Bloch-Floquet Transform

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Contents

1 Raison d'être

1.1 Fourier Transform

Recall that $-\Delta : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an unbounded, self-adjoint operator whose spectral decomposition is well-known. The "generalised" eigenfunctions[1] are the *plane or Fourier waves* $e^{\imath\xi\cdot x}$, for each $\xi \in \mathbb{R}^n$, and $|\xi|^2$ is an eigenvalue, for each $\xi \in \mathbb{R}^n$, giving the spectrum to be $[0,\infty)$. Further, $-\Delta(e^{\imath x\cdot\xi}) = |\xi|^2 e^{\imath x\cdot\xi}$.

Theorem 1.1. *Given any $f \in L^2(\mathbb{R}^n)$ there is a unique $\hat{f} \in L^2(\mathbb{R}^n)$ such that*

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{\imath\xi\cdot x}\, d\xi.$$

Also, for any $f, g \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi.$$

In particular, the Fourier transform $f \mapsto \hat{f}$ is an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

[1] For each $\xi \in \mathbb{R}^n$, $e^{\imath\xi\cdot x}$ are not elements of $L^2(\mathbb{R}^n)$ but they span $L^2(\mathbb{R}^n)$

The Fourier transform will change a differential equation in to an algebraic equation. For instance, $-\Delta u = f$ tranforms to, on applying Fourier transform,

$$\begin{aligned}
\hat{f}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-\imath x\cdot\xi}\,dx = -\frac{1}{(2\pi)^{n/2}} \sum_{j=1}^{n} \int_{\mathbb{R}^n} \frac{\partial^2 u(x)}{\partial x_j^2} e^{-\imath x\cdot\xi}\,dx \\
&= \frac{1}{(2\pi)^{n/2}} \sum_{j=1}^{n} (-\imath\xi_j) \int_{\mathbb{R}^n} \frac{\partial u(x)}{\partial x_j} e^{-\imath x\cdot\xi}\,dx \quad \text{(Integration by parts)} \\
&= -\sum_{j=1}^{n} (-\imath\xi_j)^2 \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-\imath x\cdot\xi}\,dx \quad \text{(Integration by parts)} \\
&= |\xi|^2 \hat{u}(\xi).
\end{aligned}$$

More generally, any m-th order linear differential equation with constant coefficients $P(D)u = f$ where $P(D) = \sum_{|\alpha|\le m} a_\alpha D^\alpha$ will transform in to an algebraic eqaution $P(\imath\xi)\hat{u}(\xi) = \hat{f}(\xi)$.

The Laplacian is a particular case of the elliptic operator $-\Delta + c(x)$ with $c \equiv 0$. For $c(x) \neq 0$ (without loss of generality assume $c(x) \ge 0$), the Bloch theorem gives the generalised eigenfunction for $-\Delta + c(x)$ when c is Y-periodic, for any given reference cell $Y \subset \mathbb{R}^n$.

1.2 Schrödinger Operator with Periodic Potential

Definition 1.2. *Let $\{e_i\}$ be the canonical basis for $\mathbb{R}^n$. Let $Y = \Pi_{i=1}^n [0, \ell_i)$ be a reference cell (or period) in $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be* Y-periodic *if $f(x + e_i p_i \ell_i) = f(x)$ for a.e. $x \in \mathbb{R}^n$ and all $p \in \mathbb{Z}^n$, for all $i = 1, 2, \ldots, n$.*

Consider the Schrödinger operator $-\Delta + c(x)$ where c is a periodic function, i.e., for some $\ell = (\ell_i) \in \mathbb{R}^n$ and $p \in \mathbb{Z}^n$, $c(x+e_i\ell_i p_i) = c(x)$. Let $L : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ be the operator $L := -\Delta + c(x)$.

To begin, let us consider the one dimension situation with $c \in C_c^\infty(\mathbb{R})$ with bounded derivatives and $L : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ defined as

$$L := -\frac{d^2}{dx^2} + c(x).$$

If c is 2π-periodic and, hence, c admits a uniformly convergent Fourier series

$$c(x) = \sum_{\eta\in\mathbb{Z}} c_\eta e^{\imath\eta x}$$

where

$$c_\eta = \frac{1}{2\pi} \int_{-\pi}^{\pi} c(x) e^{-\imath\eta x}\,dx.$$

If $u \in \mathcal{S}(\mathbb{R})$ then

$$\begin{aligned}
\widehat{Lu(x)}(\xi) &= \xi^2 \hat{u}(\xi) + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} c(x)u(x)e^{-\imath\xi x}\, dx \\
&= \xi^2 \hat{u}(\xi) + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\sum_{\eta\in\mathbb{Z}} c_\eta e^{\imath\eta x} \right) u(x)e^{-\imath\xi x}\, dx \\
&= \xi^2 \hat{u}(\xi) + \sum_{\eta\in\mathbb{Z}} c_\eta \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x)e^{-\imath(\xi-\eta)x}\, dx \\
&= \xi^2 \hat{u}(\xi) + \sum_{\eta\in\mathbb{Z}} c_\eta \hat{u}(\xi-\eta).
\end{aligned}$$

Thus, $\widehat{Lu}(\xi)$ depends only on the values $\hat{u}(\xi-\eta)$ for all $\eta \in \mathbb{Z}$. But recall that $\hat{u}(\xi-\eta) = \widehat{e^{\imath x\eta}u(x)}(\xi)$. This suggests that the operator L depends on the modulation by all $\eta \in \mathbb{Z}$.

1.3 Direct Integral Decomposition

Let H be a separable Hilbert space and (X, μ) be a σ-finite measure space. Let $L^2(X, \mu; H)$ is the Hilbert space of square integrable H-valued functions. If μ is a sum of point measures at finite set of points $x_1, \ldots, x_k$ then, any $f \in L^2(X, \mu; H)$, is determined by the k-tuple $(f(x_1), \ldots, f(x_k))$. Thus, $L^2(X, \mu; H)$ is isomorphic to the direct sum $\oplus_{i=1}^k H$. For more general μ, one may define a kind of "continuous direct sum" called the *constant fiber direct integral* and write

$$L^2(X, \mu; H) = \int_X^{\oplus} H\, d\mu.$$

Definition 1.3. *A function $T(\cdot) : X \to L(H)$ is measurable iff, for each $\phi, \psi \in H$, $\langle \phi, T(\cdot)\psi \rangle$ is measurable. $L^\infty(X, \mu; L(H))$ denotes the equivalence class (with a.e.) of measurable functions from X to $L(H)$ with*

$$\|T\|_\infty = \text{ess sup} \|T(x)\|_{L(H)} < \infty.$$

Definition 1.4. *A bounded operator T on $\mathcal{H} = \int_X^{\oplus} H\, d\mu$ is said to be decomposed by the direct integral decomposition iff there is $T(\cdot) \in L^\infty(X, \mu; L(H))$ such that, for all $\psi \in \mathcal{H}$,*

$$(T\psi)(x) = T(x)\psi(x).$$

We then say T is decomposable and

$$T = \int_X^{\oplus} T(x)\, d\mu(x).$$

The $T(x)$ are called the fibers *of T.*

Theorem 1.5. *Let $H = l_2$ and*

$$\mathcal{H} = \int_{(-\frac{1}{2}, \frac{1}{2}]}^{\oplus} H\, dx.$$

For $\eta \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, *let* $L_\eta : l_2 \to l_2$ *be defined as*

$$(L_\eta(z))_k = (\eta + k)^2 z_k + \sum_{m\in\mathbb{Z}} c_m z_{k-m}.$$

Define $T : L^2(\mathbb{R}) \to \mathcal{H}$ *by*

$$[(Tf)(\eta)]_k = \hat{f}(\eta + k).$$

For $L = -\frac{d^2}{dx^2} + c(x)$ *on* $L^2(\mathbb{R})$,

$$TLT^{-1} = \int_{(-\frac{1}{2},\frac{1}{2}]}^{\oplus} L_\eta \, d\eta.$$

When $c \equiv 0$, the eigenvalues and eigenfunctions of L_η are $(\eta + k)^2$ and the Fourier transform of $e^{\imath(\eta+k)x}$, respectively. This suggests that L_η is related to $-\frac{d^2}{dx^2}$ on $[0, 2\pi)$ with the boundary condition $u(2\pi) = e^{\imath 2\pi\eta}u(0)$ and $u'(2\pi) = e^{\imath 2\pi\eta}u'(0)$.

