BANACH SPACES WITH PROPERTY (M) AND THEIR SZLENK INDICES

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Abstract. We show that a separable Banach space with property (M^*) has a Szlenk index equal to ω_0, and a norm with an optimal modulus of asymptotic uniform smoothness. From this we derive a condition on the Szlenk functions of the space and its dual which characterizes embeddability into c_0 or an ℓ_p-sum of finite dimensional spaces. We also prove that two Lipschitz-isomorphic Orlicz sequence spaces contain the same ℓ_p-spaces.

1. Introduction

Following Kalton [9], we say that a Banach space X has property (M) if whenever $u, v \in S_X$ and $(x_n) \subseteq X$ is a w-null sequence in X, then

$$\limsup_n \|u + x_n\| = \limsup_n \|v + x_n\|.$$

Property (M) has the following dual version which was studied by Kalton and Werner in [10].

A Banach space is said to have Property (M^*) if whenever $u^*, v^* \in S_{X^*}$ and $(x_n^*) \subseteq X^*$ is a w^*-null sequence then

$$\limsup_n \|u^* + x_n^*\| = \limsup_n \|v^* + x_n^*\|.$$

It was shown in [10] that if X is a separable Banach space having Property (M^*), then X^* is separable and X has Property (M). Conversely, if X is a separable Banach space not containing ℓ_1 and has Property (M), then X has Property (M^*) and hence X^* is separable.

Suppose X is separable Banach space. Let $\mathcal{K}(X)$ and $\mathcal{L}(X)$ denote, respectively, the space of compact linear operators and the space of bounded linear operators on X. By [9, 10], it follows that if $\mathcal{K}(X)$ is an M-ideal in $\mathcal{L}(X)$ then X has Property (M^*) and the converse holds true if the metric compact approximation property is assumed (see [7, Chapter 6] for a detailed exposition of related results). While this gives a general class of examples of Banach spaces with Property (M), [9] also shows

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that a Banach space with a subsymmetric basis and separable dual has Property (M) if and only if X is isomorphic to an Orlicz sequence space h_F.

The purpose of this article is to study separable Banach spaces with Property (M) and their asymptotic modulus of uniform smoothness and asymptotic modulus of uniform convexity. We note that Property (M) is an isometric notion, and we will denote an M-norm on X by $\|\cdot\|_M$.

We recall that for a Banach space X the modulus of asymptotic uniform smoothness ρ_X and modulus of asymptotic uniform convexity δ_X are defined as follows.

Definition 1.1.

(a) For $x \in S_X$, $t > 0$, and $Y \subseteq X$ a closed subspace of finite codimension denote

$$\rho_X(x,t,Y) = \sup_{y \in Y, \|y\| \le t} \|x+y\| - 1$$

and

$$\rho_X(x,t) = \inf_{Y, \dim(X/Y) < \infty} \rho_X(x,t,Y).$$

Then define

$$\rho_X(t) = \sup_{x \in S_X} \rho_X(x,t).$$

X is said to be asymptotically uniformly smooth if $\lim\limits_{t \to 0} \dfrac{\rho_X(t)}{t} = 0$.

(b) For $x \in S_X$, $t > 0$, and $Y \subseteq X$ a closed subspace of finite codimension denote

$$\delta_X(x,t,Y) = \inf_{y \in Y, \|y\| \ge t} \|x+y\| - 1$$

and

$$\delta_X(x,t) = \sup_{Y, \dim(X/Y) < \infty} \delta_X(x,t,Y).$$

Then define

$$\delta_X(t) = \inf_{x \in S_X} \delta_X(x,t).$$

X is said to be asymptotically uniformly convex if $\delta_X(t) > 0$ for all $t > 0$.

(c) We define the modulus of w^*-asymptotic convexity of X by taking w^*-closed finite codimensional subspaces Y of X^* in the definition of δ_{X^*}, and we denote it by δ_X^*.

Note that the quantity δ_X^* corresponds to a property of X which is read on the dual space (through the w^* topology). If X is reflexive then the modulus of asymptotic uniform convexity of X^* coincide with the modulus of w^*-asymptotic convexity of X. It is known (see [4, Proposition 2.8]) that ρ_X and δ_X^* are equivalent to dual Young functions of each other. We are interested in power type estimates of these quantities. We say that ρ_X (respectively δ_X^*) has power type p if there exists a constant K such that $\rho_X(t) \le Kt^p$ (respectively $\delta_X^*(t) \ge Kt^p$) for all $t > 0$.

