Strongly Proximinal Subspaces in Banach Spaces

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Abstract. We give descriptions of SSD- and QP-points in $C(K)$-spaces and use this to characterize strongly proximinal subspaces of finite codimension in $L_1(\mu)$. We provide some natural class of examples of strongly proximinal subspaces which are not necessarily finite codimensional. We also study transitivity of strong proximinal subspaces of finite codimension.

1. Introduction

Let X be a Banach space and Y closed subspace of X. The metric projection onto Y is the set valued map defined by $P_Y(x) = \{y \in Y : \|x - y\| = \text{dist}(x, Y)\}$ for $x \in X$. Y is said to be proximinal if $P_Y(x) \neq \emptyset$ for all $x \in X$.

For a Banach space X, we denote the closed unit ball and the unit sphere by B_X and S_X respectively. We restrict ourselves to real scalars. All subspaces we consider are assumed to be closed.

In [**9, 10**] the following stronger version of proximinality was considered:

Definition 1.1. Let Y be a closed subspace in a Banach space X and $x \in X$. For $\delta > 0$, consider the following set.

$$P_Y(x, \delta) = \{y \in Y : \|x - y\| < d(x, Y) + \delta\}.$$

A proximinal subspace Y is said to be strongly proximinal at $x \in X$ if given $\varepsilon > 0$ there exists a $\delta > 0$ such that

$$P_Y(x, \delta) \subseteq P_Y(x) + \varepsilon B_Y.$$

It is to mention here that Vlasov studied the same notion under the name H-set (see [**17**]).

In this paper we put together some new results related to strongly proximinal subspaces in Banach spaces. We divide the main contents of this paper in three sections. Section 2 contains descriptions of SSD-points and QP-points (see Definition 1.2 below) in $C(K)$-spaces and characterization of strongly proximinal subspaces of finite codimension in $L_1(\mu)$. In section 3 we show that the notion of

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local U-proximinality studied by Lau in [**15**] is equivalent to strong proximinality. This answers a question raised in [**8**, Remark 2.2]. We then present a general class of example of strongly proximinal subspaces as subspaces which have $1\frac{1}{2}$-ball property (see below for details). In section 4 we discuss transitivity questions related to strongly proximinal subspaces of finite codimension.

We now describe motivation and content of each section in detail.

In [**8**] strongly proximinal subspaces of finite codimension in subspaces of c_0 were described. Similar description was given for $\mathcal{K}(\ell_2)$ - the space of all compact operators on ℓ_2 in [**10**]. Strongly proximinal subspaces of finite codimension in $C(K)$-spaces were characterized recently in [**4**].

In all three examples mentioned above, the strongly proximinal subspaces of finite codimension are described in terms of SSD-points and QP-points of the dual.

Definition 1.2. Let X be a Banach space.

(a) The norm $\|\cdot\|$ is said to be strongly subdifferentiable (in short SSD) at $x \in X$ if the one-sided limit

$$\lim_{t\to 0^+} \frac{\|x+th\|-\|x\|}{t}$$

exists uniformly for $h \in S_X$.

We say that x is an SSD-point of X if the norm is SSD at x. Recall that the duality map J_{X^*} of X is defined as

$$J_{X^*}(x) = \{g \in B(X^*) : \ g(x) = \|x\|\} \text{ for } x \in X.$$

In [**5**], it was shown that x is an SSD-point if and only if the duality map J_{X^*} is (norm-norm) upper semi-continuous at x, that is,

for every $\varepsilon > 0$, there exists $\delta > 0$ such that

$$J_{X^*}(z) \subseteq J_{X^*}(x) + \varepsilon B_{X^*} \text{ for every } z, \|z-x\| < \delta, \ \|z\| = \|x\|.$$

(b) We say that x is a QP-point of X if there exists $\delta > 0$ such that

$$J_{X^*}(z) \subseteq J_{X^*}(x) \text{ for every } z, \|z-x\| < \delta, \ \|z\| = \|x\|.$$

By [**5**], SSD-points of a dual Banach space X^* attain their norms on X. It was shown in [**9**, Lemma 3.3] that QP-points are SSD-points but the converse is not true.

