Second Order Elliptic PDE

T. Muthukumar
tmk@iitk.ac.in

December 16, 2014

Contents

1 A Quick Introduction to PDE

A *multi-index* $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a n-tuple where α_i, for each $1 \leq i \leq n$, is a non-negative integer. Let $|\alpha| := \alpha_1 + \ldots + \alpha_n$. If α and β are two multi-indices, then $\alpha \leq \beta$ means $\alpha_i \leq \beta_i$, for all $1 \leq i \leq n$, and $\alpha \pm \beta = (\alpha_1 \pm \beta_1, \ldots, \alpha_n \pm \beta_n)$. Also, $\alpha! = \alpha_1! \ldots \alpha_n!$ and, for any $x \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$. The multi-index notation, introduced by L. Schwartz, is quite handy in representing multi-variable equations in a concise form. For instance, a k-degree polynomial in n-variables can be written as

$$\sum_{|\alpha| \leq k} a_\alpha x^\alpha.$$

The partial differential operator of *order* α is denoted as

$$D^\alpha = \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \ldots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}} = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \ldots \partial {x_n}^{\alpha_n}}.$$

One adopts the convention that between same components of α the order in which differentiation is performed is irrelevant. This is not a restrictive convention because the independence of order of differentiation is valid for smooth[1] functions. For each $k \in \mathbb{N}$, $D^k := \{D^\alpha \mid |\alpha| = k\}$.

Definition 1.1. *Let Ω be an open subset of $\mathbb{R}^n$. A k-th* order *partial differential equation of an unknown function $u : \Omega \to \mathbb{R}$ is of the form*

$$F\left(D^k u(x), D^{k-1}u(x), \dots Du(x), u(x), x\right) = 0, \tag{1.1}$$

for each $x \in \Omega$, where $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given map.

A general first order PDE has the form $F(Du(x), u(x), x) = 0$ and a general second order PDE has the form $F(D^2u(x), Du(x), u(x), x) = 0$.

Definition 1.2. *(i) A PDE is* linear *if F in (1.1) has the form*

$$\sum_{|\alpha| \le k} a_\alpha(x) D^\alpha u(x) = f(x)$$

for given functions f and a_α's. In addition, if $f \equiv 0$ then the PDE is linear and homogeneous.

(ii) A PDE is semilinear *if F is linear only in the highest order, i.e., F has the form*

$$\sum_{|\alpha| = k} a_\alpha(x) D^\alpha u(x) + a_0(D^{k-1}u(x), \dots, Du(x), u(x), x) = 0.$$

(iii) A PDE is quasilinear *if F has the form*

$$\sum_{|\alpha| = k} a_\alpha(D^{k-1}u(x), \dots, u(x), x) D^\alpha u + a_0(D^{k-1}u(x), \dots, u(x), x) = 0,$$

i.e., coefficient of the highest order derivative depends on u and its derivative only upto the previous orders.

(iv) A PDE is fully nonlinear *if it depends nonlinearly on the highest order derivatives.*

[1]smooth, usually, refers to as much differentiability as required

Observe that, for a semilinear PDE, a_0 is never linear in u, otherwise it reduces to being linear. For a quasilinear PDE, a_α (with $|\alpha| = k$), cannot be independent of u or its derivatives, otherwise it reduces to being semilinear or linear.

Definition 1.3. *We say $u : \Omega \to \mathbb{R}$ is a* classical solution *to the k-th order PDE (1.1),*

- *if $u \in C^k(\Omega)$, i.e., u is k-times differentiable with the continuous k-th derivative*
- *and u satisfies the equation (1.1).*

2 Classification of Second Order PDE

A general second order PDE is of the form $F\left(D^2u(x), Du(x), u(x), x\right) = 0$, for each $x \in \Omega \subset \mathbb{R}^n$ and $u : \Omega \to \mathbb{R}$ is the unknown. Consider the general second order semilinear PDE with n independent variable

$$F(x, u, Du, D^2u) := A(x) \cdot D^2u - D(\nabla u, u, x), \tag{2.1}$$

where $A = A_{ij}$ is an $n \times n$ matrix with entries $A_{ij}(x, u, \nabla u)$, D^2u is the Hessian matrix. The dot product in LHS is in $\mathbb{R}^{n^2}$, i.e,