Lemma 1.6. *Let* $H = L^2[0, 2\pi)$ *and*

$$\mathcal{H} = \int_{(-\frac{1}{2},\frac{1}{2}]}^{\oplus} H \, d\eta.$$

Then $T : \mathcal{S}(\mathbb{R}) \to \mathcal{H}$ *given by*

$$(Tf)_\eta(x) = \sum_{m\in\mathbb{Z}} e^{\imath 2\pi m\eta} f(x + 2\pi m) \quad \eta \in (-\frac{1}{2}, \frac{1}{2}] \, x \in [0, 2\pi)$$

which extends uniquely to an unitary operator on $L^2(\mathbb{R})$. *Moreover,*

$$T\left(-\frac{d^2}{dx^2}\right)T^{-1} = \int_{(-\frac{1}{2},\frac{1}{2}]}^{\oplus} \left(-\frac{d^2}{dx^2}\right)_\eta d\eta \tag{1.1}$$

where $\left(-\frac{d^2}{dx^2}\right)_\eta$ *is the operator* $-\frac{d^2}{dx^2}$ *on* $L^2[0, 2\pi)$ *with boundary condition*

$$u(2\pi) = e^{\imath 2\pi\eta}u(0) \quad u'(2\pi) = e^{\imath 2\pi\eta}u'(0).$$

Proof. Let us note that T is well defined. For any $f \in \mathcal{S}(\mathbb{R})$, the series in RHS is convergent. For any $f \in \mathcal{S}(\mathbb{R})$, $Tf \in \mathcal{S}(\mathbb{R})$ because

$$\begin{aligned}
&\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_0^{2\pi} \left| \sum_{m=-\infty}^{\infty} e^{-\imath 2\pi m\eta} f(x + 2\pi m) \right|^2 dx \right) d\eta \\
= \;& \int_0^{2\pi} \left[\left(\sum_{m,p\in\mathbb{Z}} \overline{f(x + 2\pi m)} f(x + 2\pi p) \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\imath 2\pi(p-m)\eta} \, d\eta \right] dx \\
& (\text{ by Fubini's Theorem}) \\
= \;& \int_0^{2\pi} \left(\sum_{m\in\mathbb{Z}} |f(x + 2\pi m)|^2 \right) dx = \int_{\mathbb{R}} |f(x)|^2 \, dx.
\end{aligned}$$

Thus, T is well defined and admits a unique isometry extension. To see that T is onto $\mathcal{H}$, we compute $T^\star$. For any $g \in \mathcal{H}$, $x \in [0, 2\pi]$ and $m \in \mathbb{Z}$

$$(T^\star g)(x + 2\pi m) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{\imath 2\pi m\eta} g_\eta(x)\, d\eta.$$

Further,

$$\begin{aligned}
\|T^\star g\|_2^2 &= \int_{\mathbb{R}} |(T^\star g)(y)|^2\, dy \\
&= \int_0^{2\pi} \left(\sum_{m\in\mathbb{Z}} |(T^\star g)(2\pi m + x)|^2 \right) dx \\
&= \int_0^{2\pi} \left(\sum_{m\in\mathbb{Z}} \left| \int_0^{2\pi} e^{\imath 2\pi m\eta} g_\eta(x)\, d\theta \right|^2 \right) dx \\
&= \int_0^{2\pi} \left(\int_0^{2\pi} |g_\eta(x)|^2\, d\theta \right) dx \quad \text{(Parseval's Identity)} \\
&= \|g\|^2.
\end{aligned}$$

Finally, to prove (1.1), let G be the operator on the right-hand side of (1.1). We shall show that if $f \in \mathcal{S}(\mathbb{R})$, then $Tf \in D(G)$ and $T(-f'') = G(Tf)$. Since $-d^2/dx^2$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R})$ and G is self-adjoint, (1.1) will follow. So, suppose $f \in \mathcal{S}(\mathbb{R}^n)$, then Tf is given by the convergent sum as in the statement. Thus, $Tf \in C^\infty(0, 2\pi)$ with $(Tf)'_\eta(x) = (Tf'_\eta(x)$ and similarly for higher derivatives. Further, it is clear that

$$\begin{aligned}
(Tf)_\theta(2\pi) &= \sum_{m\in\mathbb{Z}} e^{-\imath 2\pi m\eta} f(2\pi(m+1)) \\
&= \sum_{m\in\mathbb{Z}} e^{-\imath 2\pi(m-1)\eta} f(2\pi m) = e^{\imath 2\pi\eta} (Tf)_\eta(0).
\end{aligned}$$

Similarly, $(Tf)'_\eta(2\pi) = e^{\imath 2\pi\eta}(Tf_\eta)'(0)$. Thus, for each η, $(Tf)_\eta \in D((-\frac{d^2}{dx^2})_\eta)$ and

$$\left(-\frac{d^2}{dx^2}\right)_\eta (Tf) = U(-f'')_\eta.$$

We conclude that $Tf \in D(G)$ and $G(Tf) = U(-f'')$. This proves (1.1). $\square$

Theorem 1.7 (Direct Integral Decomposition of Periodic Schrödinger operator)**.** *Let c be a bounded measurable function on $\mathbb{R}$ with period 2π. For $\eta \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, let*

$$L_\eta = \left(-\frac{d^2}{dx^2}\right)_\eta + c(x)$$

be an operator on $L^2[0, 2\pi]$. Let T be given by

$$(Tf)_\eta(x) = \sum_{m\in\mathbb{Z}} e^{\imath 2\pi m\eta} f(x + 2\pi m) \quad \eta \in \left(-\frac{1}{2}, \frac{1}{2}\right] \; x \in [0, 2\pi).$$

Then

$$T\left(-\frac{d^2}{dx^2}+c\right)T^{-1}=\int_{(-\frac{1}{2},\frac{1}{2}]}^{\oplus} L_\eta \, d\eta.$$

Proof. Let c be the η-independent operator acting on the fiber $H = L^2[0,2\pi)$ by $(c_\eta f)(x) = c(x)f(x)$ for $0 \le x \le 2\pi$. It is sufficient to prove that

$$TcT^{-1}=\int_{(-\frac{1}{2},\frac{1}{2}]}^{\oplus} c_\eta \, d\eta.$$

For $f \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}(Tcf)_\eta(x) &= \sum_{m\in\mathbb{Z}} e^{-\imath 2\pi m\eta} c(x+2\pi m) f(x+2\pi m)\\ &= c(x)\sum_{m\in\mathbb{Z}} e^{-\imath 2\pi m\eta} f(x+2\pi m)\\ &= c_\eta (Tf)_\eta(x).\end{aligned}$$

The second last equality is due to the periodicity of c. □

1.4 Bloch Periodic Functions

The Bloch transform is a generalization of Fourier transform that leaves the periodic functions invariant, in some sense. Let us begin by considering a generalization of periodic functions.

Definition 1.8. *Let $Y = \Pi_{i=1}^{n}[0,\ell_i)$ be a reference cell (or period) in $\mathbb{R}^n$. For each $\eta \in \mathbb{R}^n$, a function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be (η, Y)-*Bloch periodic *if $f(x + \ell \cdot p) = e^{\imath 2\pi p\cdot\eta} f(x)$ for a.e. $x \in \mathbb{R}^n$ and for all $p \in \mathbb{Z}^n$.*

Note that the case $\eta = 0$ corresponds to the usual notion of Y-periodic functions. Note that the boundary condition remains unchanged if η is replaced with $\eta + k$, for any $k \in \mathbb{Z}^n$. Hence, it is sufficient to consider $\eta \in Y^\star$ where $Y^\star = (-\frac{1}{2}, \frac{1}{2}]^n$. The cell $Y^\star$ is called the *reciprocal cell* and, in Physics literature, $Y^\star$ is known as the *first Brillouin zone*.

We shall assume that $Y = [0, 2\pi)^n$ and, for $j, k = 1, 2, \ldots, n$, $a_{jk} : Y \to \mathbb{R}$ is such that $a_{jk} \in L^\infty_{\text{per}}(Y)$. Let $A(y) = (a_{jk}(y)) \in M(\alpha, \beta, Y)$ and is a symmetric matrix, i.e., $a_{jk}(y) = a_{kj}(y)$. One can extend a_{jk} to entire $\mathbb{R}^n$ as a Y-periodic function. Also, c is a Y-periodic function such that $c(y) \ge c_3 > 0$. We are interested in the spectral resolution of closure of the operator $\mathcal{A} = -\text{div}(A(y)\nabla) + c(y)$ in $L^2(\mathbb{R}^n)$.