We now define the Szlenk index and Szlenk power type and explain how they relate to ρ and δ^*.

Let X be an infinite dimensional separable Banach space and $K \subseteq X^*$ a weak*-compact set. The Szlenk derivation on K is given by the following procedure. Let $\varepsilon > 0$ and set $F_0(\varepsilon) = K$. If $F_\alpha(\varepsilon)$ is defined, take

$$F_{\alpha+1}(\varepsilon) = \{x^* \in F_\alpha(\varepsilon) : \text{ for any weak*-neighborhood } V \text{ of } x^*, \operatorname{diam}(V \cap F_\alpha(\varepsilon)) \geq \varepsilon\}.$$

If $\alpha < \omega_1$ is a limit ordinal define $F_\alpha(\varepsilon) = \cap_{\beta<\alpha} F_\beta(\varepsilon)$.

Let $F_0(\varepsilon) = B_{X^*}$. We define $Sz(X, \varepsilon)$, the Szlenk index of X at ε to be the least ordinal $\alpha < \omega_1$ such that $F_\alpha(\varepsilon) = \emptyset$ if such an ordinal exists. Otherwise we put $Sz(X, \varepsilon) = \omega_1$. The Szlenk index of X is defined by $Sz(X) = \sup_{\varepsilon>0} Sz(X, \varepsilon)$. If $Sz(X, \varepsilon) < \omega_0$ for each $\varepsilon > 0$, we have $Sz(X) = \omega_0$ and then we say X has minimal Szlenk index.

It is a classical result of W. Szlenk that $Sz(X) < \omega_1$ if and only if X^* is separable. By compactness, $Sz(X, \varepsilon)$ is always a successor ordinal.

When X has minimal Szlenk index, the Szlenk power type of X is defined as

$$p(X) = \inf\{p : \sup_{\varepsilon>0} \varepsilon^p Sz(X, \varepsilon) < \infty\}.$$

A submultiplicativity argument shows that every space X with minimal Szlenk index has a finite Szlenk power type (see [4]).

The convex Szlenk index $Cz(X, \varepsilon)$ is defined in [4] along the same lines as the Szlenk index, except that in the derivation each time we take the w^*-closed convex hull of the sets. It is obvious that $Sz(X, \varepsilon) \leq Cz(X, \varepsilon)$. However it can be shown that for every space X with minimal Szlenk index, the Szlenk power types for Sz and Cz are equal [4, Corollary 4.6].

Up to the notation, the following result was established in [4].

Theorem 1.2. [4, Theorem 4.7] *Let X be a separable Banach space such that $Sz(X) = \omega_0$. There exists an absolute constant $C < 19200$ such that for any fixed $t \in (0,1)$ there is a 2-equivalent norm $|||\cdot|||$ on X such that $\delta^*_{X,|||\cdot|||}(t) \geq Cz(X, t/C)^{-1}$.*

This result yields to a renorming (obtained through a series of dual norms) which works for every $t \in (0,1]$.

Theorem 1.3. [4, Theorem 4.8] *Let X be a separable Banach space such that $Sz(X) = \omega_0$ and let $p(X)$ be the Szlenk power type of X. Then for any $q > p(X)$, there exists an equivalent norm $|||\cdot|||$ on X and a constant $C > 0$ such that for all $t \in (0,1]$, $\delta^*_{X,|||\cdot|||}(t) \geq Ct^q$.*

In Proposition 2.4 of this paper we show that the homogeneity provided by property (M^*) implies that the norms with this property have optimal modulus of asymptotic uniform smoothness and in the dual optimal modulus of w^*-asymptotic uniform convexity.

It was shown in [5] (see also [8]) that a separable Banach space X embeds in c_0 if and only if $\rho_X(t) = 0$ for some $t \in (0,1)$ or equivalently if and only if $\delta^*_X(t) \geq ct$ for some constant c. Note that the later condition implies X has summable Szelenk index (see [4, Theorem 4.10]). However, the example of the classical Tsirelson space T shows that summability of the Szlenk power type is not sufficient for embeddability in c_0. Similarly by [8], if X is a separable reflexive space such that X has renorming with both modulus of asymptotic uniform smoothness and modulus of asymptotic uniform convexity of power type p for some $1 < p < \infty$ then X embeds in ℓ_p-sum of finite dimensional spaces and consequently $p(X) = p$.