$$A(x) \cdot D^2u = \sum_{i,j=1}^{n} a_{ij}(x) u_{x_i x_j}(x).$$

Since we demand the solution to be in C^2, the mixed derivatives are equal and we can assume, without loss of generality that, A is symmetric. In fact if A is not symmetric, we can replace A with $A^s := \frac{1}{2}(A + A^t)$, which is symmetric since $A \cdot D^2u = A^s \cdot D^2u$. Since $A(x)$ is a real symmetric matrix, it is diagonalisable. There is a coordinate transformation $T(x)$ such that the matrix $T(x)A(x)T^t(x)$ is diagonal with diagonal entries, say $\lambda_1(x), \lambda_2(x), \ldots, \lambda_n(x)$, for each $x \in \Omega$. Thus, we classify a PDE, at $x \in \Omega$, based on the eigenvalues of the matrix $A(x)$. Let P denote the number of strictly positive eigenvalues and Z denote the number of zero eigenvalues.

Definition 2.1. *We say a PDE is* hyperbolic *at a point $x \in \Omega$, if $Z = 0$ and either $P = 1$ or $P = n - 1$. We say it is* parabolic *if $Z > 0$. We say it is* elliptic, *if $Z = 0$ and either $P = n$ or $P = 0$. If $Z = 0$ and $1 < P < n - 1$ then the PDE is said to be* ultra hyperbolic.

One may, equivalently, define a linear second order PDE to be *elliptic* at x if

$$\sum_{i,j=1}^{n} A_{ij}(x)\xi_i\xi_j \neq 0 \quad \forall \xi \in \mathbb{R}^n \setminus \{0\}.$$

If $A(x)$ is a constant matrix (independent of x) then with a suitable transformation T one can rewrite

$$\sum_{i,j=1}^{n} a_{ij}u_{x_ix_j}(x) = \Delta v(x)$$

where $v(x) := u(Tx)$.

3 Linear Second Order Elliptic Operators

The elliptic operators come in two forms, divergence and non-divergence form, and we shall see that a notion of weak solution can be defined for elliptic operator in divergence form.

Let Ω be an open subset of $\mathbb{R}^n$. Let $A = A(x) = (a_{ij}(x))$ be any given $n \times n$ matrix of functions, for $1 \leq i, j \leq n$. Let $\boldsymbol{b} = \boldsymbol{b}(x) = (b_i(x))$ be any given n-tuple of functions and let $c = c(x)$ be any given function.

Definition 3.1. *A second order operator L is said to be in* divergence form*, if L acting on some u has the form*

$$Lu := -\operatorname{div}(A(x)\nabla u) + \boldsymbol{b}(x) \cdot \nabla u + c(x)u.$$

On the other hand, a second order operator L is said to be in non-divergence form*, if L acting some u has the form*

$$Lu := -\sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i x_j} + \boldsymbol{b}(x) \cdot \nabla u + c(x)u.$$

Observe that the operator L makes sense, in the divergence form, only if $a_{ij}(x) \in C^1(\Omega)$. Thus, if $a_{ij}(x) \in C^1$, then a divergence form equation can be rewritten in to a non-divergence form because

$$\nabla \cdot (A(x)\nabla u) = \sum_{i,j=1}^{n} a_{ij}(x)\frac{\partial^2 u}{\partial x_i \partial x_j} + \left(\sum_{j=1}^{n} \frac{\partial a_{ij}}{\partial x_j}\right)_i \cdot \nabla u.$$

Now, by setting $\widetilde{b}_i(x) = b_i(x) - \sum_{j=1}^n \frac{\partial a_{ij}}{\partial x_j}$, we have written a divergence L in non-divergence form.

Definition 3.2. *We say a second order operator L is* elliptic *or* coercive *if there is a positive constant $\alpha > 0$ such that*

$$\alpha|\xi|^2 \leq A(x)\xi.\xi \qquad a.e. \text{ in } x, \quad \forall \xi = (\xi_i) \in \mathbb{R}^n.$$

The second order operator L is said to be degenerate elliptic *if*

$$0 \leq A(x)\xi.\xi \qquad a.e. \text{ in } x, \quad \forall \xi = (\xi_i) \in \mathbb{R}^n.$$

We remark that for the integrals in the defintion of weak solution to make sense, the minimum hypotheses on $A(x), \boldsymbol{b}$ and c is that $a_{ij}, b_i, c \in L^\infty(\Omega)$.