Following two propositions describe the connections between strongly proximinal subspaces of finite codimension to QP- and SSD-points of the dual.

Proposition 1.3. [**9**] *Let Y be a finite codimensional subspace of a Banach space X. If Y is strongly proximinal then $Y^\perp$ is contained in the SSD-points of X^*.*

It remains an open question if the converse of Proposition 1.3 is true. The following proposition gives sufficient condition for strong proximinality of subspaces of finite codimension.

Proposition 1.4. [**9**] *Let Y be a finite codimensional subspace of a Banach space X such that $Y^\perp$ is contained in the QP points of X^*. Then Y is strongly proximinal.*

In section 2 of this paper we show that SSD- and QP-points are same in $C(K)$-spaces. As a corollary, we obtain description of all finite codimensional strongly proximinal subspaces in $L_1(\mu)$.

If Y is not of finite codimension in X, there exists no general criterion to check for proximinality or strong proximinality of Y in X. One sufficient condition, studied by Lau in [**15**] is the notion of locally U-proximinal and U-proximinal subspaces.

Definition 1.5. Let Y be a proximinal subspace of a Banach space X. We say that Y is locally U-proximinal if there exists a function $\varepsilon : (X \setminus Y) \times \mathbb{R}^+ \to \mathbb{R}^+$ such that for each fixed x, $\varepsilon(x,.)$ is continuous, increasing on δ and $\varepsilon(x,\delta) \to 0$ as $\delta \to 0$ and

$$(1+\delta)B_X \cap (B_{Y_x} + Y) \subseteq B_X + \varepsilon(x,\delta)B_Y \qquad \text{for all } \delta > 0$$

where $Y_x = \text{span }(Y, x)$.

Y is called U proximinal if the function ε can be chosen independent of x and

$$(1+\delta)B_X \cap (B_X + Y) \subseteq B_X + \varepsilon(\delta)B_Y$$

In section 3, we actually show that the notion of local U-proximinality studied in [**15**] is equivalent to strong proximinality. Our next class of examples of strongly proximinal subspaces are subspaces with $1\frac{1}{2}$-ball property.

Definition 1.6. $Y \subseteq X$ is said to have the $1\frac{1}{2}$-ball property if $x \in X$, $y \in Y$, $B(x,r) \cap Y \neq \emptyset$ and $\|x - y\| < r + s$, then the intersection $Y \cap B(x,r) \cap B(y,s) \neq \emptyset$.

For example of subspaces with $1\frac{1}{2}$-ball property see [**18**].

We show that if Y has $1\frac{1}{2}$-ball property in X then Y is strongly proximinal and the metric projection P_Y, considered as a single valued map from X to 2^Y, is 2-Lipschitz continuous with respect to the Hausdorff metric defined on 2^Y. Recall that for $A, B \in 2^Y$, the Hausdorff metric is given by

$$d_H(A,B) = \inf\{r > 0 : A \subseteq B + rB_Y \text{ and } B \subseteq A + rB_Y\}.$$

To the end of section 3 we consider stability result related to function modules.

Definition 1.7. A function module is a triple $(K, (X_k)_{k\in K}, X)$, where K is a non void compact Hausdorff space, $(X_k)_{k\in K}$ a family of Banach spaces and X, a closed subspace of the space $\Pi^{\infty}_{k\in K} X_k$ such that:

(a) X is a C(K)-module,
(b) For every $x \in X$, the map $k \to \|x(k)\|_k$ is upper semi continuous,
(c) $X_k = \{x(k) : x \in X\}$ for every $k \in K$,
(d) $\overline{\{k \in K \ : X_k \neq \{0\}\}} = K$.

Let $Y \subseteq X$. A triple $(K, (Y_k)_{k\in K}, Y)$ is called a sub-module of $(K, (X_k)_{k\in K}, X)$ if it is a function module over K and for each $k \in K$, $Y_k \subseteq X_k$.

