(t_0)\| + \|f_i(t_0) - f_i(t)\| \leq r_i + \varepsilon/2$. Therefore, $x_0 \in \cap_{i=1}^n B_X[f_i(t), r_i + \varepsilon/2]$. By Lemma 4.8, there exists $z \in F(t) = \cap_{i=1}^n B_X[f_i(t), r_i]$ such that $\|x_0 - z\| \leq \varepsilon/2 < \varepsilon$. Then $z \in F(t) \cap U$, and hence, $t \in V$. This completes the proof of the claim.

Now since T is paracompact, by Michael's selection theorem, there exists $g \in Z$ such that $g(t) \in F(t)$ for all $t \in T$. It follows that $g \in \cap_{i=1}^n B_Z[f_i, r_i]$. □

REMARK 4.10. For T compact Hausdorff, this result follows from [**13**, Corollary 2, p 43]. But our proof is simpler.

5. Stability Results

In this section we consider some stability results. With a proof similar to [**3**, Proposition 14], we first observe that

PROPOSITION 5.1. *$\mathcal{K}$-IP is a separably determined property, i.e., if every separable subspace of a Banach space X have $\mathcal{K}$-IP, then X also has $\mathcal{K}$-IP.*

DEFINITION 5.2. [**10**] A subspace Y of a Banach space X is called a semi-L-summand if there exists a (nonlinear) projection $P : X \longrightarrow Y$ such that

$$\begin{aligned} P(\lambda x + Py) &= \lambda Px + Py, \text{ and} \\ \|x\| &= \|Px\| + \|x - Px\| \end{aligned}$$

for all $x, y \in X$, λ scalar.

In [**3**], it was shown that semi-L summands are $\mathcal{F}$-C-subspaces. Basically the same proof actually shows that

PROPOSITION 5.3. *A semi-L-summand is an $\mathcal{A}$-C-subspace for any $\mathcal{A}$.*

Our next result concerns proximinal subspaces.

DEFINITION 5.4. A subspace Z of a Banach space X is called proximinal if for every $x \in X$, there exists $z_0 \in Z$ such that $\|x - z_0\| = d(x, Z) = \inf_{z \in Z} \|x - z\|$.

The map $P_Z(x) = \{z_0 \in Z : \|x - z_0\| = \inf_{z \in Z} \|x - z\|\}$ is called the metric projection.

PROPOSITION 5.5. *Let $Z \subseteq Y \subseteq X$, Z proximinal in X.*

(*a*) *Let $\mathcal{A}$ be a family of subsets of Y/Z. Let $\mathcal{A}'$ be a family of subsets of Y such that for any $x \in X$ and $A \in \mathcal{A}$, there exists $A' \in \mathcal{A}'$ such that for any $a + Z \in A$, $\{a + P_Z(x - a)\} \cap A' \neq \emptyset$. Suppose Y is a $\mathcal{A}'$-C-subspace of X. Then Y/Z is a $\mathcal{A}$-C-subspace of X/Z.*

Let $\mathcal{S}$ be any of the families $\mathcal{F}$, $\mathcal{B}$ or $\mathcal{P}$.

(*b*) *If Y is a $\mathcal{S}(Y)$-C-subspace of X, then Y/Z is a $\mathcal{S}(Y/Z)$-C-subspace of X/Z.*

(*c*) *Suppose the metric projection has a continuous selection. Then, if Y is a $\mathcal{K}(Y)$-C-subspace of X, Y/Z is a $\mathcal{K}(Y/Z)$-C-subspace of X/Z.*

(*d*) *Let $Z \subseteq Y \subseteq X^*$, Z w^*-closed in X^*. If Y is a $\mathcal{S}(Y)$-C-subspace of X^*, then Y/Z is a $\mathcal{S}(Y/Z)$-C-subspace of X^*/Z, and hence, has the $\mathcal{S}(Y/Z)$-IP.*

(*e*) *Let X have the $\mathcal{S}(X)$-IP. Let $M \subseteq X$ be a reflexive subspace. Then X/M has the $\mathcal{S}(X/M)$-IP.*

PROOF. (*a*). Let $A \in \mathcal{A}$ and $x + Z \in X/Z$. Choose A' as above. Then, for $a + Z \in A$, there exists $z \in P_Z(x - a)$ (depending on x and a) such that $a + z \in A'$. Since Y is a $\mathcal{A}'$-C-subspace of X, there exists $y_0 \in Y$ such that $\|y_0 - a - z\| \leq \|x - a - z\|$ for all $a + Z \in A$. Clearly then $\|y_0 - a + Z\| \leq \|y_0 - a - z\| \leq \|x - a - z\| = \|x - a + Z\|$.