By Bloch Theorem, it is enough to study the (η, Y)-Bloch periodic eigenvalue problem, for each $\eta \in \mathbb{R}^n$, i.e.,

Definition 1.9. *For any fixed (momentum) vector $\eta \in Y^\star$, consider the eigenvalue problem: given a symmetric $A \in M(\alpha, \beta, Y)$, find $\lambda(\eta) \in \mathbb{C}$ and non-zero $\psi(\cdot;\eta) : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\left\{\begin{array}{lll} \mathcal{A}\psi(y;\eta)) & = \lambda(\eta)\psi(y;\eta) & \text{in } \mathbb{R}^n \\ \psi(y+2\pi\ell) & = e^{2\pi\imath\ell\cdot\eta}\psi(y) & \ell \in \mathbb{Z}^n, y \in \mathbb{R}^n. \end{array}\right. \tag{1.2}$$

The eigenvalues ψ are known as Bloch waves *associated with $\mathcal{A}$ and the eigenvalues λ are called* Bloch eigenvalues.

Suppose $\eta \in Y^\star$ have rational components and $\eta = (\eta_1, \ldots, \eta_n)$. Recall that there is a homeomorphism from $Y^\star$ to S^1. Thus, $e^{\imath 2\pi \eta_j} \in S^1$. In this sense, the Bloch periodicity condition has the form $e^{2\pi \imath p \cdot \eta} = \omega^p$ where $\omega \in [S^1]^n$ and $\omega^p = \omega_1^{p_1}\omega_2^{p_2}\ldots\omega_n^{p_n}$. For any $m \in \mathbb{Z}^n$, let $D_m \subset [S^1]^n$ be the collection of all $\omega \in [S^1]^n$ such that its j-th component is the m_j-th root of unity. Thus, $\omega^m = 1$ for all $\omega \in D_m$. The spectral problem (1.2) may be seen as a sequence of spectral problems, i.e., for each $m \in \mathbb{Z}^n$, we define ψ_m as

$$\begin{cases} \mathcal{A}\psi_m(y) &= \lambda_m \psi_m(y) \quad \text{in } \mathbb{R}^n \\ \psi_m(y + 2\pi m) &= \psi(y) \qquad\quad y \in \mathbb{R}^n. \end{cases}$$

Note that in the above boundary condition ψ is Y_m-periodic where $Y_m = \prod_{i=1}^n [0, 2\pi m_i)$. The space of spectral decomposition is $L^2_{\text{per}}(Y_m)$ which admits the orthogonal decomposition $L^2_{\text{per}}(Y_m) = \oplus_{\omega \in D_m} L^2_{\text{per}}(\omega, Y)$ where

$$L^2_{\text{per}}(\omega, Y) = \{\psi \in L^2_{\text{loc}}(\mathbb{R}^n) \mid \psi(y + 2\pi\ell) = \omega^\ell \psi(y)\ \forall \ell \in \mathbb{Z}^n, y \in \mathbb{R}^n\}.$$

Thus, we observe that the above space consists of (η, Y)-Bloch Periodic functions. For any irrational η can be approximated by rationals by varying m and noting that the sets of roots of unity is dense in S^1.

2 Bloch Transform

Theorem 2.1 (Bloch Decomposition)**.** *Let $Y = [0, 2\pi)^n$ and $Y^\star = \left(-\frac{1}{2}, \frac{1}{2}\right]^n$. Given a $f \in L^2(\mathbb{R}^n)$ there is a unique function, called* Bloch Transform*, $f_b \in L^2(Y \times Y^\star)$ such that*

$$f(y) = \int_{Y^\star} f_b(y, \eta) e^{\imath \eta \cdot y}\, d\eta.$$

Also, for any $f, g \in L^2(\mathbb{R}^n)$, the Plancherel formula holds, i.e.,

$$\int_{\mathbb{R}^n} f(y)\overline{g(y)}\, dy = \int_Y \int_{Y^\star} f_b(y, \eta)\overline{g_b(y, \eta)}\, dy\, d\eta.$$

In particular, the Bloch transform $f \mapsto f_b$ is an isometry from $L^2(\mathbb{R}^n)$ to $L^2(Y \times Y^\star)$.

Proof. For any $f \in \mathcal{D}(\mathbb{R}^n)$ and for each $\eta \in Y^\star$, define

$$f_b(y; \eta) := \sum_{p \in \mathbb{Z}^n} f(y + 2\pi p) e^{-\imath (y + 2\pi p)\cdot \eta}.$$

The sum is well defined because it has finite number of terms because f has compact support. Note that $f_b(y; \eta)$ is Y-periodic in y variable because

$$f_b(y + 2\pi; \eta) := \sum_{p+1 \in \mathbb{Z}^n} f(y + 2\pi p) e^{-\imath (y + 2\pi p)\cdot \eta} = f_b(y; \eta).$$

Similarly, $e^{\imath y\cdot\eta} f_b(y;\eta)$ is $Y^\star$-periodic in η variable because, for $k \in \mathbb{Z}^n$,

$$\begin{aligned}
e^{\imath y\cdot(\eta+k)} f_b(y;\eta+k) &= e^{\imath y\cdot(\eta+k)} \sum_{p\in\mathbb{Z}^n} f(y+2\pi p) e^{-\imath(y+2\pi p)\cdot(\eta+k)} \\
&= e^{\imath y\cdot(\eta+k)} \sum_{p\in\mathbb{Z}^n} f(y+2\pi p) e^{-\imath(y+2\pi p)\cdot\eta} e^{-\imath(y+2\pi p)\cdot k} \\
&= e^{\imath y\cdot(\eta+k)} e^{-\imath y\cdot k} \sum_{p\in\mathbb{Z}^n} f(y+2\pi p) e^{-\imath(y+2\pi p)\cdot\eta} e^{-\imath 2\pi p\cdot k} \\
&= e^{\imath y\cdot\eta} f_b(y;\eta).
\end{aligned}$$

In the above relation we have used the fact that $e^{\imath 2\pi p\cdot k} = 1$. Observe that

$$e^{\imath y\cdot\eta} f_b(y;\eta) = \sum_{p\in\mathbb{Z}^n} f(y+2\pi p) e^{-2\imath\pi p\cdot\eta}.$$

Thus,

$$\begin{aligned}
\int_{Y^\star} e^{\imath y\cdot\eta} f_b(y;\eta)\, d\eta &= f(y) + \sum_{\substack{p\in\mathbb{Z}^n\\ p\neq 0}} f(y+2\pi p) \int_{Y^\star} e^{-2\imath\pi p\cdot\eta}\, dy \\
&= f(y) - \sum_{\substack{p\in\mathbb{Z}^n\\ p\neq 0}} f(y+2\pi p) \left[\frac{e^{-\imath\pi p} - e^{\imath\pi p}}{2\imath\pi p_1 \dots p_n}\right] dy \\
&= f(y).
\end{aligned}$$

Therefore, we have proved the results for all functions in $\mathcal{D}(\mathbb{R}^n)$. Similarly, one can prove the Plancherel's formula for functions in $\mathcal{D}(\mathbb{R}^n)$. The Bloch transform is a linear map on $\mathcal{D}(\mathbb{R}^n)$ bounded on $L^2(\mathbb{R}^n)$. Thus, by density of $\mathcal{D}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, the Bloch transform extends to $L^2(\mathbb{R}^n)$ and Plancherel's formula holds true. □

Remark 2.2. Note that, for each fixed $\eta \in Y^\star$, $y \mapsto f_b(y,\eta)$ is extended Y-periodic to $\mathbb{R}^n$ and, for each fixed $y \in Y$, $\eta \mapsto e^{\imath\eta\cdot y} f_b(y,\eta)$ is extended $Y^\star$-periodic to $\mathbb{R}^n$. Thus, the Bloch transform may be seen as an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n \times \mathbb{R}^n)$.

Remark 2.3. The Bloch transform is a "modulation" of Zak transform. The Zak transform for any $f \in \mathcal{D}(\mathbb{R}^n)$ is defined as

$$f_z(y;\eta) := \sum_{p\in\mathbb{Z}^n} f(y+2\pi p) e^{-\imath 2\pi p\cdot\eta}$$

and extended unitarily to to $L^2(\mathbb{R}^n)$. Further, $f_b(y;\eta) = e^{-\imath y\cdot\eta} f_z(y;\eta)$ for all $f \in \mathcal{D}(\mathbb{R}^n)$.

The following theorem explains the sense in which the Bloch transform leaves the periodic functions invariant.