Definition 3.3. *Let $a_{ij}, b_i, c \in L^\infty(\Omega)$ and let $f \in H^{-1}(\Omega)$, we say $u \in H_0^1(\Omega)$ is a* weak solution *of the homogeneous Dirichlet problem*

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) + \boldsymbol{b}(x) \cdot \nabla u + c(x)u & = f \quad \text{in } \Omega \\ u & = 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.1}$$

whenever, for all $v \in H_0^1(\Omega)$,

$$\int_\Omega A\nabla u \cdot \nabla v \, dx + \int_\Omega (\boldsymbol{b} \cdot \nabla u) v \, dx + \int_\Omega cuv \, dx = \langle f, v \rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \, . \tag{3.2}$$

We define the map $a(\cdot, \cdot) : H_0^1(\Omega) \times H_0^1(\Omega) \to \mathbb{R}$ as

$$a(v, w) := \int_\Omega A\nabla v \cdot \nabla w \, dx + \int_\Omega (\boldsymbol{b} \cdot \nabla v) w \, dx + \int_\Omega cvw \, dx.$$

It is easy to see that $a(\cdot, \cdot)$ is bilinear.

Lemma 3.4. *If $a_{ij}, b_i, c \in L^\infty(\Omega)$ then the bilinear map $a(\cdot, \cdot)$ is continuous on $H_0^1(\Omega) \times H_0^1(\Omega)$, i.e., there is a constant $c_1 > 0$ such that*

$$|a(v, w)| \leq c_1 \|v\|_{H_0^1(\Omega)} \|w\|_{H_0^1(\Omega)}.$$

Also, in addition, if Ω is bounded and L is elliptic, then there are constants $c_2 > 0$ and $c_3 \geq 0$ such that

$$c_2 \|v\|_{H_0^1(\Omega)}^2 \leq a(v, v) + c_3 \|v\|_2^2.$$

Proof. Consider,

$$\begin{aligned}
|a(v,w)| &\le \int_\Omega |A(x)\nabla v(x)\cdot\nabla w(x)|\,dx + \int_\Omega |(\boldsymbol{b}(x)\cdot\nabla v(x))w(x)|\,dx \\
&\quad + \int_\Omega |c(x)v(x)w(x)|\,dx \\
&\le \max_{i,j}\|a_{ij}\|_\infty\|\nabla v\|_2\|\nabla w\|_2 + \max_i\|b_i\|_\infty\|\nabla v\|_2\|w\|_2 \\
&\quad +\|c\|_\infty\|v\|_2\|w\|_2 \\
&\le m\|\nabla v\|_2(\|\nabla w\|_2+\|w\|_2) + m\|v\|_2\|w\|_2, \\
&\qquad \text{where } m = \max\left(\max_{i,j}\|a_{ij}\|_\infty, \max_i\|b_i\|_\infty, \|c\|_\infty\right) \\
&\le c_1\|v\|_{H_0^1(\Omega)}\|w\|_{H_0^1(\Omega)}.
\end{aligned}$$

Since L is elliptic, we have

$$\begin{aligned}
\alpha\|\nabla v\|_2^2 &\le \int_\Omega A(x)\nabla v\cdot\nabla v\,dx \\
&= a(v,v) - \int_\Omega(\boldsymbol{b}\cdot\nabla v)v\,dx - \int_\Omega cv^2\,dx \\
&\le a(v,v) + \max_i\|b_i\|_\infty\|\nabla v\|_2\|v\|_2 + \|c\|_\infty\|v\|_2^2.
\end{aligned}$$

If $\boldsymbol{b}=0$, then we have the result with $c_2=\alpha$ and $c_3=\|c\|_\infty$. If $\boldsymbol{b}\neq 0$, we choose a $\gamma>0$ such that

$$\gamma < \frac{2\alpha}{\max_i\|b_i\|_\infty}.$$

Then, we have

$$\begin{aligned}
\alpha\|\nabla v\|_2^2 &\le a(v,v) + \max_i\|b_i\|_\infty\|\nabla v\|_2\|v\|_2 + \|c\|_\infty\|v\|_2^2 \\
&= a(v,v) + \max_i\|b_i\|_\infty\gamma^{1/2}\|\nabla v\|_2\frac{\|v\|_2}{\gamma^{1/2}} \\
&\quad +\|c\|_\infty\|v\|_2^2 \\
&\le a(v,v) + \frac{\max_i\|b_i\|_\infty}{2}\left(\gamma\|\nabla v\|_2^2 + \frac{\|v\|_2^2}{\gamma}\right) \\
&\quad +\|c\|_\infty\|v\|_2^2 \quad (\text{using } ab\le a^2/2+b^2/2) \\
\left(\alpha - \frac{\gamma}{2}\max_i\|b_i\|_\infty\right)\|\nabla v\|_2^2 &\le a(v,v) + \left(\frac{1}{2\gamma}\max_i\|b_i\|_\infty + \|c\|_\infty\right)\|v\|_2^2.
\end{aligned}$$