Suppose X is a $C(K)$ module over a compact Hausdorff space K and Y is a closed subspace of X which is a sub-$C(K)$ module. Suppose further that each fiber Y_k, $k \in K$ has $1\frac{1}{2}$-ball property in X_k. We show that in this case Y is strongly proximinal in X and P_Y is 2-Lipschitz continuous with respect to the Hausdorff metric on 2^Y. As a corollary it follows that if Y has $1\frac{1}{2}$-ball property in X, then for every compact Hausdorff space K, $C(K,Y)$ is strongly proximinal in $C(K,X)$ and the metric projection is 2-Lipschitz continuous in Hausdorff metric.

In section 4 we consider transitivity of the relation 'strongly proximinal subspace of finite codimension'. The transitivity question in this context is the following.

Suppose Y and M are finite codimensional subspaces of X such that $Y \subseteq M \subseteq X$; Y is strongly proximinal in M and M is strongly proximinal in X. When is Y strongly proximinal in X?.

By [**13**] it follows that for strongly proximinal subspaces of finite codimension in c_0 transitivity holds. Similarly it was shown in [**4**] that transitivity holds for strongly proximinal subspaces of finite codimension in $C(K)$. Here we give an example to show that the same fails in ℓ_1. We then consider transitivity of proximinal and strongly proximinal subspaces of finite codimension in $C(K)$. In particular we show that there exist $Y,\ M$ subspaces of finite codimension in $C(K)$ such that Y is a strongly proximinal in M and M is proximinal in X but Y is not proximinal in $C(K)$.

2. Strongly proximinal subspaces in $L_1(\mu)$

We first show that the SSD and QP-points coincide in $C(K)$-spaces.

Let $f \in C(K)$. Consider the following set.

$$P_f = \{k \in\ K : |f(k)| = 1\,\}.$$

Theorem 2.1. *Let $f \in C(K)$ with $\|f\| = 1$. Then the following assertions are equivalent.*

(a) *P_f is a clopen set.*
(b) *f is a QP-point.*
(c) *f is an SSD-point.*

Proof. $(a) \Rightarrow (b)$ We first note that for any $g \in C(K)$ we can write,

$$\text{ext}J_{C(K)^*}(g) = E_1(g) \cup E_2(g),$$

where $E_1(g) = \{\delta_k : g(k) = 1\}$ and $E_2(g) = \{-\delta_k : g(k) = -1\}$. Also note that $J_{C(K)^*}(g)$ is the w^*-closed convex hull of $\text{ext}J_{C(K)^*}(g)$.

For $n \in \mathbb{N}$ we take $A_n = \{k \in K : |f(k)| > 1 - \frac{1}{n}\}$. Then $P_f = \cap_{n\geq 1} A_n$. Since P_f is a clopen set, there exists n_0 such that for all $n \geq n_0,\ A_n = P_f$.

Take $\delta < \frac{1}{n_0}$. Suppose $g \in S_{C(K)}$ is such that $\|f - g\| < \delta$. If $\delta_k \in E_1(g)$ then $g(k) = 1$ and hence $f(k) > 1-\delta > 1-\frac{1}{n_0}$. Thus $k \in P_f$. Similarly if $\delta_k \in E_2(g)$ then $f(k) < -1 + \delta < -1 + \frac{1}{n_0}$ and hence $k \in P_f$. This gives $\text{ext}J_{C(K)^*}(g) \subseteq J_{C(K)^*}(f)$ and hence $J_{C(K)^*}(g) \subseteq J_{C(K)^*}(f)$. This shows f is a QP-point.

$(b) \Rightarrow (c)$ Follows from [**9**, Lemma 3.3].

$(c) \Rightarrow (a)$ It is straightforward to verify that if f is an SSD-point of $C(K)$ then so is $|f|$. Since $P_f = P_{|f|}$, without loss of generality we assume $f \geq 0$.

By continuity of f, P_f is a closed set. To show P_f is also open, we claim that there exists n_0 such that $A_n = P_f$ for all $n \geq n_0$.