Theorem 2.4 (Invariance of Periodic Functions)**.** *Let $Y = [0,2\pi)^n$ and $c : Y \to \mathbb{C}$ be such that $c \in L^\infty(Y)$ extended Y-periodically to $\mathbb{R}^n$. For any $f \in L^2(\mathbb{R}^n)$, $(cf)_b(y;\eta) = c(y) f_b(y;\eta)$.*

Proof. It is enough to prove the result for $f \in \mathcal{D}(\mathbb{R}^n)$. Consider

$$
\begin{aligned}
(cf)_b(y;\eta) &= \sum_{p\in\mathbb{Z}^n} c(y+2\pi p) f(y+2\pi p) e^{-\imath(y+2\pi p)\cdot\eta} \\
&= c(y) \sum_{p\in\mathbb{Z}^n} f(y+2\pi p) e^{-\imath(y+2\pi p)\cdot\eta} \\
&= c(y) f_b(y;\eta).
\end{aligned}
$$

By density the result is true for any $f \in L^2(\mathbb{R}^n)$. □

Theorem 2.5. *For any $f \in H^1(\mathbb{R}^n)$, $(\nabla_y f)_b(y;\eta) = (\nabla_y + \imath\eta) f_b(y;\eta)$.*

Proof. It is enough to prove the result for $f \in \mathcal{D}(\mathbb{R}^n)$. Consider

$$
\begin{aligned}
(\nabla_y f)_b(y;\eta) &= \sum_{p\in\mathbb{Z}^n} \left[\nabla_y f(y+2\pi p)\right] e^{-\imath(y+2\pi p)\cdot\eta} \\
&= \sum_{p\in\mathbb{Z}^n} \nabla_y \left[f(y+2\pi p) e^{-\imath(y+2\pi p)\cdot\eta}\right] \\
&\quad + \imath\eta \sum_{p\in\mathbb{Z}^n} f(y+2\pi p) e^{-\imath(y+2\pi p)\cdot\eta} \\
&= \left[\nabla_y + \imath\eta\right] f_b(y;\eta).
\end{aligned}
$$

□

For any $f \in L^2(\mathbb{R}^n)$, consider the equation $\mathcal{A}u = f$ in $\mathbb{R}^n$. Applying Bloch transform to this equation, using Theorems 2.4 and 2.5, we obtain a family of equations, indexed by $\eta \in Y^\star$, with periodic boundary conditions:

$$
\begin{cases}
\mathcal{A}(\eta) u_b(y;\eta) &= f_b(y;\eta) \quad \text{in } \mathbb{R}^n \\
u_b(y+2\pi\ell;\eta) &= u_b(y;\eta) \quad \ell \in \mathbb{Z}^n \; y \in \mathbb{R}^n,
\end{cases}
\tag{2.1}
$$

where $\mathcal{A}(\eta)$ is the *shifted operator*, denoted as

$$
\mathcal{A}(\eta) := -\sum_{j,k=1}^{n} \left(\frac{\partial}{\partial y_j} + \imath\eta_j\right) \left[a_{jk}(y) \left(\frac{\partial}{\partial y_k} + \imath\eta_k\right)\right] + c(y).
$$

The shifted operator equation admits a solution (being a periodic problem) in $H^1_{\text{per}}(Y)$ and a corresponding Poincaré inequality holds true, i.e., for all $u \in H^1_{\text{per}}(Y)$ and $\eta \in Y^\star$,

$$
c\left(\|\nabla u\|_{2,Y} + |\eta|\|u\|_{2,Y}\right) \leq \|\nabla u + \imath u\eta\|_{2,Y} \leq \|\nabla u\|_{2,Y} + |\eta|\|u\|_{2,Y}.
$$

2.1 Spectrum of Elliptic Operator

The spectral decomposition of $\mathcal{A}$, in one dimension periodic media, was first studied by Floquet (1883) and much later, in crystal lattice, by Bloch (1928). We shall compute the spectral decomposition of $\mathcal{A}$ in $L^2(\mathbb{R}^n)$ via the spectral decomposition of the shifted operator $\mathcal{A}(\eta)$. Consider the eigenvalue problem

$$
\begin{cases}
\mathcal{A}(\eta)\phi(y;\eta) &= \lambda(\eta)\phi(y;\eta) \quad \text{in } \mathbb{R}^n \\
\phi(y+2\pi\ell) &= \phi(y) \quad \ell \in \mathbb{Z}^n, y \in \mathbb{R}^n
\end{cases}
\tag{2.2}
$$

Theorem 2.6 (Periodic Eigen Value problem)**.** *There exists a sequence of pairs* (λ_m, ϕ_m) *satisfying*

$$\begin{cases} \mathcal{A}\phi(y) &= \lambda\phi(y) \quad \text{in } \mathbb{R}^n \\ \phi(y+2\pi\ell) &= \phi(y) \quad\quad \ell \in \mathbb{Z}^n, y \in \mathbb{R}^n \end{cases} \tag{2.3}$$

where $\{\lambda_m\}$ *are positive real eigenvalues and* $\{\phi_m(y)\}$ *are the corresponding eigenvectors, for each* $m \in \mathbb{N}$*, such that* $\{\phi_m\}$ *form an orthonormal basis of* $L^2_{per}(Y)$ *and* $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ *diverges and each eigenvalue has finite multiplicity.*

Remark 2.7. By Theorem 2.6, for each fixed $\eta \in Y^\star$, there exists a sequence of pairs (λ_m, ϕ_m) satisfying (2.2) where $\{\lambda_m(\eta)\}$ are positive real eigenvalues and $\{\phi_m(y;\eta)\}$ are the corresponding eigenvectors, for each $m \in \mathbb{N}$, such that $\{\phi_m(\cdot;\eta)\}$ form an orthonormal basis of $L^2_{\text{per}}(Y)$ and $0 \leq \lambda_1(\eta) \leq \lambda_2(\eta) \leq \dots$ diverges and each eigenvalue has finite multiplicity. By varying $\eta \in Y^\star$, we obtain the spectral resolution of $\mathcal{A}$ in $L^2(\mathbb{R}^n)$. The set $\{e^{\imath y\cdot\eta}\phi_m(y,\eta); m \in \mathbb{N}, \eta \in Y^\star\}$forms a 'generalised' basis of $L^2(\mathbb{R}^n)$. As a consequence, $L^2(\mathbb{R}^n)$ can be identified with $L^2(Y^\star;\ell^2(\mathbb{N}))$. $\mathcal{A}$ acts as a multiplication operator: $\mathcal{A}[e^{\imath y\cdot\eta}\phi_m(y,\eta)] = \lambda_m(\eta)e^{\imath y\cdot\eta}\phi_m(y,\eta)$. The spectrum of $\mathcal{A}$, denoted as $\sigma(\mathcal{A})$, coincides with the *Bloch spectrum* and denoted as σ_b. The Bloch spectrum is defined as the union of the images of all the mappings $\lambda_m(\eta)$, i.e.,

$$\sigma_b := \cup_{m=1}^\infty \left[\inf_{\eta\in Y^\star} \lambda_m(\eta), \sup_{\eta\in Y^\star} \lambda_m(\eta)\right].$$

The spectrum has a band structure. In contrast to the homogeneous case, $\sigma(\mathcal{A})$ need not fill up the entire $[0,\infty)$ and there may be gaps.

Theorem 2.8. *For any* $f \in L^2(\mathbb{R}^n)$*, its Bloch transform is given as*

$$f_b(y;\eta) = \sum_{m=1}^\infty f_b^m(\eta)\phi_m(y;\eta)$$

where, $\{\phi_m\}$ *are the eigenfunctions corresponding to the shifted operator* $\mathcal{A}(\eta)$ *and* $f_b^m(\eta)$*, for each* $\eta \in Y^\star$*, is the* m*-th Bloch coefficient of* f *defined as*

$$f_b^m(\eta) := \int_{\mathbb{R}^n} f(y)e^{-\imath y\cdot\eta}\overline{\phi_m(y;\eta)}\,dy.$$

Proof. It is enough to prove the result for $f \in \mathcal{D}(\mathbb{R}^n)$. Recall that, for each $\eta \in Y^\star$, $f_b(\cdot;\eta) \in L^2_{\text{per}}(Y)$. Hence, by spectral decomposition of $\mathcal{A}(\eta)$,

$$f_b(y;\eta) = \sum_{m=1}^\infty f_b^m(\eta)\phi_m(y;\eta),$$

where

$$f_b^m(\eta) = \int_Y f_b(y;\eta)\overline{\phi_m(y;\eta)}\,dy.$$

But,

$$\begin{aligned} f_b^m(\eta) &= \int_Y \sum_{p\in\mathbb{Z}^n} f(y+2\pi p)e^{-\imath(y+2\pi p)\cdot\eta}\overline{\phi_m(y;\eta)}\,dy \\ &= \int_Y \sum_{p\in\mathbb{Z}^n} f(y+2\pi p)e^{-\imath(y+2\pi p)\cdot\eta}\overline{\phi_m(y+2\pi p);\eta)}\,dy \\ &= \int_{\mathbb{R}^n} f(y)e^{-\imath y\cdot\eta}\overline{\phi_m(y;\eta)}\,dy. \end{aligned}$$