If $\mathcal{S}$ is the family under consideration in (b) and (c) above and $\mathcal{A} = \mathcal{S}(Y/Z)$, then for any choice of $\mathcal{A}'$ as above, $\mathcal{S}(Y) \subseteq \mathcal{A}'$. Hence, ($b$) and ($c$) follows from ($a$). For ($d$), we simply observe that any w*-closed subspace of a dual space is proximinal. And (e) follows from (d). □

As in [**3**, Corollary 4.6], we observe

Proposition 5.6. *Let $Z \subseteq Y \subseteq X$, Z proximinal in Y and Y is a semi-L-summand in X. Then Y/Z is a $\mathcal{P}$-C-subspace of X/Z.*

Let us now consider the c_0 or ℓ_p sums.

Theorem 5.7. *Let Γ be an index set. For all $\alpha \in \Gamma$, let Y_α be a subspace of X_α. Let X and Y denote resp. the c_0 or ℓ_p $(1 \leq p \leq \infty)$ sum of X_α's and Y_α's.*

(a) *For each $\alpha \in \Gamma$, let $\mathcal{A}_\alpha$ be a family of subsets of Y_α such that $\{0\} \in \mathcal{A}_\alpha$ and for any $A \in \mathcal{A}_\alpha$, there exists $B \in \mathcal{A}_\alpha$ such that $A \cup \{0\} \subseteq B$.*

Let $\mathcal{A}$ be a family of subsets of Y such that for any $\alpha \in \Gamma$, the α-section of any $A \in \mathcal{A}$ belongs to $\mathcal{A}_\alpha$.

Then Y is an $\mathcal{A}$-C-subspace of X if and only if for each $\alpha \in \Gamma$, Y_α is an $\mathcal{A}_\alpha$-C-subspace of X_α.

Let $\mathcal{S}$ be any of the families $\mathcal{F}$, $\mathcal{K}$, $\mathcal{B}$ or $\mathcal{P}$.

(b) *Y is a $\mathcal{S}(Y)$-C-subspace of X if and only if for any $\alpha \in \Gamma$, Y_α is a $\mathcal{S}(Y_\alpha)$-C-subspace of X_α.*

(c) *The $\mathcal{S}$-IP is stable under ℓ_p-sums $(1 \leq p \leq \infty)$.*

Proof. (a). The proof is very similar that of to [**3**, Theorem 4.7]. We omit the details.

(c). X_α has $\mathcal{S}$-IP if and only if X_α is a $\mathcal{S}$-C-subspace of some dual space Y_α^*. Now the ℓ_p-sum $(1 \leq p \leq \infty)$ of Y_α^*'s is a dual space. □

Remark 5.8. The result for $\mathcal{F}$-IP has already been noted by [**18**] with a much different proof. The stability of the $\mathcal{P}$-IP under ℓ_1-sums is noted in [**15**] again with a different proof.

[**18**] also notes that $\mathcal{F}$-IP is stable under c_0-sum. And Corollary 4.3 shows that c_0 has the $\mathcal{K}$-IP. However, we do not know if the $\mathcal{K}$-IP is stable under c_0-sums. As for the $\mathcal{B}$-IP or $\mathcal{P}$-IP, we now show that c_0-sum of any infinite family of Banach spaces lacks the $\mathcal{B}$-IP, and therefore, also the $\mathcal{P}$-IP. This is quite similar to Example 2.9.