If not, for each n we can find $k_n \in U_n = f^{-1}(1 - \frac{1}{n}, 1)$. Choose $h_n \in S_{C(K)}$ such that $h_n(k_n) = 1,\ h \geq 0$ and $h_n|_{U_n^c} = 0$. Taking $g_n = f + (1 - f)h_n$ we obtain that $g_n(k_n) = 1, 1 \geq g_n(k) \geq f(k)$ for all $k \in K$ and $g_n \to f$ in norm.

Let $\mu \in J_{C(K)^*}(f)$. Since $f \geq 0$, μ is supported on the set $\{k \in K : f(k) = 1\}$. Thus for any n, $\text{dist}(\delta_{k_n}, J_{C(K)^*}(f)) = 1$. But $\delta_{k_n} \in J_{C(K)^*}(g_n)$ and $g_n \to f$. This contradicts f is an SSD-point. $\square$

REMARK 2.2. In [**3**] Contreras, Paya and Werner obtained a characterization of SSD-points in a C^*-algebra as elements which has clopen spectrum. Theorem 2.1 is a special case of their results. However, the proof presented here is simple and straightforward.

For a finite measure space (T, μ), identifying $L_\infty(\mu)$ as a $C(K)$ space, and applying Theorem 2.1 we have that SSD and QP points coincide in $L_\infty(T, \mu)$. This immediately provides the following characterization of strongly proximinal subspaces on $L_1(\mu)$.

COROLLARY 2.3. *Let $Y \subseteq L_1(\mu)$ be of finite codimension. Then the following assertions are equivalent.*

(a) *Y is strongly proximinal.*

(b) *Every closed subspace Z of finite codimension with $Y \subset Z \subseteq X$ is strongly proximinal.*

(c) *Every hyperplane containing Y is strongly proximinal.*

(d) $Y^\perp \subseteq \{f \in L_\infty(\mu) : f$ *is an SSD point* $\} = \{f \in L_\infty(\mu) : f$ *is a QP point* $\}$.

REMARK 2.4. It was shown in [**12**] that for any $n \geq 2$, there exists a closed subspace $Y \subseteq L_1(T, \mu)$ of codimension n such that every closed subspace Z satisfying $Y \subseteq Z \subseteq L_1(\mu)$, $Y \neq Z$, is proximinal but Y is not proximinal in $L_1(\mu)$. Corollary 2.3 shows, however, that the situation is completely different for strongly proximinal subspaces.

3. More Examples of Strongly Proximinal Subspaces

In this section we provide sufficient conditions for strong proximinality for subspaces which are not necessarily of finite codimension.

We begin by showing that locally U-proximinal subspaces considered in [**15**] are same as strongly proximinal subspaces. This answers a question raised in [**9**].

PROPOSITION 3.1. *A subspace Y of a Banach space X is locally U-proximinal if and only if Y is strongly proximinal.*

PROOF. It was noted in [**9**, Remark 2.2] that locally U-proximinal subspaces are strongly proximinal. We need to show the converse.

Suppose Y is strongly proximinal. Let $x \in X \setminus Y$ be such that $\operatorname{dist}(x, Y) = 1 = \|x\|$. Choose $f \in S_{Y_x^*}$ such that $f(x) = 1$ and $Y = \ker f$.

For $\delta > 0$ define,

$$\varepsilon(x, \delta) = d_H((1+\delta)B_{Y_x} \cap f^{-1}(1), B_{Y_x} \cap f^{-1}(1)).$$

It follows by [**15**, Proposition 2.2] that

$$\varepsilon(x, \delta) = \inf\{r > 0 : \ (1+\delta)B_X \cap (B_{Y_x} + Y) \subseteq B_X + rB_Y\}.$$

Also, by [**9**] (see pages 110 and 111) one gets

$$x - P_Y(x) = B_{Y_x} \cap f^{-1}(1); \ x - P_Y(x, \delta) = (1+\delta)B_{Y_x} \cap f^{-1}(1)$$

so that

$$\varepsilon(x, \delta) = d_H(x - P_Y(x), \ x - P_Y(x, \delta)) = \inf\{r > 0 : \ P_Y(x, \delta) \subseteq P_Y(x) + rB_Y\}.$$