By Poincaré inequality there is a constant $C > 0$ such that $1/C\|v\|^2_{H^1_0(\Omega)} \leq \|\nabla v\|^2_2$. Thus, we have

$$c_2\|v\|^2_{H^1_0(\Omega)} \leq a(v,v) + c_3\|v\|^2_2.$$

□

Theorem 3.5 (Lax-Milgram)**.** *Let H be a Hilbert space. Let $a(\cdot,\cdot)$ be a coercive bilinear form on H and $f \in H^\star$. Then there exists a unique solution $x^\star \in H$ such that $a(x^\star, y) = \langle f, y\rangle_{H^\star,H}$ for all $y \in H$.*

Theorem 3.6. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $a_{ij}, c \in L^\infty(\Omega)$, $\boldsymbol{b} = 0$, $c(x) \geq 0$ a.e. in Ω and $f \in H^{-1}(\Omega)$. Also, let A satisfy ellipticity condition. Then there is a unique weak solution $u \in H^1_0(\Omega)$ satisfying*

$$\int_\Omega A\nabla u \cdot \nabla v \, dx + \int_\Omega cuv \, dx = \langle f, v\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}, \quad \forall v \in H^1_0(\Omega).$$

Further, if A is symmetric then u minimizes the functional $J : H^1_0(\Omega) \to \mathbb{R}$ defined as,

$$J(v) := \frac{1}{2}\int_\Omega A\nabla v \cdot \nabla v \, dx + \frac{1}{2}\int_\Omega cv^2 \, dx - \langle f, v\rangle_{H^{-1}(\Omega),H^1_0(\Omega)}$$

in $H^1_0(\Omega)$.

Proof. We define the bilinear form as

$$a(v,w) := \int_\Omega A\nabla v \cdot \nabla w \, dx + \int_\Omega cvw \, dx.$$

It follows from Lemma 3.4 that a is a continuous. Now ,consider

$$\begin{aligned}
\alpha\|\nabla v\|^2_2 &\leq \int_\Omega A(x)\nabla v \cdot \nabla v \, dx \\
&\leq \int_\Omega A(x)\nabla v \cdot \nabla v \, dx + \int_\Omega cv^2 \, dx \quad \text{(since } c(x) \geq 0) \\
&= a(v,v).
\end{aligned}$$

Thus, a is coercive in $H^1_0(\Omega)$, by Poincaré inequality. Hence, by Lax Milgram theorem (cf. Theorem 3.5), $u \in H^1_0(\Omega)$ exists. Also, if A is symmetric, then u minimizes the functional J on $H^1_0(\Omega)$. □

Theorem 3.7. *Let $\langle X, Y\rangle$ be a dual system and $S : X \to X$, $T : Y \to Y$ be compact adjoint operators. Then*

$$dim(N(I-S)) = dim(N(I-T)) < \infty.$$

Theorem 3.8. *Let $\langle X, Y\rangle$ be a dual system and $S : X \to X$, $T : Y \to Y$ be compact adjoint operators. Then*

$$R(I-S) = \{x \in X \mid \langle x, y\rangle = 0, \forall y \in N(I-T)\}$$

and

$$R(I-T) = \{y \in Y \mid \langle x, y\rangle = 0, \forall x \in N(I-S)\}.$$

Theorem 3.9. *Let Ω be a bounded open subset of $\mathbb{R}^n$, $a_{ij}, b_i, c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$. Also, let A satisfy ellipticity condition. Consider L as in (3.1). The space of solutions $\{u \in H_0^1(\Omega) \mid Lu = 0\}$ is finite dimensional. For non-zero $f \in L^2(\Omega)$, there exists a finite dimensional subspace $S \subset L^2(\Omega)$ such that (3.1) has solution iff $f \in S^\perp$, the orthogonal complement of S.*

Proof. It is already noted in Lemma 3.4 that one can find a $c_3 > 0$ such that $a(v, v) + c_3\|v\|_2^2$ is coercive in $H_0^1(\Omega)$. Thus, by Theorem 3.6, there is a unique $u \in H_0^1(\Omega)$ such that

$$a(u, v) + c_3 \int_\Omega uv\, dx = \int_\Omega fv\, dx \quad \forall v \in H_0^1(\Omega).$$