Proposition 5.9. *Let Γ be an infinite index set. For any family of Banach spaces X_α, $\alpha \in \Gamma$, $X = \oplus_{c_0} X_\alpha$ lacks the $\mathcal{B}$-IP.*

Proof. For each $\alpha \in \Gamma$, let x_α be an unit vector in X_α and define $e_\alpha \in X$ by

$$(e_\alpha)_\beta = \begin{cases} x_\alpha & \text{if} \quad \beta = \alpha \\ 0 & \text{otherwise} \end{cases}$$

Then the set $A = \{e_\alpha : \alpha \in \Gamma\}$ is bounded and the balls $B_{X^{**}}[e_\alpha, 1/2]$ intersect at the point $(1/2 x_\alpha) \in X^{**}$, but the balls $B_X[e_\alpha, 1/2]$ cannot intersect in X. □

Remark 5.10. As before, taking $A \cup \{0\}$, it follows that X lacks the $\mathcal{A}$-IP even for $\mathcal{A}$ = weakly compact sets.

Coming to function spaces, we note the following general result.

PROPOSITION 5.11. *Let Y be a subspace of a Banach space X and $\mathcal{A}$ be a family of subsets of Y.*

(a) *For any topological space T, if $C(T,Y)$ is a $\mathcal{A}$-C-subspace of $C(T,X)$, then Y is a $\mathcal{A}$-C-subspace of X. Moreover, if $C(T,X)$ has $\mathcal{A}$-IP, X has $\mathcal{A}$-IP.*

(b) *Let (Ω,Σ,μ) be a probability space. If for some $1 \le p < \infty$, $L^p(\mu,Y)$ is a $\mathcal{A}$-C-subspace of $L^p(\mu,X)$, then Y is a $\mathcal{A}$-C-subspace of X. Moreover, if $L^p(\mu,X)$, has $\mathcal{A}$-IP, then X has $\mathcal{A}$-IP.*

PROOF. For (a) and (b), let $F(X)$ denote the corresponding space of functions and identify X with the constant functions. In (a), point evaluation and in (b), integral over Ω gives us a norm 1 projection from $F(X)$ onto X. Thus X inherits $\mathcal{A}$-IP from $F(X)$.

Now suppose $F(Y)$ is an $\mathcal{A}$-C-subspace of $F(X)$. Let $P : F(Y) \to Y$ be the above norm 1 projection. Let $x \in X$ and $A \in \mathcal{A}$. Then, there exists $g \in F(Y)$ such that $\|g-a\| \le \|x-a\|$ for all $a \in A$. Let $y = Pg$. Then, $\|y-a\| \le \|g-a\| \le \|x-a\|$ for all $a \in A$. □

The following Proposition was proved in [**3**].

PROPOSITION 5.12. (a) *Let X has Radon Nikodym Property and is 1-complemented in Z^* for some Banach space Z. Then for $1 < p < \infty$, $L^P(\mu,X)$ is 1-complemented in $L^q(\mu,Y)^*$ $(1/p+1/q=1)$, and hence has the $\mathcal{P}$-IP.*

(b) *Suppose X is separable and 1-complemented in X^{**} by a projection P that is w^*-w universally measurable. Then for $1 \le p < \infty$ $L^P(\mu,X)$ is 1-complemented in $L^q(\mu,X^*)^*$ $(1/p+1/q=1)$, and hence has the $\mathcal{P}$-IP.*

Since the $\mathcal{B}$-IP or $\mathcal{P}$-IP is inherited by 1-complemented subspaces and c_0 lacks the $\mathcal{B}$-IP, the next result follows essentially from the arguments of [**17**].

PROPOSITION 5.13. (a) *Let X be a Banach space containing c_0 and let Y be any infinite dimensional Banach space. Then $X \otimes_\varepsilon Y$ fails the $\mathcal{B}$-IP and $\mathcal{P}$-IP.*

(b) *If $C(K,X)$ has the $\mathcal{B}$-IP, then either K is finite or X is finite dimensional. $C(K,X)$ has the $\mathcal{P}$-IP if and only if either (i) K is finite and X has the $\mathcal{P}$-IP or (ii) X is finite dimensional and K is extremally disconnected.*

(c) *For any nonatomic measure space (Ω,Σ,μ) and a Banach space X containing c_0, $L^1(\mu,X)$ fails the $\mathcal{B}$-IP.*

In the next Proposition, we prove a partial converse of Proposition 5.11(a) when Y is finite dimensional and K is compact and extremally disconnected.