Strongly proximinality of Y now implies $\varepsilon(x, \delta)$ is a continuous increasing function of δ such that $\varepsilon(x, \delta) > 0$ for every $x \in X \setminus Y$, $\delta > 0$ and $\varepsilon(x, \delta) \to 0$ as $\delta \to 0$. The proof is complete. ∎

It was shown in [**15**, Theorem 3.3] that if Y is a locally U-proximinal subspace in X and for each $\delta > 0$, $\varepsilon(\cdot, \delta)$ is an upper semi-continuous function on $X \setminus Y$, then P_Y is continuous in Hausdorff metric. It follows from this that if Y is U-proximinal then P_Y is continuous in Hausdorff metric and has continous selection $s : X \to Y$ (see [**15**, Theorem 3.4 and Corollary 3.5]).

Here are some examples of locally U-proximinal and U-proximinal subspaces. See [**15**] for details.

EXAMPLE 3.2. (*a*) *$M \subseteq X$, M finite dimensional. Then M is locally U-proximinal.*

(*b*) *Let X be a locally uniformly convex Banach space. Then every proximinal subspace of X is locally U-proximinal.*

(*c*) *Let X be a uniformly convex Banach space. Then every proximinal subspace is U-proximinal.*

Our next class of examples of strongly proximinal subspaces are subspaces with the $1\frac{1}{2}$-ball property.

PROPOSITION 3.3. *Let X be a Banach space and $Y \subseteq X$ a subspace with $1\frac{1}{2}$-ball property. Then Y is strongly proximinal and P_Y is 2-Lipschitz continuous in Hausdorff metric.*

PROOF. The argument for strong proximinality of Y is essentially same as in the proof of [**11**, Proposition II.1.1]. For completeness, we present the proof here.

Let $x \in X$, $\operatorname{dist}(x, Y) = d$. Let $\varepsilon > 0$ be given. Take $y \in P_Y(x, \varepsilon)$. We will produce an element $y_0 \in P_Y(x)$ such that $\|y - y_0\| < \varepsilon$. To do this consider the balls $B(x, d + \varepsilon/2)$ and $B(y, \varepsilon/2)$. Applying $1\frac{1}{2}$-ball property of Y we get $y_1 \in Y$ such that $\|x - y_1\| < d + \varepsilon/2$ and $\|y_1 - y\| < \varepsilon/2$. Now consider the balls $B(x, d + \varepsilon/2^2)$ and $B(y, \varepsilon/2^2)$ and find $y_2 \in Y$ such that $\|x - y_2\| < d + \varepsilon/2^2$ and $\|y_2 - y_1\| < \varepsilon/2^2$.

Continuing, we get a a sequence $(y_n) \in Y$ such that $\|x - y_n\| < d + \varepsilon/2^n$ and $\|y_n - y_{n-1}\| < \varepsilon/2^n$. Thus the sequence (y_n) is Cauchy and converges to some $y_0 \in Y$. Clearly $y_0 \in P_Y(x)$ and $\|y_0 - y\| < \sum_{n=1}^{\infty} \varepsilon/2^n = \varepsilon$.

That P_Y is 2-Lipschitz continuous in Hausdorff metric follows from the same line of argument as in [**11**, Proposition II.1.8 and Theorem II.1.9] (see also Remarks on $1\frac{1}{2}$-ball property in page 95 of [**11**]). □

REMARK 3.4. As mentioned in [**11**, Page 55], the example of c_0 in ℓ_∞ shows that the constant 2 is optimal in Proposition 3.3 (recall that c_0 is an M-ideal in ℓ_∞ and therefore satisfies $1\frac{1}{2}$-ball property). Also as illustrated in [**19**], (see also [**11**, Page 95]), there exists a Banach space X and closed subspace $Y \subseteq X$ satisfying $1\frac{1}{2}$-ball property but P_Y has no Lipschitz selection. Note that P_Y being 2-Lipschitz continuous in Hausdorff metric, there always exists a continuous selection.

We now consider function modules. It is a remarkable result by E. Behrends [**1**] that every Banach space has a representation as a function module over a choice of a compact set K. The following theorem describes a general class of stability result for subspaces with $1\frac{1}{2}$-ball property.