Set the map $T : L^2(\Omega) \to H_0^1(\Omega)$ as $Tf = u$. The map T is a compact operator on $L^2(\Omega)$ because it maps u into $H_0^1(\Omega)$ which is compactly contained in $L^2(\Omega)$. Note that (3.1) is equivalent to $u = T(f + c_3u)$. Set $v := f + c_3u$. Then $v - c_3Tv = f$. Recall that T is compact and $c_3 > 0$. Thus, $I - c_3T$ is invertible except when c_3^{-1} is an eigenvalue of T. If c_3^{-1} is not an eigenvalue then there is a unique solution v for all $f \in L^2(\Omega)$. If c_3^{-1} is an eigenvalue then it has finite geometric multiplicity (T being compact). Therefore, by Fredhölm alternative (cf. Theorems 3.7 and 3.8), solution exists iff $f \in N(I - c_3T^*)^\perp$ and the dimension of $S := N(I - c_3T^*)$ is same as $N(I - c_3T)$. □

Theorem 3.10 (Regularity of Weak Solution)**.** *Let Ω be an open subset of class C^2. Let $a_{ij}, b_i, c \in L^\infty(\Omega)$ and $f \in L^2(\Omega)$. Let $u \in H_0^1(\Omega)$ be such that it satisfies (3.2). If $a_{ij} \in C^1(\overline{\Omega})$, $b_i \in C(\overline{\Omega})$ and $f \in L^2(\Omega)$ then $u \in H^2(\Omega)$. More generally, for $m \geq 1$, if $a_{ij} \in C^{m+1}(\overline{\Omega})$, $b_i \in C^m(\overline{\Omega})$ and $f \in H^m(\Omega)$ then $u \in H^{m+2}(\Omega)$.*

Theorem 3.11 (Weak Maximum Principle). *Let Ω be a bounded open subset of $\mathbb{R}^n$ with sufficient smooth boundary $\partial\Omega$. Let $a_{ij}, c \in L^\infty(\Omega)$, $c(x) \geq 0$ and $f \in L^2(\Omega)$. Let $u \in H^1(\Omega) \cap C(\overline{\Omega})$ be such that it satisfies (3.2) with $\boldsymbol{b} \equiv 0$. Then the following are true:*

(i) If $f \geq 0$ on Ω and $u \geq 0$ on $\partial\Omega$ then $u \geq 0$ in Ω.

(ii) If $c \equiv 0$ and $f \geq 0$ then $u(x) \geq \inf_{y\in\partial\Omega} u(y)$ for all $x \in \Omega$.

(iii) If $c \equiv 0$ and $f \equiv 0$ then $\inf_{y\in\partial\Omega} u(y) \leq u(x) \leq \sup_{y\in\partial\Omega} u(y)$ for all $x \in \Omega$.

Proof. Recall that if $u \in H^1(\Omega)$ then $|u|$, u^+ and u^- are also in $H^1(\Omega)$.

(i) If $u \geq 0$ on $\partial\Omega$ then $u = |u|$ on $\partial\Omega$. Hence, $u^- \in H^1_0(\Omega)$. Thus, using $v = u^-$ in (3.2), we get

$$-\int_\Omega A\nabla u^- \cdot \nabla u^- \, dx - \int_\Omega c(x)(u^-)^2 \, dx = \int_\Omega f(x)u^-(x) \, dx$$

because u^+ and u^- intersect on $\{u = 0\}$ and, on this set, $u^+ = u^- = 0$ and $\nabla u^+ = \nabla u^- = 0$ a.e. Note that RHS is non-negative because both f and u^- are non-negative. Therefore,

$$0 \geq \int_\Omega A\nabla u^- \cdot \nabla u^- \, dx + \int_\Omega c(x)(u^-)^2 \, dx \geq \alpha\|\nabla u^-\|_2^2.$$

Thus, $\|\nabla u^-\|_2 = 0$ and, by Poincarè inequality, $\|u^-\|_2 = 0$. This implies $u^- = 0$ a.e and, hence, $u = u^+$ a.e. on Ω.

(ii) Let $m = \inf_{y\in\partial\Omega} u(y)$. Then $u - m \geq 0$ on $\partial\Omega$. Further, $c \equiv 0$ implies that $u - m$ satisfies (3.2) with $\boldsymbol{b} = 0$. By previous case, $u - m \geq 0$ on Ω.