PROPOSITION 5.14. *Let $\mathcal{S}$ be any of the families $\mathcal{F}$, $\mathcal{K}$, $\mathcal{B}$ or $\mathcal{P}$. Let Y be a finite dimensional a $\mathcal{S}(Y)$-C-subspace of a Banach space X. Then for any extremally disconnected compact space K, $C(K,Y)$ is a $\mathcal{S}(C(K,Y))$-C-subspace of $C(K,X)$.*

PROOF. We argue similar to the proof of [**3**, Proposition 4.11]. Let K be homeomorphically embedded in the Stone-Cech compactification $\beta(\Gamma)$ of a discrete set Γ and let $\phi : \beta(\Gamma) \to K$ be a continuous retract. Let $A \in \mathcal{S}(C(K,Y))$ and $g \in C(K,X)$. Note that since Y is finite dimensional, by the defining property of $\beta(\Gamma)$, any Y-valued bounded function on Γ has a norm preserving extension in $C(\beta(\Gamma), Y)$. Thus $C(\beta(\Gamma), Y)$ can be identified with $\bigoplus_{\ell_\infty(\Gamma)} Y$. Lift A to this space. In view of Theorem 5.7, this space is $\mathcal{S}(Y)$-C-subspace of $\oplus_\infty X$. This latter space contains $C(\beta(\Gamma), X)$. Thus by composing the functions with ϕ, we get a $f \in C(K, Y)$ such that $\|f - h\| \leq \|g - h\|$ for all $h \in A$. Hence the result. □

And now for a partial converse of Proposition 5.11(b).

Theorem 5.15. *Let Y be a separable subspace of X. If Y is a $\mathcal{P}$-C-subspace of X, then for any standard Borel space Ω and any σ-finite measure μ, $L_p(\mu, Y)$ is a $\mathcal{P}$-C-subspace of $L_p(\mu, X)$.*

Proof. Let $f \in L_p(\mu, X)$. Since Y is a $\mathcal{P}$-C-subspace of X, for each $x \in X$, $\cap_{y \in Y} B_Y[y, \|x - y\|] \neq \emptyset$.

Define a multi-valued map $F : \Omega \longrightarrow Y$, by

$$F(t) = \begin{cases} \bigcap_{y \in Y} B_Y[y, \|f(t) - y\|] & \text{if} \quad f(t) \in X \setminus Y \\ \{f(t)\} & \text{if} \quad f(t) \in Y \end{cases}$$

Let $G = \{(t, z) : z \in F(t)\}$ be the graph of F.

Claim : G is a measurable subset of $\Omega \times Y$.

To establish the claim, we show that G^c is measurable. Since Y is separable, let $\{y_n\}$ be a countable dense set in Y. Observe that $z \notin F(t)$ if and only if either $f(t) \in Y$ and $z \neq f(t)$ or $f(t) \in X \setminus Y$ and there exists y_n such that $\|z - y_n\| > \|f(t) - y_n\|$. And hence,

$$G^c = \Big\{ f(t) \in Y \text{ and } z \neq f(t) \Big\} \cup \bigcup_{n \geq 1} \Big\{ f(t) \in X \setminus Y \text{ and } \|z - y_n\| > \|f(t) - y_n\| \Big\}$$

is a measurable set.

By von Neumann selection theorem, there is a measurable function $g : \Omega \longrightarrow Y$ such that $(t, g(t)) \in G$ for almost all $t \in \Omega$.

Observe that $\|g(t)\| \leq \|f(t)\|$ for almost all t. Hence $g \in L_p(\mu, Y)$. Also for any $h \in L_p(\mu, Y)$ we have $\|g(t) - h(t)\| \leq \|f(t) - h(t)\|$ for almost all t. Thus, $\|g - h\|_p \leq \|f - h\|_p$ for all $h \in L_p(\mu, Y)$. □

Question 5.16. *Suppose Y is a separable $\mathcal{K}$-C-subspace of X. Let (Ω, Σ, μ) be a probability space. Is $L^p(\mu, Y)$ a $\mathcal{K}$-C-subspace of $L^p(\mu, X)$?*

Remark 5.17. This question was answered in positive in [**3**] for $\mathcal{F}$-C-subspaces and we did it for $\mathcal{P}$-C-subspaces. Both the proofs are applications of von Neumann selection Theorem. The problem here is for a compact set A in $L^p(\mu, Y)$ and $\omega \in \Omega$ the set $\{f(\omega) : f \in A\}$ need not be compact in Y.

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