As an application of Proposition 2.4 we show that for a separable Banach space X with Property (M^*) the embeddability in c_0 or ℓ_p-sum of finite dimensional Banach spaces is completely characterized by the Szlenk functions of X and X^*. Namely we show that if X is a separable Banach space with property (M^*) then X embeds in c_0 or ℓ_p-sum of finite dimensional Banach spaces if and only if for some renorming of X the Szlenk power types $p(X)$ and $p(X^*)$ are actually attained and $p(X)^{-1} + p(X^*)^{-1} = 1$ (where $p(X^*)$ is to interpreted as ∞ when $p(X) = 1$ and this is exactly the case where X embeds in c_0).

A natural class of examples of separable Banach spaces with Property (M^*) are Orlicz sequence spaces h_F. As a second application of Proposition 2.4 we show that if two Orlicz sequence spaces h_F and h_G are Lipschitz isomorphic then they contain the same ℓ_p-spaces.

2. Main Results

We first show that if a Banach space X has Property (M^*), then for all $x \in S_X$, $\rho(x,t) = \rho(t)$. Similarly for all $x^* \in S_{X^*}$, $\delta^*(x^*,t) = \delta^*(t)$. The following lemma also gives alternative descriptions of these two quantities which are more useful for applications. We first introduce the following constants, as in section 2 of [4].

$\lambda_X(t)$ is the least constant such that

$$\limsup \|x + x_n\| \leq 1 + \lambda_X(t)$$

whenever $\|x\| = 1,\ \|x_n\| \leq t,\ x_n \to 0$ weakly.

$\theta_X(t)$ is the greatest constant such that

$$\liminf \|x^* + x^*_n\| \geq 1 + \theta_X(t)$$

whenever $\|x^*\| = 1,\ \|x_n^*\| \geq t,\ x_n^* \to 0$ weak*.

Lemma 2.1. *Let X be a separable Banach space with Property (M^*) and $\|\cdot\|_M$ be a norm with Property (M). Then for all $x \in S_{X,\|\cdot\|_M}$ we have $\rho_{\|\cdot\|_M}(x,t) = \lambda_{\|\cdot\|_M}(t)$ and for all $x^* \in S_{X^*,\|\cdot\|_M}$, $\delta^*_{\|\cdot\|_M}(x^*,t) = \theta_{\|\cdot\|_M}(t)$.*

Proof. We prove it for ρ. The proof for δ^* is similar.

We first show that $\lambda(t) \geq \rho(x,t)$ for all $x \in S_X$ in any given norm.

Choose and fix $x \in S_X$. Since X^* is separable there exists an increasing sequence (F_n) of finite dimensional subspaces of X^* such that $X^* = \overline{\cup_n F_n}$. Fix (Z_n) in X, finite codimensional such that,

$$\rho(x,t) = \lim_n \sup_{y \in Z_n, \|y\| \leq t} \|x+y\| - 1$$

Since $\rho(x,t)$ is infimum over all finite codimensional subspaces, we have

$$\rho(x,t) = \lim_n \sup_{y \in Z_n \cap (F_n)^\perp, \|y\| \leq t} \|x+y\| - 1$$

Fix $y_n \in Z_n \cap (F_n)^\perp$, $\|y_n\| \leq t$ such that

$$\|x+y_n\| + 1/n \geq \sup_{y \in Z_n \cap (F_n)^\perp, \|y\| \leq t} \|x+y\|$$

Note that $y_n \xrightarrow{w} 0$. Taking lim sup we have $\lambda(t) \geq \rho_X(x,t)$.