(iii) If $f \equiv 0$ then $-u$ satisfies (3.2) with $\boldsymbol{b} = 0$. By previous case, we have the result.

□

Definition 3.12. *Let H be a Hilbert space with scalar product $\langle\cdot,\cdot\rangle$. A linear continuous operator $T : H \to H$ is said to be:*

(i) positive *if, for all $x \in H$, $\langle Tx, x\rangle \geq 0$.*

(ii) self-adjoint *if, for all* $x, y \in H$, $\langle Tx, y\rangle = \langle x, Ty\rangle$.

(iii) compact *if the image of any bounded set in* H *is relatively compact (i.e. has compact closure) in* H.

Theorem 3.13. *Let* H *be a separable Hilbert space of infinite dimension and* $T : H \to H$ *is a self-adjoint, compact and positive operator. Then, there exists a sequence of real positive eigenvalues* $\{\mu_m\}$, *for* $m \geq 1$, *converging to* 0 *and a sequence of eigenvectors* $\{x_m\}$, *for* $m \geq 1$, *forming a basis of* H *such that, for all* $m \geq 1$, $Tx_m = \mu_m x_m$.

Theorem 3.14 (Dirichlet Spectral Decomposition)**.** *Let* A *be a symmetric matrix, i.e.,* $a_{ij}(x) = a_{ji}(x)$, *and* $c(x) \geq 0$. *There exists a sequence of positive real eigenvalues* $\{\lambda_m\}$ *and corresponding orthonormal basis* $\{\phi_m\} \subset C^\infty(\Omega)$ *of* $L^2(\Omega)$, *with* $m \in \mathbb{N}$, *such that*

$$\begin{cases} -\text{div}[A(x)\nabla\phi_m(x)] + c(x)\phi_m(x) &= \lambda_m\phi_m(x) \quad \text{in } \Omega \\ \phi_m &= 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.3}$$

and $0 < \lambda_1 \leq \lambda_2 \leq \ldots$ *diverges.*

Proof. Let $T : L^2(\Omega) \to H^1_0(\Omega)$ defined as $Tf = u$ where u is the solution of

$$\begin{cases} -\text{div}[A(x)\nabla u(x)] + c(x)u(x) &= f(x) \quad \text{in } \Omega \\ \phi_m &= 0 \quad \text{on } \partial\Omega. \end{cases}$$

Thus,

$$\int_\Omega A(x)\nabla(Tf)\cdot\nabla v(x)\,dx + \int_\Omega c(x)(Tf)(x)v(x)\,dx = \int_\Omega f(x)v(x)\,dx.$$

Note that T is a compact operator on $L^2(\Omega)$ and T is self-adjoint because, for every $g \in L^2(\Omega)$,

$$\begin{aligned} \int_\Omega (Tf)(x)g(x)\,dx &= \int_\Omega A(x)\nabla(Tg)\cdot\nabla(Tf)\,dx \\ &\quad + \int_\Omega c(x)(Tg)(x)(Tf)(x)\,dx \\ &= \int_\Omega (Tg)(x)f(x)\,dx. \end{aligned}$$

Further, T is positive definite because, for $f \not\equiv 0$,

$$\begin{aligned}\int_\Omega (Tf)(x)f(x)\,dx &= \int_\Omega A(x)\nabla(Tf)\cdot\nabla(Tf)\,dx + \int_\Omega c(x)(Tf)^2(x)\,dx \\ &\geq \alpha\|\nabla Tf\|_2^2 > 0.\end{aligned}$$

Thus, there exists an orthonormal basis of eigenfunctions $\{\phi_m\}$ in $L^2(\Omega)$ and a sequence of positive eigenvalues μ_m decreasing to zero such that $T\phi_m = \mu_m\phi_m$. Set $\lambda_m = \mu_m^{-1}$. Then $\phi_m = \lambda_m T\phi_m = T(\lambda_m\phi_m)$. Thus, $\phi_m \in H_0^1(\Omega)$ because range of T is $H_0^1(\Omega)$. Hence, ϕ_m satisfies (3.3). It now only remains to show that $\phi_m \in C^\infty(\Omega)$. For any $x \in \Omega$, choose $B_r(x) \subset \Omega$. Since $\phi_m \in L^2(B_r(x))$ and solves the eigen value problem, by interior regularity (cf. Theorem 3.10), $\phi_m \in H^2(B_r(x))$. Arguing similarly, we obtain $\phi_m \in H^k(B_r(x))$ for all k. Thus, by Sobolev imbedding results, $\phi_m \in C^\infty(B_r(x))$. Since $x \in \Omega$ is arbitrary, $\phi_m \in C^\infty(\Omega)$. □