THEOREM 3.5. *Let $(K, (X_k)_{k \in K}, X)$ be a function module and $(K, (Y_k)_{k \in K}, Y)$ is a sub-module such that for each $k \in K$ for which $X_k \neq \{0\}$, Y_k has $1\frac{1}{2}$-ball property in X_k. Then Y is a strongly proximinal subspace of X and the metric projection P_Y is 2-Lipschitz continuous in Hausdorff metric.*

Proof. By Proposition 3.3 it follows that for every $k \in K$ whenever $X_k \neq \{0\}$, Y_k is a strongly proximinal subspace of X_k. Let $x \in X$ be such that $\text{dist}(x, Y) = 1$ and $\varepsilon > 0$ be given. We show that $P_Y(x, \varepsilon) \subseteq P_Y(x) + \varepsilon B_Y$.

Let $y \in P_Y(x, \varepsilon)$. For each $k \in K$, consider the balls $B(x(k), 1 + \frac{\varepsilon}{2})$ and $B(y(k), \frac{\varepsilon}{2})$. Since $\text{dist}(x, Y) = 1$, $B(x(k), 1 + \frac{\varepsilon}{2}) \cap Y \neq \emptyset$ and thus applying $1\frac{1}{2}$-ball property of Y_k, there exists $y'(k) \in Y_k$ such that $\|x(k) - y'(k)\|_k < 1 + \frac{\varepsilon}{2}$ and $\|y(k) - y'(k)\|_k < \frac{\varepsilon}{2}$. Let $y_k \in Y$ be such that $y_k(k) = y'(k)$. Take,

$$V_k = \{s \in K : \|x(s) - y_k(s)\|_s < 1 + \frac{\varepsilon}{2}\}.$$

By upper semi-continuity of the map $s \to \|(x - y_k)\|_s$, it follows V_k is an open set containing k. The collection $\{V_k : k \in K\}$ forms an open covering of the compact set K. Hence we can find $\{k_1, k_2, \cdots, k_n\}$ such that $K = \cup_{i=1}^n V_{k_i}$. Let $\{f_i\}_{i=1}^n$ be a partition of unity subordinate to $\{V_{k_i}\}_{i=1}^n$. Put $y_1 = \sum_{i=1}^n f_i y_{k_i}$. Then y_1 satisfies $\|x - y_1\| < 1 + \frac{\varepsilon}{2}$ and $\|y - y_1\| < \frac{\varepsilon}{2}$.

Thus $y_1 \in P_Y(x, \frac{\varepsilon}{2})$. Repeating the argument above with the balls $B(x(k), 1 + \frac{\varepsilon}{2^2})$ and $B(y_1(k), \frac{\varepsilon}{2^2})$, we can find $y_2 \in P_Y(x, 1 + \frac{\varepsilon}{2^2})$ such that $\|y_1 - y_2\| < \frac{\varepsilon}{2^2}$. Continuing, we obtain a sequence $(y_n) \subseteq Y$ such that $y_n \in P_Y(x, 1 + \frac{\varepsilon}{2^n})$ and $\|y_{n-1} - y_n\| < \frac{\varepsilon}{2^n}$. Thus the sequence (y_n) is Cauchy and it has a limit point $y_0 \in Y$. Clearly $y_0 \in P_Y(x)$ and $\|y - y_0\| < \sum_n \frac{\varepsilon}{2^n} = \varepsilon$. This shows Y is strongly proximinal.

To show P_Y is 2-Lipschitz continuous in Hausdorff metric let $x_1, x_2 \in X$ and $\varepsilon > 0$. Choose $z \in P_Y(x_1)$. We construct a Cauchy sequence $(y_n) \subseteq Y$ such that $\|x_2 - y_n\| \leq \text{dist}(x_2, Y) + \frac{\varepsilon}{2^n}$ and $\|z - y_n\| < 2\|x_1 - x_2\| + \varepsilon$. Note that for $k \in K$, $\|x_2(k) - z(k)\| \leq \|x_1 - x_2\| + \text{dist}(x_1, Y) \leq 2\|x_1 - x_2\| + \text{dist}(x_2, Y)$. Thus we can apply $1\frac{1}{2}$-ball property of Y_k in X_k to the balls $B(x_2(k), \text{dist}(x_2, Y) + \frac{\varepsilon}{2})$ and $B(z(k), 2\|x_1 - x_2\| + \frac{\varepsilon}{2})$ to get a $y'(k) \in Y_k$ such that