□

Remark 2.9. The Bloch inversion formula can rewritten as:

$$f(y) = \int_{Y^\star} e^{\imath y\cdot\eta} f_b(y;\eta)\,d\eta = \int_{Y^\star} e^{\imath y\cdot\eta}\sum_{m=1}^{\infty} f_b^m(\eta)\phi_m(y;\eta)\,d\eta.$$

Further, the Parseval formula holds:

$$\int_{\mathbb{R}^n} |f(y)|^2\,dy = \int_{Y^\star}\sum_{m=1}^{\infty} |f_b^m(\eta)|^2\,d\eta. \tag{2.4}$$

Remark 2.10 (Algebraic Formula for Solution)**.** For each $m \in \mathbb{N}$ and $\eta \in Y^\star$, multiply $\phi_m(y;\eta)$ on both sides of (2.1) to obtain

$$\begin{aligned} \int_Y \mathcal{A}(\eta)\left[\sum_{k=1}^{\infty} u_b^k(\eta)\phi_k(y;\eta)\right]\phi_m(y;\eta)\,dy &= \int_Y \sum_{k=1}^{\infty} f_b^k(\eta)\phi_k(y;\eta)\phi_m(y;\eta)\,dy \\ \int_Y \sum_{k=1}^{\infty} u_b^k(\eta)\phi_k(y;\eta)\lambda_m(\eta)\phi_m(y;\eta)\,dy &= f_b^m(\eta) \\ u_b^m(\eta)\lambda_m(\eta) &= f_b^m(\eta) \\ u_b^m(\eta) &= \frac{f_b^m(\eta)}{\lambda_m(\eta)}. \quad \square \end{aligned}$$

Set $\psi_m(y;\eta) := \{e^{\imath y\cdot\eta}\phi_m(y;\eta)\}$. Then, for each $\eta \in Y^\star$, $\psi_m(\cdot;\eta)$ forms a basis of $L^2(\mathbb{R}^n)$. Thus, $L^2(\mathbb{R}^n)$ can be identified with $L^2(Y^\star;\ell^2(\mathbb{N}))$. Let us compute $\psi(y+2\pi\ell)$:

$$\begin{aligned} \psi_m(y+2\pi\ell) &= e^{\imath y\cdot\eta}e^{2\pi\imath\ell\cdot\eta}\phi_m(y+2\pi\ell) \\ &= e^{\imath y\cdot\eta}e^{2\pi\imath\ell\cdot\eta}\phi_m(y) \\ &= e^{2\pi\imath\ell\cdot\eta}\psi_m(y). \end{aligned}$$

2.2 Regularity of $\lambda_m(\eta)$ and $\phi_1(\cdot,\eta)$

Theorem 2.11. *For all $m \geq 1$, $\eta \mapsto \lambda_m(\eta)$ is a Lipschitz function.*

Proof. Consider the quadratic form associated with $\mathcal{A}(\eta)$:

$$a(v,v;\eta) = \int_Y a_{jk}(y)\left(\frac{\partial v}{\partial y_k} + \imath\eta_k v\right)\overline{\left(\frac{\partial v}{\partial y_j} + \imath\eta_j v\right)}\,dy.$$

The quadratic form admits a decomposition as follows:

$$a(v,v;\eta) = a(v,v;\eta^0) + R(v,v;\eta,\eta^0)$$

where

$$\begin{aligned} R = \int_Y a_{jk}(y)\frac{\partial v}{\partial y_k}\overline{(\imath\eta_j - \imath\eta_j^0)v}\,dy \quad &+ \quad \int_Y a_{jk}(y)(\imath\eta_k - \imath\eta_k^0)v\overline{\frac{\partial v}{\partial y_j}}\,dy \\ &+ \quad \int_Y a_{jk}(y)(\eta_k\eta_j - \eta_k^0\eta_j^0)|v|^2\,dy. \end{aligned}$$

By Cauchy-Schwarz's inequality,

$$|R| \le C_0|\eta - \eta^0|\int_Y (|\nabla v|^2 + |v|^2)\,dy.$$

By min-max principle,

$$\lambda_m(\eta) = \min_{W\subset H^1_{\text{per}}(Y)}\max_{v\in W}\frac{a(v,v;\eta)}{\|v\|^2_{2,Y}}$$

where W is a m-dimensional subspace of $H^1_{\text{per}}(Y)$. Using the estimate on R, we deduce that

$$\lambda_m(\eta) \le \lambda_m(\eta^0) + C_0|\eta - \eta^0|$$

for a suitable constant C_0. Interchanging η and η^0, we obtain

$$|\lambda_m(\eta) - \lambda_m(\eta^0)| \le C_0|\eta - \eta^0|.$$

□

Theorem 2.12 (Analyticity)**.** *There is a $\delta > 0$ such that $\lambda_1(\eta)$ is analytic in the open ball $B_\delta(0)$ centred at origin and radius δ. Further, one can choose a corresponding unit eigenvector $\phi_1(y;\eta)$ satisfying*

(i) $\eta \mapsto \phi_1(\cdot;\eta)$ from $Y^\star$ to $H^1_{per}(Y)$ is analytic on $B_\delta(0)$.

(ii) $\phi_1(y;0) := |Y|^{-1/2} := (2\pi)^{-n/2}$.

(iii) $\|\phi_1(\cdot;\eta)\|_{2,Y} = 1$ and $\int_Y \phi_1(y;\eta)\,dy = 0$ for each $\eta \in B_\delta$.

2.3 Taylor Expansion of Ground State

Observe that (2.1) is a polynomial of degree two w.r.t η variable. Let $T_m(\eta) : L^2(Y) \to L^2(Y)$ be defined as

$$T_m(\eta)(\phi) = \mathcal{A}(\eta)\phi - \lambda_m\phi.$$

For a fixed $m \in \mathbb{N}$, let us compute the j-th first partial derivative of (2.2) w.r.t η to get

$$\mathcal{A}(\eta)\frac{\partial \phi_m}{\partial \eta_j} + \frac{\partial \mathcal{A}(\eta)}{\partial \eta_j}\phi_m = \lambda_m\frac{\partial \phi_m}{\partial \eta_j} + \phi_m\frac{\partial \lambda_m}{\partial \eta_j}.$$

Thus,

$$
\begin{aligned}
T_m(\eta)\frac{\partial \phi_m}{\partial \eta_j} &= -\frac{\partial \mathcal{A}(\eta)}{\partial \eta_j}\phi_m + \phi_m \frac{\partial \lambda_m}{\partial \eta_j} \\
&= \imath e_j A(\nabla_y + \imath\eta)\phi_m + (\nabla_y + \imath\eta)\cdot(\imath A e_j \phi_m) + \phi_m \frac{\partial \lambda_m}{\partial \eta_j}.
\end{aligned}
$$

There exists a solution to the above equation which is unique upto an additive multiple of ϕ_m. Hence, the RHS satisfies the compatibility condition or Fredhölm alternative. Therefore,

$$\int_Y T_m(\eta)\frac{\partial \phi_m}{\partial \eta_j}\overline{\phi}_m\,dy = 0$$

yields a formula for $\nabla_\eta \lambda_m(\eta^m)$ in terms of ϕ_m. Thus,

$$\frac{\partial \lambda_m}{\partial \eta_j}(\eta) = \left\langle \frac{\partial \mathcal{A}(\eta)}{\partial \eta_j}\phi_m(\cdot;\eta), \phi_m(\cdot;\eta)\right\rangle.$$

Similarly, by computing the j-th second partial derivative of (2.2) w.r.t η, we get

$$
\begin{aligned}
T_m(\eta)\frac{\partial^2 \phi_m}{\partial \eta_j \partial \eta_k} &= \imath e_j A(\nabla_y + \imath\eta)\frac{\partial \phi_m}{\partial \eta_k} + (\nabla_y + \imath\eta)\cdot\left(\imath A e_j \frac{\partial \phi_m}{\partial \eta_k}\right) \\
&\quad + \imath e_k A(\nabla_y + \imath\eta)\frac{\partial \phi_m}{\partial \eta_j} + (\nabla_y + \imath\eta)\cdot\left(\imath A e_k \frac{\partial \phi_m}{\partial \eta_j}\right) \\
&\quad + \frac{\partial \lambda_m}{\partial \eta_j}\frac{\partial \lambda_m}{\partial \eta_k} + \frac{\partial \lambda_m}{\partial \eta_k}\frac{\partial \lambda_m}{\partial \eta_j} - e_j A e_k \phi_m - e_k A e_j \phi_m \\
&\quad + \frac{\partial^2 \lambda_m}{\partial \eta_k \partial \eta_j}\phi_m.
\end{aligned}
$$