To show the reverse inequality let $\varepsilon > 0$ be given and $x_n \xrightarrow{w} 0$ and $\|x_n\|_M \leq t$. For any $Y \subseteq X$ with codimension of Y finite, there exists n_0 such that whenever $n > n_0$, we have $y_n \in Y$ satisfying $\|x_n - y_n\|_M < \varepsilon$. Note that this also implies $\|y_n\|_M \leq t + \varepsilon$. Now for $n > n_0$, $\|x + x_n\| \leq \|x + y_n\| + \varepsilon$. Taking infimum over all Y of finite codimension we have $\limsup \|x + x_n\|_M \leq \rho_{\|\cdot\|_M}(x, t+\varepsilon) + \varepsilon + 1$. But by Property (M), the left hand side is same for all $x \in S_{X,\|\cdot\|_M}$. Hence $\lambda_{\|\cdot\|_M}(t) \leq \rho_{\|\cdot\|_M}(x, t+\varepsilon) + \varepsilon$. Since ε is arbitrary and $\rho(x,\cdot)$ is a continuous function in t, it follows that $\lambda_{\|\cdot\|_M}(t) \leq \rho_{\|\cdot\|_M}(x,t)$. □

As an immediate application we obtain that if X is a separable Banach space with Property (M^*), then $Sz(X) = \omega_0$.

Proposition 2.2. *Let X be a separable Banach space with Property (M^*). Then X is asymptotically uniformly smooth in $\|\cdot\|_M$-norm and consequently $Sz(X) = \omega_0$.*

Proof. Since X^* is separable X is an Asplund space. Choosing for Y the kernel of the derivative, it is easy to observe that if $x_0 \in S_X$ is a Frechet smooth point of the norm then $\frac{\rho(x_0,t)}{t} \underset{t \to 0}{\longrightarrow} 0$. But in $\|\cdot\|_M$-norm $\rho_{\|\cdot\|_M}(t) = \rho_{\|\cdot\|_M}(x,t)$ for any $x \in S_X$. Thus X is asymptotically uniformly smooth and by [4] it follows that $Sz(X) = \omega_0$. □

We now proceed to show that $\rho_{\|\cdot\|_M}$ is essentially optimal among all equivalent norms.

Proposition 2.3. *Let X be a separable Banach space with Property (M^*) and $\|\cdot\|$ an equivalent norm on X. Let d be the Banach-Mazur distance between the two norms. Then for any $t>0$, we have*

$$\rho_{\|\cdot\|_M}(t) \;\le\; \rho_{\|\cdot\|}(dt) \qquad \textit{and} \qquad \delta^*_{\|\cdot\|_M}(t) \;\ge\; \delta^*_{\|\cdot\|}(t/d)\ .$$

Proof. Let $\varepsilon>0$ be given. Up to an isometric change of norms, we may assume the inequalities $\|\cdot\|_M \le \|\cdot\| \le (d+\varepsilon)\|\cdot\|_M$ hold and the existence of x in X satisfying $\|x\|_M = 1$ and $\|x\| \le 1+\varepsilon$. Lemma 2.1 and the inequality

$$\|x+x_n\|_M - 1 \;\le\; \|x\|\,(\|x'+x_n'\|-1) + \|x\| - 1$$

with $x' = \dfrac{x}{\|x\|}$ and $x_n' = \dfrac{x_n}{\|x\|}$, shows that whenever (x_n) is a weak-null sequence satisfying $\|x_n\|_M \le t$, we have

$$\limsup \|x+x_n\|_M - 1 \;\le\; (1+\varepsilon)\,\rho_{\|.\|}((d+\varepsilon)\,t) + \varepsilon.$$

Using Property (M) and Lemma 2.1, this yields

$$\rho_{\|\cdot\|_M}(t) \;\le\; (1+\varepsilon)\,\rho_{\|.\|}((d+\varepsilon)\,t) + \varepsilon.$$

which proves the result.

The proof for δ^* is similar. □

We now relate Property (M) with the quantitative behavior of the Szlenk index. We refer to [4, Proposition 2.8] for the definition and use of Young duality in this context. If $Sz(X)=\omega_0$, we denote by $\rho_0(t)$ the dual Young function to the function $Cz(X,t/C)^{-1}$. We note that if $p>p(X)$ and $p^{-1}+q^{-1}=1$, then the function $t^{-q}\rho_0(t)$ is bounded.

Proposition 2.4. *Let X be a separable Banach space with Property (M^*) and let $\|\cdot\|_M$ be a norm with Property (M). There exist constants $K, K'>0$ such that $\rho_{\|\cdot\|_M}(t) \le K\rho_0(K't)$. Similarly there exist constant $C>0$ such that $\delta^*_{\|\cdot\|_M}(t) \ge Cz(X,t/C')$.*

Proof. By Theorem 1.2 and [4, Proposition 2.8], there exist constants K and L such that given $t\in(0\,,1)$, there exists a 2-equivalent norm $\|\cdot\|_t$ on X such that $\rho_{\|\cdot\|_t}(t) \le K\rho_0(Lt)$. By Proposition 2.3, we have $\rho_{\|\cdot\|_M}(t) \le \rho_{\|\cdot\|_t}(2t) \le K\rho_0(2Lt)$. It suffices now to take $K'=2L$. □

In the following theorem we show that for a separable Banach space X with Property (M^*) the embeddability in c_0 or ℓ_p-sum of finite dimensional Banach spaces is completely characterized by the Szlenk functions of X and X^*.