$$\|x_2 - y'(k)\|_k < \text{dist}(x_2, Y) + \frac{\varepsilon}{2}$$

and

$$\|z(k) - y'(k)\|_k < 2\|x_1 - x_2\| + \frac{\varepsilon}{2}.$$

A similar argument as in the first part of the proof shows that there exists $y_1 \in Y$ such that $\|x_2 - y_1\| < \text{dist}(x_2, Y) + \frac{\varepsilon}{2}$ and $\|z - y\| < 2\|x_1 - x_2\| + \frac{\varepsilon}{2}$. Now we repeat the argument with balls $B(x_2(k), \text{dist}(x_2, Y) + \frac{\varepsilon}{2^2})$ and $B(y_1, \frac{\varepsilon}{2^2})$. This gives us $y_2 \in Y$ such that $\|x_2 - y_2\| < \text{dist}(x_2, Y) + \frac{\varepsilon}{2^2}$ and $\|y_1 - y_2\| < \frac{\varepsilon}{2^2}$. Note that this also implies

$$\begin{aligned} \|z - y_2\| &\leq \|z - y_1\| + \|y_1 - y_2\| \\ &\leq 2\|x_1 - x_2\| + \frac{\varepsilon}{2} + \frac{\varepsilon}{2^2} \\ &\leq 2\|x_1 - x_2\| + \varepsilon. \end{aligned}$$

An induction argument as in the first part of the proof gives us the desired conclusion. □

For a Banach space X and a compact Hausdorff space K, the space $C(K, X)$ is a natural function module over K where for each $k \in K$, $X_k = X$. The following corollary is immediate from Theorem 3.5.

Corollary 3.6. *Let $Y \subseteq X$ has $1\frac{1}{2}$ ball property and K be a compact Hausdorff space. Then $C(K, Y)$ is strongly proximinal in $C(K, X)$ and the metric projection in 2-Lipschitz continuous in Hausdorff metric.*

Remark 3.7. It is proved in [**2**, Theorem 2.7] that even if Y is finite dimensional, $C(K, Y)$ need not be proximinal in $C(K, X)$. Thus in Corollary 3.6 the assumption that $1\frac{1}{2}$-ball property in X cannot be replaced by the assumption that Y is strongly proximinal in X.

4. Transitivity of strongly proximinal subspaces of finite codimension

In this section we consider transitivity properties of strongly proximinal subspaces of finite codimension. We say, in a Banach space X the relation 'strongly proximinal subspace of finite codimension' is transitive if for any two subspaces Y, M, both of finite codimension and $Y \subseteq M \subseteq Z$, the property that Y is strongly proximinal in M and M is strongly proximinal in X implies that Y is strongly proximinal in X.

The following definition is from [**13**].

Definition 4.1. Let $f, g \in X^*$. We say f is strongly orthogonal to g if f attains its norm on $\ker g$.

A set $S \subseteq X^*$ is called orthogonally linear if for $f, g \in S$, f strongly orthogonal to g, we have $\text{span}\{f, g\} \subseteq S$.

Proposition 4.2. *Suppose in a Banach space X the set of QP-points is same as the set of SSD-points of X^*. If the relation 'strongly proximinal subspace of finite codimension' is transitive in X then the set of QP-points of X^* is orthogonally linear.*

Proof. Let $f, g \in X^*$ be QP-points of X^* such that f is strongly orthogonal to g. Take $M = \ker g$ and $Y = \ker g \cap \ker f$. Then M is a strongly proximinal subspace of X and since f attains its norm on M and its a QP-point X, $f|_M$ is a QP-point of M as well. Thus Y is also strongly proximinal in M. By transitivity of strong proximinality, Y is strongly proximinal in X. By Proposition 1.3 it follows that $Y^\perp \subseteq \{$ SSD points of $X^*\}$. Since by our assumption, SSD and QP-points coincide and $\text{span}\{f, g\} \subseteq Y^\perp$ we have the set of QP-points of X^* is orthogonally linear. $\square$

In the following we give an example that on ℓ_∞, the set of QP-points is not orthogonally linear. By Theorem 2.1 it follows that in ℓ_∞ SSD and QP-points coincide. Thus by Proposition 4.2 the transitivity of strong proximinality will fail in ℓ_1.