Remark 3.15. Observe that if $H_0^1(\Omega)$ is equipped with the inner product $\int_\Omega \nabla u\cdot\nabla v\,dx$, then $\lambda_m^{-1/2}\phi_m$ is an orthonormal basis for $H_0^1(\Omega)$ where (λ_m, ϕ_m) is the eigen pair corresponding to $A(x) = I$ and $c \equiv 0$. With the usual inner product

$$\int_\Omega uv\,dx + \int_\Omega \nabla u\cdot\nabla v\,dx$$

in $H_0^1(\Omega)$, $(\lambda_m + 1)^{-1/2}\phi_m$ forms an orthonormal basis of $H_0^1(\Omega)$. The set $\{\phi_m\}$ is dense in $H_0^1(\Omega)$ w.r.t both the norms mentioned above. Suppose $f \in H_0^1(\Omega)$ is such that $\langle f, \phi_m\rangle = 0$ in $H_0^1(\Omega)$, for all m. Then, from the eigenvalue problem, we get $\lambda_m \int_\Omega \phi_m f\,dx = 0$. Since ϕ_m is a basis for $L^2(\Omega)$, $f = 0$.

Theorem 3.16 (Krein-Rutman)**.** *Let X be a Banach space and C be a closed convex cone in X with vertex at O, Int$(C) \neq \emptyset$ and satisfying $C \cap (-C) = \{O\}$. Let $T : E \to E$ be a compact operator such that $T(C \setminus \{O\}) \subset$ Int(C). Then the greatest eigenvalue of T is simple, and the corresponding eigenvector is in Int(C) (or in $-$Int(C)).*

Theorem 3.17. *Let Ω be a regular connected open set. Then the first eigenvalue $\lambda_1(\Omega)$ is simple and the first eigenfunction ϕ_1 has a constant sign on Ω. Usually, we choose it to be positive on Ω.*

Proof. In the Krein-Rutman theorem, let $X = C(\overline{\Omega})$, $T = L^{-1}$ and $C = \{v \in C(\overline{\Omega}) \mid v(x) \geq 0\}$. Then, by strong maximum principle, T satisfies $T(C \setminus \{O\}) \subset \mathrm{Int}(C)$. □

Theorem 3.18 (Neumann Spectral Decomposition)**.** *Let Ω be a bounded open subset of $\mathbb{R}^n$ with Lipschitz boundary. Let A be such that $a_{ij}(x) = a_{ji}(x)$, i.e., is a symmetric matrix and $c(x) \geq 0$. There exists a sequence of positive real eigenvalues $\{\lambda_m^{(N)}\}$ and corresponding orthonormal basis $\{\phi_m^{(N)}\} \subset C^\infty(\Omega)$ of $L^2(\Omega)$, with $m \in \mathbb{N}$, such that*

$$\left\{ \begin{array}{rll} -\text{div}[A(x)\nabla\phi_m^{(N)}(x)] + c(x)\phi_m^{(N)}(x) & = \lambda_m^{(N)}\phi_m^{(N)}(x) & \text{in } \Omega \\ A(x)\nabla\phi_m^{(N)} \cdot \nu & = 0 & \text{on } \partial\Omega \end{array} \right. \tag{3.4}$$

and $0 \leq \lambda_1^{(N)} \leq \lambda_2^{(N)} \leq \ldots$ diverges.

Remark 3.19. The case when $c(x) \equiv 0$, the first eigenvalue $\lambda_1^{(N)} = 0$ and $\phi_1^{(N)}$ is a non-zero constant on a connected component of Ω. The Lipschitz condition on Ω is required for the compactness of $H^1(\Omega)$ imbedding in $L^2(\Omega)$.

Definition 3.20. *The* Rayleigh quotient *map $R : H_0^1(\Omega) \setminus \{0\} \to [0, \infty)$ is defined as*

$$R(v) = \frac{\int_\Omega A(x)\nabla v \cdot \nabla v \, dx + \int_\Omega c(x)v^2(x)\, dx}{\|v\|_{2,\Omega}^2}.$$

Remark 3.21 (Min-Max Principle)**.** The eigenvalues satisfy the formula

$$\lambda_m = \min_{W_m \subset H_0^1(\Omega)} \max_{\substack{v \in W_m \\ v \neq 0}} R(v)$$

and

$$\lambda_m^{(N)} = \min_{W_m \subset H^1(\Omega)} \max_{\substack{v \in W_m \\ v \neq 0}} R(v)$$

where W_m is a m-dimensional subspace. The minimum is achieved for the subspace W_m spanned by the first m eigenfunctions.