Theorem 2.5. *(a) Let X be a Banach space with Property (M^*). Then the following assertions are equivalent:*

(i) X has a summable Szlenk index.

(ii) X is isomorphic to a subspace of c_0.

(b) Let X be a separable reflexive Banach space with property (M). Then the following assertions are equivalent:

(i) There exists a constant $K > 0$ and $1 < p < \infty$ such that $Sz(X, \varepsilon) \leq K\varepsilon^{-p^*}$ and $Sz(X^*, \varepsilon) \leq K\varepsilon^{-p}$ for all $0 < \varepsilon \leq 1$, where $p^{-1} + (p^*)^{-1} = 1$.

(ii) X is isomorphic to a subspace of an ℓ_p-sum of finite dimensional spaces where $p = p(X^*)$.

Proof. (a) By [4, Theorem 4.10], we have $Cz(X, t)^{-1} \geq Kt$ for some constant K. It follows from Proposition 2.4 that in $\|\cdot\|_M$-norm $\rho_{\|\cdot\|_M}(t) = 0$ for $t > 0$ small enough. By [5] X is isomorphic to a subspace of c_0.

(b) Now Proposition 2.4 shows that X has an equivalent norm whose modulus of asymptotic uniform smoothness has power type p, and by duality an equivalent norm whose modulus of asymptotic uniform convexity also has power type p. It now follows from [8, Proposition 2.11] that X is isomorphic to a subspace of an ℓ_p-sum of finite dimensional spaces (see also [12, Theorem 4.1]). □

Remark 2.6. (a) Note that in any norm $\delta(t) \leq \rho(t)$. It follows from this inequality that if X is a reflexive Banach space such that X and X^* have minimal Szlenk index, then $p(X)^{-1} + p(X^*)^{-1} \leq 1$. The space X with Property (M) is isomorphic to an ℓ_p-sum of finite dimensional spaces if and only if the Szlenk power types are actually attained and $p(X)^{-1} + p(X^*)^{-1} = 1$.

(b) It was shown in [4, Theorem 4.9 and Theorem 4.10] that if X has summable Szlenk index or both X and X^* have minimal Szlenk indices then the functions $Cz(X, t)$ and $Sz(X, t)$ are equivalent. If X is separable reflexive and has Property (M) then its dual space shares these properties and thus it follows by Proposition 2.2 that X and X^* have minimal Szlenk index. Hence there is $L > 0$ such that $Cz(X, \varepsilon) \leq Sz(X, \varepsilon/L)$ for all $\varepsilon \in (0, 1]$. Similarly for X^*, $Cz(X^*, t)$ and $Sz(X^*, t)$ are equivalent.

Our next application is to show that if two Orlicz sequence spaces h_F and h_G are Lipschitz isomorphic then they contain the same ℓ_p-spaces.

Recall [11] that an Orlicz function F is a continuous non-decreasing and convex function defined on $\mathbb{R}_+$ such that $F(0) = 0$. We will only consider non-degenerate Orlicz functions, that is Orlicz functions which vanish only at 0.

To any Orlicz function F is associated the Banach space ℓ_F of all sequences of scalars (a_n) such that $\sum_{i=1}^{+\infty} F(|a_i|/r)$ is finite for some $r > 0$ equipped with the Luxemburg norm

$$\|a\| = \inf\left\{ r > 0\,;\, \sum_{i=1}^{+\infty} F\left(\frac{|a_i|}{r}\right) \le 1 \right\}.$$

The subspace h_F consisting of those sequences $(a_n) \in \ell_F$ for which $\sum_{i=1}^{+\infty} F(|a_i|/r)$ is finite for every $r > 0$ is the closed subspace of ℓ_F generated by the unit vectors. In general these two spaces are different but if ℓ_F is separable they coincide. By the Luxemburg norm on h_F we mean the standard norm given by F as above.