Example 4.3. Let $f = (0, 1, 1, 1, \cdots)$ and $g = (1, -\frac{1}{2}, -\frac{1}{3}, -\frac{1}{4}, \cdots) \in S_{\ell_\infty}$. It is easy to verify that f and g are QP-points of ℓ_∞. Also g attains its norm on $\ker f$. But $f + g = (1, 1 - \frac{1}{2}, 1 - \frac{1}{3}, 1 - \frac{1}{4}, \cdots)$ is clearly not a QP-point of ℓ_∞.

We now consider transitivity of proximinal and strongly proximinal subspaces of finite codimension in $C(K)$. It was shown in [**4**] that strong proximinality is transitive for finite codimensional subspaces in $C(K)$. We first give an example to show that this is not true for proximinal subspaces. We will need the following result by Garkavi which characterizes finite codimensional proximinal subspaces in $C(K)$. For $\mu \in C(K)^*$, we denote by $S(\mu)$ the support of the measure μ.

Theorem 4.4. [**6**] *Let Y be a closed subspace of finite codimension in $C(K)$. Then Y is proximinal if and only if the annihilator space $Y^\perp$ satisfies the following three conditions:*

(*a*) $S(\mu^+) \cap S(\mu^-) = \emptyset$ *for each* $\mu \in Y^\perp \setminus \{0\}$,

(*b*) μ *is absolutely continuous with respect to* ν *on* $S(\nu)$ *for every pair* μ, ν *in* $Y^\perp \setminus \{0\}$,

(*c*) $S(\nu) \setminus S(\mu)$ *is closed for each pair* μ, ν *in* $Y^\perp \setminus \{0\}$.

Let $NA(X)$ denote the set of norm attaining functionals on X. It is well known that (see [**7**]) if Y is a proximinal subspace of finite codimension in X, then $Y^\perp \subseteq NA(X)$.

Example 4.5. *There exist* Y, M *subspaces of finite codimension in* $C(K)$ *such that* Y *is a proximinal in* M *and* M *is proximinal in* X *but* Y *is not proximinal in* $C(K)$.

Select a sequence (k_n) in K with $k_n \neq k_m$ for $n \neq m$, such that $k_n \to k_0$. Let k' be a point different from the k_n's and k_0. Consider $\mu = \sum_{n=1}^{\infty} \frac{1}{2^n}\delta_{k_n}$ and $\nu = \frac{1}{2}(\delta_{k_0} - \delta_{k'})$ in $C(K)^*$. Take $M = \ker\mu$ in $C(K)$ and $Y = \ker\nu|_M$. We have $\mu \in NA(C(K))$ and $\nu \in NA(M)$ with $\|\mu\| = \|\nu\| = 1$. Thus M is proximinal in $C(K)$ and Y is proximinal in M. But ν is not absolutely continuous with respect to μ on $S(\mu)$. So by Theorem 4.4, Y is not proximinal in $C(K)$.

The following example show that we cannot mix proximinality and strong proximinality to obtain transitivity.

Example 4.6. *There exist* Y, M *subspaces of finite codimension in* $C(K)$ *such that* Y *is a strongly proximinal in* M *and* M *is proximinal in* X *but* Y *is not proximinal in* $C(K)$.

In Example 4.5 $S(\nu)$ is a finite set. By [**4**] it follows that ν is a QP-point of $C(K)^*$. Since $\nu|_M$ attains its norm over M. Hence $\nu|_M$ is a QP-point M^* as well. So Y is strongly proximinal in M but Y is not proximinal in $C(K)$.

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