There exists a solution to the above equation which is unique upto an additive multiple of ϕ_m. Hence, the RHS satisfies the compatibility condition or Fredhölm alternative. Therefore,

$$\int_Y T_m(\eta)\frac{\partial^2 \phi_m}{\partial \eta_j \partial \eta_k}\overline{\phi}_m\,dy = 0$$

yields a formula for the Hessian matrix $D^2_\eta \lambda_m(\eta^m)$ in terms of ϕ_m. Thus,

$$
\begin{aligned}
\frac{1}{2}\frac{\partial^2 \lambda_m}{\partial \eta_j \partial \eta_k}(\eta) &= \langle a_{jk}\phi_m, \phi_m\rangle + \frac{1}{2}\left\langle \left[\frac{\partial \mathcal{A}(\eta)}{\partial \eta_j} - \frac{\partial \lambda_m}{\partial \eta_j}\right]\frac{\partial \phi_m}{\partial \eta_k}, \phi_m\right\rangle \\
&\quad + \frac{1}{2}\left\langle \left[\frac{\partial \mathcal{A}(\eta)}{\partial \eta_k} - \frac{\partial \lambda_m}{\partial \eta_k}\right]\frac{\partial \phi_m}{\partial \eta_j}, \phi_m\right\rangle.
\end{aligned}
$$

Let us summarise the properties of the eigenvalues $\lambda_m(\eta)$ and eigenvectors $\phi_m(y;\eta)$.

(a) All odd order derivatives of $\lambda_1(\eta)$ at $\eta = 0$ vanish.

(b) All odd order derivatives of $\phi_1(\cdot,\eta)$ at $\eta = 0$ are purely imaginary. For instance, the first order derivatives at $\eta = 0$ are given by

$$\frac{\partial \phi_1}{\partial \eta_j}(y;0) = \imath |Y|^{-1/2} w_j(y),$$

where $w_j \in H^1_{\text{per}}(Y)$ is the unique solution of the cell problem

$$\begin{cases} \mathcal{A}w_j & = \sum_{k=1}^n \frac{\partial a_{jk}}{\partial y_k} \quad \text{in } \mathbb{R}^n, \\ \frac{1}{|Y|}\int_Y w_j(y)\,dy & = 0. \end{cases}$$

(c) All even order derivatives of $\phi_1(\cdot;\eta)$ at $\eta = 0$ are real.

(d) Second order derivatives of $\lambda_1(\eta)$ at $\eta = 0$ are given by

$$\frac{1}{2}\frac{\partial^2 \lambda_1}{\partial \eta_j \partial \eta_k}(0) = a^0_{jk}, \quad \forall j,k = 1,...,n,$$

where a^0_{jk} are the homogenized coefficients defined by

$$\frac{1}{|Y|}\int_Y \left[a_{jk} + \sum_{m=1}^n a_{jm}\frac{\partial w_m}{\partial y_m}\right].$$

Theorem 2.13. *The origin is a critical point of the first Bloch eigenvalue, i.e.,* $\frac{\partial \lambda_1}{\partial \eta_j}(0) = 0$ *for all* $j = 1,...,n$*.Further, the Hessian of* λ_1 *at* $\eta = 0$ *is given by*

$$\frac{1}{2}\frac{\partial^2 \lambda_1}{\partial \eta_j \partial \eta_k}(0) = a^0_{jk} \quad \forall j,k = 1,...,n.$$

The derivatives of the first Bloch mode can also be calculated and they are as follows:

$$\frac{\partial \phi_1}{\partial \eta_j}(y;0) = \imath |Y|^{-\frac{1}{2}} w_j(y) \quad \forall j = 1,...,n.$$

Proof. Use the information $\lambda_1(0) = 0$ and $\phi_1(y;0) = |Y|^{-\frac{1}{2}}$ in the Taylor expansion with $\eta = 0$. □

3 Homogenization of Second order Elliptic Operator

Let $\mathcal{A}_\varepsilon = -\text{div}_x(A(x/\varepsilon)\nabla_x)$ be the elliptic opertor with periodically oscillating coefficients. If ξ corresponds to the Fourier variable corresponding to x then $\varepsilon\xi$ corresponds to the Fourier variable corresponding to x/ε. Recall that, for each $m \in \mathbb{N}$, $\{\lambda_m(\eta)\}$ and $\{e^{\imath y\cdot\eta}\phi_m(y;\eta)\}$ are the eigenvalues and eigenvectors, respectively, of $\mathcal{A} = -\text{div}_y(A(y)\nabla_y)$. We employ the change of variables, $y = x/\varepsilon$ and $\eta = \varepsilon\xi$, in the equation $\mathcal{A}[e^{\imath y\cdot\eta}\phi_m(y;\eta)] = \lambda_m(\eta)e^{\imath y\cdot\eta}\phi_m(y;\eta)$ to obtain

$$\varepsilon^2\mathcal{A}_\varepsilon\left[e^{\imath x\cdot\xi}\phi_m\left(\frac{x}{\varepsilon};\varepsilon\xi\right)\right] = \lambda_m(\varepsilon\xi)e^{\imath x\cdot\xi}\phi_m\left(\frac{x}{\varepsilon};\varepsilon\xi\right).$$

Thus, the eigenvalues and eigenvectors of $\mathcal{A}_\varepsilon$ are $\varepsilon^{-2}\lambda_m(\varepsilon\xi)$ and $e^{\imath x\cdot\xi}\phi_m(x/\varepsilon;\varepsilon\xi)$. Set $\lambda^\varepsilon_m(\xi) := \varepsilon^{-2}\lambda_m(\varepsilon\xi)$ and $\phi^\varepsilon_m(x;\xi) := \phi_m(x/\varepsilon;\varepsilon\xi)$. Hence, the Bloch transform of $f \in L^2(\mathbb{R}^n)$, for each $x \in \mathbb{R}^n$ and $\varepsilon > 0$, is

$$f^\varepsilon_b(x;\xi) = \sum_{m=1}^\infty f^{m,\varepsilon}_b(\xi)\phi^\varepsilon_m(x;\xi)$$

where, for each $m \in \mathbb{N}$, $\varepsilon > 0$ and $\xi \in \varepsilon^{-1}Y^\star$, the m-th Bloch coefficient of f is

$$f_b^{m,\varepsilon}(\xi) = \varepsilon^{-n/2} \int_{\mathbb{R}^n} f(x) e^{-\imath x\cdot\xi} \overline{\phi_m^\varepsilon(x;\xi)}\, dx.$$

Thus, the inverse formula is

$$f(x) = \varepsilon^{n/2} \int_{\varepsilon^{-1}Y^\star} \sum_{m=1}^{\infty} f_b^{m,\varepsilon}(\xi) e^{\imath x\cdot\xi} \phi_m^\varepsilon(x;\xi)\, d\xi.$$

The $\varepsilon^{n/2}$ is a normalising factor appearing because the Lebesgue measure of $\varepsilon^{-1}Y^\star$ is ε^{-n}. The Plancherel identity holds: for any $f, g \in L^2(\mathbb{R}^n)$

$$\varepsilon^{-n} \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx = \int_{\varepsilon^{-1}Y^\star} \sum_{m=1}^{\infty} f_b^{m,\varepsilon}(\xi) \overline{g_b^{m,\varepsilon}(\xi)}\, d\xi.$$

Applying the Bloch transform, the equation $\mathcal{A}_\varepsilon u_\varepsilon = f$ transforms in to a set of algebraic equations, indexed by $m \geq 1$, $\lambda_m^\varepsilon(\xi) u_b^{m,\varepsilon}(\xi) = f_b^{m,\varepsilon}(\xi)$ for all $\xi \in \varepsilon^{-1}Y^\star$ (cf. Remark 2.10). Our aim is to pass to the limit in the system of algebraic equations. We first claim that one can neglect all the equations corresponding to $m \geq 2$.

Proposition 3.1. *Let*

$$v_\varepsilon(x) = \varepsilon^{n/2} \int_{\varepsilon^{-1}Y^\star} \sum_{m=2}^{\infty} u_b^{m,\varepsilon}(\xi) e^{\imath x\cdot\xi} \phi_m^\varepsilon(x;\xi)\, d\xi.$$

Then $\|v_\varepsilon\|_{2,\mathbb{R}^n} \leq C_0\varepsilon$.