Associated to an Orlicz function F are the following two quantities.

$$\alpha_F = \sup\left\{ q\ ;\ \sup_{0<\lambda,\, t\le 1} \frac{F(\lambda t)}{t^q F(\lambda)} < \infty \right\}$$

and

$$\beta_F = \inf\left\{ q\ ;\ \inf_{0<\lambda,\, t\le 1} \frac{F(\lambda t)}{t^q F(\lambda)} > 0 \right\}.$$

It is easily checked that we always have $1 \le \alpha_F \le \beta_F \le \infty$. It is a classical result (see [11, Theorem I. 4.a.9]) that the space ℓ_p or c_0 if $p = \infty$ is isomorphic to a subspace of h_F if and only if $p \in [\alpha_F, \beta_F]$. Also, $\beta_F < \infty$ if and only $\ell_F = h_F$ and in this case the sequence of unit vectors of ℓ_F is a boundedly complete basis and ℓ_F is isomorphic to a separable dual space. Thus $\beta_F < \infty$ if and only if h_F has Radon Nikodym Property (RNP for short). Also $\alpha_F > 1$ if and only if h_F has separable dual.

We now show that the value α_F determines almost exactly the best modulus of asymptotic uniform smoothness of equivalent norms on h_F.

Theorem 2.7. *Let F be an Orlicz function. Then for any $p < \alpha_F$ the Luxemburg norm on h_F has modulus of asymptotic uniform smoothness of power type p. If $q > \alpha_F$, no equivalent norm on h_F has a modulus of asymptotic uniform smoothness of power type q.*

Proof. It was established in [6, Theorem 1] that for all $p < \alpha_F$, the Luxemburg norm of h_F has a modulus of asymptotic uniform smoothness of power type p. The space h_F contains a copy of ℓ_{α_F}, and thus the power type of asymptotic uniform smoothness of any of its renormings is at most equal to the power type of asymptotic uniform smoothness of the restriction of the norm to the corresponding copy. Since ℓ_p has Property (M) for its natural norm, the power type of asymptotic uniform smoothness of any of its equivalent norms is bounded above by p by Proposition 2.4, and the result follows. □

Note that unless $h_F = \ell_p$ the Luxemburg norm on h_F does not satisfy Property (M) (there is however an equivalent norm on h_F which does by [9]). Also, the result of [6] combined with [4] yields $p(h_F) = {\alpha_F}^*$.

We are now ready to prove that the quantities α_F and β_F are Lipschitz invariants.

Theorem 2.8. *Let F and G be two Orlicz functions such that h_F and h_G are Lipschitz isomorphic. Then $\alpha_F = \alpha_G$ and $\beta_F = \beta_G$. Hence two Lipschitz isomorphic Orlicz sequence spaces contain the same ℓ_p-spaces.*

Proof. We consider the following two cases.

CASE 1: Suppose $\beta_F < \infty$. Then h_F has RNP and since RNP is a Lipschitz invariant property, h_G also has RNP and therefore $\beta_G < \infty$. It is by now a classical result (see [1, Corollary 7.7]) that in this case h_F and h_G embed into each other isomorphically as complemented subspaces. Using the subsymmetry of their canonical bases, these spaces are easily seen to be isomorphic to their squares. By the Pelczynski decomposition scheme it follows that $h_F \simeq h_G$, and thus of course they contain the same ℓ_p spaces.

CASE 2: Suppose $\beta_F = \infty$. By the above one has $\beta_G = \infty$ and all that remains to be proved is $\alpha_F = \alpha_G$. If not, then for instance $\alpha_F > \alpha_G$. Pick any p with $\alpha_F > p > \alpha_G$. By Theorem 2.7, the Luxemburg norm on h_F has a modulus of asymptotic uniform smoothness of power type p. It now follows from [4, Theorem 5.4] that h_G has an equivalent norm with a modulus of asymptotic uniform smoothness of power type p (see also [3, Proposition 42, page 67]). But h_G contains an isomorphic copy of ℓ_{α_G} and since ℓ_{α_G} has Property (M) for its natural norm, the power type of asymptotic uniform smoothness of any of its equivalent norms is bounded above by α_G by Proposition 2.3. This contradiction concludes the proof. □

We recall that it is not known whether two *separable* Lipschitz isomorphic Banach spaces are necessarily linearly isomorphic.

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