4 Periodic Boundary Conditions

Let $Y = [0, 1)^n$ be the unit cell of $\mathbb{R}^n$ and let, for each $i, j = 1, 2, \ldots, n$, $a_{ij} : Y \to \mathbb{R}$ and $A(y) = (a_{ij})$. For any given $f : Y \to \mathbb{R}$, extended Y-periodically to $\mathbb{R}^n$, we want to solve the problem

$$\left\{ \begin{array}{rll} -\text{div}(A(y)\nabla u(y)) & = f(y) & \text{in } Y \\ u & \text{is} & Y-\text{periodic.} \end{array} \right. \tag{4.1}$$

The condition u is Y-periodic is equivalent to saying that u takes equal values on opposite faces of Y. One may rewrite the equation on the n-dimensional unit torus $\mathbb{T}^n$ without the periodic boundary condition.

Let us now identify the solution space for (4.1). Let $C^\infty_{\text{per}}(Y)$ be the set of all Y-periodic functions in $C^\infty(\mathbb{R}^n)$. Let $H^1_{\text{per}}(Y)$ denote the closure of $C^\infty_{\text{per}}(Y)$ in the H^1-norm. Being a second order equation, in the weak formulation, we expect the weak solution u to be in $H^1_{\text{per}}(Y)$. Note that if u solves (4.1) then $u+c$, for any constant c, also solves (4.1). Thus, the solution will be unique up to a constant in the space $H^1_{\text{per}}(Y)$. Therefore, we define the quotient space $W_{\text{per}}(Y) = H^1_{\text{per}}(Y)/\mathbb{R}$ as solution space where the solution is unique.

Solving (4.1) is to find $u \in W_{\text{per}}(Y)$, for any given $f \in (W_{\text{per}}(Y))^\star$ in the dual of $W_{\text{per}}(Y)$, such that

$$\int_Y A\nabla u \cdot \nabla v \, dx = \langle f, v\rangle_{(W_{\text{per}}(Y))^\star, W_{\text{per}}(Y)} \quad \forall v \in W_{\text{per}}(Y).$$

The requirement that $f \in (W_{\text{per}}(Y))^\star$ is equivalent to saying that

$$\int_Y f(y)\, dy = 0$$

because f defines a linear functional on $W_{\text{per}}(Y)$ and $f(0) = 0$, where $0 \in H^1_{\text{per}}(Y)/\mathbb{R}$. In particular, the equivalence class of 0 is same as the equivalence class 1 and hence

$$\int_Y f(y)\, dy = \langle f, 1\rangle = \langle f, 0\rangle = 0.$$

Theorem 4.1. *Let Y be unit open cell and let $a_{ij} \in L^\infty(\Omega)$ such that the matrix $A(y) = (a_{ij}(y))$ is elliptic with ellipticity constant $\alpha > 0$. For any $f \in (W_{per}(Y))^\star$, there is a unique weak solution $u \in W_{per}(Y)$ satisfying*

$$\int_Y A\nabla u \cdot \nabla v \, dx = \langle f, v\rangle_{(W_{per}(Y))^\star, W_{per}(Y)} \quad \forall v \in W_{per}(Y).$$

Note that the solution u we find from above theorem is an equivalence class of functions which are all possible solutions. Any representative element from the equivalence class is a solution. All the elements in the equivalence differ by a constant. Let u be an element from the equivalence class and let c be the constant

$$c = \frac{1}{|Y|}\int_Y u(y)\, dy.$$

Thus, we have $u-c$ is a solution with zero mean value in Y, i.e., $\int_Y u(y)\,dy = 0$. Therefore, rephrasing (4.1) as

$$\begin{cases} -\mathrm{div}(A(y)\nabla u(y)) & = f(y) \quad \text{in } Y \\ u & \text{is} \qquad Y - \text{periodic} \\ \frac{1}{|Y|}\int_Y u(y)\,dy & = 0 \end{cases}$$

we have unique solution u in the solution space

$$V_{\mathrm{per}}(Y) = \left\{ u \in H^1_{\mathrm{per}}(Y) \mid \frac{1}{|Y|}\int_Y u(y)\,dy = 0 \right\}.$$