Proof. Since

$$\int_{\mathbb{R}^n} \mathcal{A}_\varepsilon u_\varepsilon \overline{u_\varepsilon}\, dx = \int_{\mathbb{R}^n} f(x)\overline{u_\varepsilon}(x)\, dx.$$

The LHS is bounded and, applying Plancherel Identity, we get

$$\begin{aligned}
\beta \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\, dx &\geq \varepsilon^n \int_{\varepsilon^{-1}Y^\star} \sum_{m=1}^{\infty} f_b^{m,\varepsilon}(\xi) \overline{u_b^{m,\varepsilon}}(\xi)\, d\xi \\
&= \varepsilon^n \int_{\varepsilon^{-1}Y^\star} \sum_{m=1}^{\infty} \lambda_m^\varepsilon(\xi) |u_b^{m,\varepsilon}(\xi)|^2\, d\xi \\
&= \varepsilon^{n-2} \int_{\varepsilon^{-1}Y^\star} \sum_{m=1}^{\infty} \lambda_m(\eta) |u_b^{m,\varepsilon}(\xi)|^2\, d\xi \\
&\geq \varepsilon^{n-2} \int_{\varepsilon^{-1}Y^\star} \sum_{m=2}^{\infty} \lambda_m(\eta) |u_b^{m,\varepsilon}(\xi)|^2\, d\xi \\
&\geq \varepsilon^{n-2} \lambda_2^{(N)} \int_{\varepsilon^{-1}Y^\star} \sum_{m=2}^{\infty} |u_b^{m,\varepsilon}(\xi)|^2\, d\xi.
\end{aligned}$$

The last inequality is a consequence of the min-max principle yielding, for $m \geq 2$,

$$\lambda_m(\eta) \geq \lambda_2(\eta) \geq \lambda_2^{(N)} > 0 \quad \forall \eta \in Y^\star,$$

where $\lambda_2^{(N)}$ is the second eigenvalue of the eigenvalue problem for $\mathcal{A}$ in the cell Y with Neumann boundary condition on ∂Y. Then

$$\varepsilon^n \int_{\varepsilon^{-1}Y^\star} \sum_{m=2}^{\infty} |u_b^{m,\varepsilon}(\xi)|^2 \, d\xi \leq C_0 \varepsilon^2.$$

By Parseval's Identity, the left side is equal to $\|v_\varepsilon\|_{2,\mathbb{R}^n}$. □

Remark 3.2. Consider the algebraic equation corresponding to $m = 1$, i.e.,

$$\lambda_1^\varepsilon(\xi) u_b^{1,\varepsilon}(\xi) = f_b^{1,\varepsilon}(\xi) \quad \forall \xi \in \varepsilon^{-1} Y^\star.$$

Multiplying both sides by $\varepsilon^{n/2}$, we get

$$\varepsilon^{-2} \lambda_1(\varepsilon\xi) \varepsilon^{n/2} u_b^{1,\varepsilon}(\xi) = \varepsilon^{n/2} f_b^{1,\varepsilon}(\xi) \quad \forall \xi \in \varepsilon^{-1} Y^\star.$$

Expanding $\lambda_1(\varepsilon\xi)$ by Taylor's formula around $\xi = 0$, we get

$$\left[\frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \lambda_1}{\partial \eta_j \eta_k}(0) \xi_j \xi_k + O(\varepsilon \xi^3) \right] \varepsilon^{n/2} u_b^{1,\varepsilon}(\xi) = \varepsilon^{n/2} f_b^{1,\varepsilon}(\xi)$$

Passing to the limit as $\varepsilon \to 0$ to get

$$\frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \lambda_1}{\partial \eta_j \eta_k}(0) \xi_j \xi_k \hat{u}_0(\xi) = \hat{f}(\xi).$$

Setting

$$a_{jk}^0 = \frac{1}{2} \frac{\partial^2 \lambda_1}{\partial \eta_j \eta_k}(0)$$

Then $\sum_{j,k=1}^n a_{jk}^0 \xi_k \xi_j \hat{u_0}(\xi) = \hat{f}(\xi)$ and $\mathcal{A}_0 u_0 := -\sum_{j,k=1}^n a_{jk}^0 \frac{\partial^2 u_0}{\partial x_j \partial x_k} = f(x)$. The only flaw in the above argument is that in passing to limit we have not checked uniform compact support of the sequence. To overcome this difficulty we use cut-off function technique to localize the equation.

Proposition 3.3 (First Bloch Transform tends to Fourier Transform)**.** *Let $\{g_\varepsilon\} \subset L^2(\mathbb{R}^n)$ be a sequence such that there is a fixed compact set $K \subset \mathbb{R}^n$ such that supp$(g_\varepsilon) \subseteq K$ for all ε. If $g_\varepsilon \rightharpoonup g$ weakly in $L^2(\mathbb{R}^n)$ then $\varepsilon^{\frac{n}{2}} g_b^{1,\varepsilon} \rightharpoonup \hat{g}$ weakly in $L^2_{loc}(\mathbb{R}^n)$.*

Proof. The first Bloch transform $g_b^{1,\varepsilon}(\xi)$, *a priori* defined for

$$\xi \in \varepsilon^{-1} Y^\star = \left(-\frac{\varepsilon^{-1}}{2}, \frac{\varepsilon^{-1}}{2}\right)^n$$

can be extended by zero outside $\varepsilon^{-1} Y^\star$. We write

$$\begin{aligned} \varepsilon^{\frac{n}{2}} g_b^{1,\varepsilon}(\xi) &= \int_{\mathbb{R}^n} g_\varepsilon(x) e^{-\imath x \cdot \xi} \overline{\phi_1(\frac{x}{\varepsilon}; 0)} \, dx \\ &\quad + \int_K g_\varepsilon(x) e^{-\imath x \cdot \xi} \left(\overline{\phi_1(\frac{x}{\varepsilon}; \varepsilon\xi)} - \overline{\phi_1(\frac{x}{\varepsilon}; 0)} \right) dx. \end{aligned}$$

Since $\phi_1(y;0) = |Y|^{\frac{-1}{2}} = (2\pi)^{-n/2}$, the first term is nothing but the Fourier transform of g_ε and so it converges weakly to $\hat{g}(\xi)$ in $L^2(\mathbb{R}^n)$. By Cauchy-Schwarz inequality and the regularity of the first Bloch eigenfunction $\eta \mapsto \phi_1(\cdot,\eta) \in L^2_{\text{per}}(Y)$ at $\eta = 0$, the second term is bounded by

$$\|g_\varepsilon\|_{2,\mathbb{R}^n} \left[\int_K |\phi_1(\frac{x}{\varepsilon};\varepsilon\xi) - \phi_1(\frac{x}{\varepsilon};0)|^2\,dx\right]^{\frac{1}{2}} \le C_0\|\phi_1(y;\varepsilon\xi) - \phi_1(y;0)\|_{2,Y}.$$

By Lipschitz continuity of $\eta \mapsto \phi_1(\cdot,\eta)$, the second term in the right side is bounded above by $C_0\varepsilon\xi$. Thus, if $|\xi| \le M$ then it is bounded above by $cM\varepsilon$ and so, in particular, it converges to zero in $L^\infty_{\text{loc}}(\mathbb{R}^n)$. □

Theorem 3.4. *Let $\Omega \subset \mathbb{R}^n$ be an arbitrary, not necessarily bounded, domain. Consider a sequence $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1(\Omega)$ and $A_\varepsilon u_0 = f$ in Ω with $f \in L^2(\Omega)$. Then u_0 satisfies $A_0 u_0 = f$ in Ω. In fact, $A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_0 \nabla u_0$ weakly in $L^2(\Omega)$.*

Proof. Let $\phi \in D(\Omega)$ be arbitrary. If u_ε satisfies $\mathcal{A}_\varepsilon u_\varepsilon = f$ in Ω then consider its localization ϕu_ε satisfies

$$\mathcal{A}_\varepsilon(\phi u_\varepsilon) = \phi f + g_\varepsilon + h_\varepsilon \text{ in } \quad \mathbb{R}^n,$$

where

$$\begin{aligned}
g_\varepsilon &= -2\sum_{j=1}^n \sigma_j^\varepsilon \frac{\partial\phi}{\partial x_j} - \sum_{j,k=1}^n a_{jk}^\varepsilon \frac{\partial^2\phi}{\partial x_j \partial x_k} u_\varepsilon,\\
\sigma_j^\varepsilon(x) &= \sum_{k=1}^n a_{jk}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_k},\\
h_\varepsilon &= -\sum_{j,k=1}^n \frac{\partial a_{jk}^\varepsilon}{\partial x_j}\frac{\partial\phi}{\partial x_k} u_\varepsilon.
\end{aligned}$$

Using the arguments given in the remark above, we can pass to the limit above, since ϕu_ε is bounded in $H^1(\mathbb{R}^n)$. Neglecting all the harmonics corresponding to $m \ge 2$ and considering only the $m = 1$ yields at the limit

$$\frac{1}{2}\sum_{j,k=1}^n \frac{\partial^2\lambda_1}{\partial\eta_j\partial\eta_k}(0)\xi_j\xi_k\widehat{(\phi u_0)}(\xi) = \widehat{(\phi f)}(\xi) + \lim_{\varepsilon\to 0}\varepsilon^{\frac{n}{2}} g_b^{1,\varepsilon}(\xi) + \lim_{\varepsilon\to 0}\varepsilon^{\frac{n}{2}}\hat{h}_b^{1,\varepsilon}(\xi). \tag{3.1}$$

The sequence σ_j^ε is bounded in $L^2(\Omega)$. Therefore, we can extract a subsequence (still denoted by ε) which is weakly convergent in $L^2(\Omega)$. Let σ_j^0 denote its limit and its extension by zero outside Ω. Using this convergence and the definition of g_ε, we see that

$$g_\varepsilon \rightharpoonup g_0 := -2\sum_{j=1}^n \sigma_j^0 \frac{\partial\phi}{\partial x_j} - \sum_{j,k=1}^n \mathcal{M}(a_{jk})\frac{\partial^2\phi}{\partial x_j\partial x_k} u_0 \text{ weakly in } L^2(\mathbb{R}^n),$$

where $\mathcal{M}(a_{jk})$ is the average of a_{jk} on Y. Therefore,

$$\varepsilon^{\frac{n}{2}} g_b^{1,\varepsilon}(\xi) \rightharpoonup \hat{g}_0(\xi) \text{ weakly in } L^2_{\text{loc}}(\mathbb{R}^n).$$

A similar argument fails for $\{h_b^{1,\varepsilon}\}$ because h_ε is not bounded in $L^2(\mathbb{R}^n)$. We decompose

$$\begin{aligned}\varepsilon^{\frac{n}{2}} h_b^{1,\varepsilon}(\xi) &= \int_{\mathbb{R}^n} h_\varepsilon(x) e^{-\imath x\cdot\xi} \overline{\phi_1\left(\frac{x}{\varepsilon},0\right)}\, dx \\ &+ \int_{\mathbb{R}^n} h_\varepsilon(x) e^{-\imath x\cdot\xi} \left(\overline{\phi_1\left(\frac{x}{\varepsilon};\varepsilon\xi\right)} - \overline{\phi_1\left(\frac{x}{\varepsilon};0\right)}\right) dx.\end{aligned}$$

Using the Taylor expansion of $\phi_1(y;\eta)$ at $\eta = 0$, the second term is equal to

$$-\varepsilon^{-1}\sum_{j,k=1}^n \int_{\mathbb{R}^n} \frac{\partial a_{jk}}{\partial y_j}\left(\frac{x}{\varepsilon}\right)\frac{\partial \phi}{\partial x_k}(x) u_\varepsilon(x) e^{-\imath x\cdot\xi}\left[\varepsilon\sum_{\ell=1}^n \frac{\partial\overline{\phi_1}}{\partial \eta_\ell}\left(\frac{x}{\varepsilon};0\right)\xi_\ell + O(\varepsilon^2\xi^2)\right] dx,$$

which evidently converges to

$$-\sum_{j,k,\ell=1}^n \mathcal{M}\left(\frac{\partial a_{jk}}{\partial y_j}\frac{\partial\overline{\phi_1}}{\partial\eta_\ell}(y;0)\right)\xi_\ell\int_{\mathbb{R}^n}\frac{\partial\phi}{\partial x_k} u_0 e^{-\imath x\cdot\xi}\, dx.$$

strongly in $L^\infty_{\text{loc}}(\mathbb{R}^n)$. On the other hand, after integraing by parts, the first term in the RHS of the decomposition of $\varepsilon^{n/2}h_b^{1,\varepsilon}$ becomes

$$\sum_{j,k=1}^n\int_{\mathbb{R}^n} a_{jk}^\varepsilon\left[\frac{\partial^2\phi}{\partial x_j\partial x_k}u_\varepsilon + \frac{\partial\phi}{\partial x_k}\frac{\partial u^\varepsilon}{\partial x_j} - \imath\xi_j\frac{\partial\phi}{\partial x_k}u_\varepsilon\right] e^{-\imath x\cdot\xi}\overline{\phi_1}\left(\frac{x}{\varepsilon};0\right) dx.$$

Choosing $\phi_1(y;0) = |Y|^{-\frac{1}{2}}$, it is easily seen that the above integral converges weakly in $L^2(\mathbb{R}^n)$ to

$$\begin{aligned}&|Y|^{-\frac{1}{2}}\sum_{j,k=1}^n\int_{\mathbb{R}^n}\left[\mathcal{M}(a_{jk})\frac{\partial^2\phi}{\partial x_j\partial x_k}u_0 - \imath\xi_j\mathcal{M}(a_{jk})\frac{\partial\phi}{\partial x_k}u_0\right] e^{-\imath x\cdot\xi}\, dx \\ +\ &|Y|^{-\frac{1}{2}}\sum_{k=1}^n\int_{\mathbb{R}^n}\sigma_k^0\frac{\partial\phi}{\partial x_k}e^{-\imath x\cdot\xi}\, dx.\end{aligned}$$

Using this information in (3.1) and using Theorem 2.13, we conclude that

$$\begin{aligned}\sum_{j,k=1}^n a_{jk}^0\xi_j\xi_k\widehat{(\phi u_0)}(\xi) &= \widehat{(\phi f)}(\xi) - |Y|^{-\frac{1}{2}}\sum_{k=1}^n\int_{\mathbb{R}^n}\sigma_k^0\frac{\partial\phi}{\partial x_k}e^{-\imath x\cdot\xi}\, dx \\ &\quad -\imath\sum_{j,k=1}^n\xi_j|Y|^{-\frac{1}{2}}a_{jk}^0\int_{\mathbb{R}^n}\frac{\partial\phi}{\partial x_k}u_0 e^{-\imath x\cdot\xi}\, dx.\end{aligned}$$

This can be rewritten as

$$\begin{aligned}[\widehat{\mathcal{A}_0(\phi u_0)}](\xi) &= \widehat{(\phi f)}(\xi) - |Y|^{-\frac{1}{2}}\sum_{k=1}^n\int_{\mathbb{R}^n}\sigma_k^0\frac{\partial\phi}{\partial x_k}e^{-\imath x\cdot\xi}\, dx \\ &\quad -\imath\sum_{j,k=1}^n\xi_j|Y|^{-\frac{1}{2}}a_{jk}^0\int_{\mathbb{R}^n}\frac{\partial\phi}{\partial x_k}u_0 e^{-\imath x\cdot\xi}\, dx.\end{aligned}$$

This is the *localized homogenized equation* in the Fourier space. Taking inverse Fourier transform of the above equation, we obtain

$$\mathcal{A}_0(\phi u_0) = \phi f - \sum_{k=1}^{n} \sigma_k^0 \frac{\partial \phi}{\partial x_k} - \sum_{j,k=1}^{n} a_{jk}^0 \frac{\partial}{\partial x_j}\left(\frac{\partial \phi}{\partial x_k} u_0\right) \text{ in } \mathbb{R}^n.$$

On the other hand, we can calculate $\mathcal{A}_0(\phi u_0)$ directly:

$$\mathcal{A}_0(\phi u_0) = -\sum_{j,k=1}^{n} \left[a_{jk}^0 \frac{\partial^2 \phi}{\partial x_j \partial x_k} u_0 + 2a_{jk}^0 \frac{\partial \phi}{\partial x_j}\frac{\partial u_0}{\partial x_k}\right] + \phi \mathcal{A}_0 u_0 \text{ in } \mathbb{R}^n.$$

A comparison of the above two equation yields

$$\phi(\mathcal{A}_0 u_0 - f) = \sum_{j=1}^{n}\left(\sum_{k=1}^{n} a_{jk}^0 \frac{\partial u_0}{\partial x_k} - \sigma_j^0\right)\frac{\partial \phi}{\partial x_j} \text{ in } \mathbb{R}^n.$$

Since the above relation is true for all ϕ in $\mathcal{D}(\Omega)$, the desired conclusions follow. In fact, let us choose $\phi(x) = \phi_0(x)e^{\imath m x\cdot\nu}$, where ν is a unit vector in $\mathbb{R}^n$ and $\phi_0(x) \in \mathcal{D}(\Omega)$ is fixed. Letting $m \to \infty$ in the resuling relation and varying the unit vector ν, we can easily deduce, successively, that $\sigma_j^0 = \sum_{k=1}^{n} a_{jk}^0 \frac{\partial u_0}{\partial x_k}$ in Ω and $\mathcal{A}_0 u_0 = f$ in Ω. □