Introduction to Γ-convergence

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Contents

1 Motivation

Let Ω be an open bounded subset of $\mathbb{R}^n$ and let $\partial\Omega$ denote the boundary of Ω. For any given $0 < a < b$, let $\mathcal{M} = \mathcal{M}(a, b, \Omega)$ denote the class of all $n \times n$ matrices, $A = A(x)$, with $L^\infty(\Omega)$ entries such that,

$$a|\xi|^2 \leq A(x)\xi.\xi \leq b|\xi|^2 \quad a.e.\ x \quad \forall \xi \in \mathbb{R}^n.$$

Recall the following result on variational inequality on a Hilbert space. Refer [6] for a complete theory on variational inequality.

Theorem 1.1. *Let $a(x, y)$ be a coercive bilinear form on H, $K \subset H$ be a closed and convex subset of H and $f \in H'$. Then there exists a unique solution $x \in K$ to*

$$a(x, y - x) \geq \langle f, y - x\rangle, \quad \forall y \in K. \tag{1.1}$$

The case $K = H$ in the above result is popularly known as Lax-Milgram result. In fact, by choosing $y = x + z$ and $y = x - z$ for any $z \in H$ in (1.1), we have $a(x, z) = \langle f, z\rangle$ for all $z \in H$ and for every given $f \in H'$.

The Lax-Milgram result implies the existence of a weak solution to the following second order elliptic equation with Dirichlet boundary condition,

$$\begin{cases} -\text{div}(A\nabla u) &= f \quad \text{in } \Omega \\ \qquad\quad u &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where $A \in \mathcal{M}(a, b, \Omega)$ and let $f \in H^{-1}(\Omega)$. In fact, one also has the estimate

$$\|u\|_{H_0^1(\Omega)} \leq \frac{1}{a}\|f\|_{H^{-1}(\Omega)}. \tag{1.3}$$

The bounded elliptic operator $\mathcal{A} = -\text{div}(A\nabla)$ from $H_0^1(\Omega)$ into $H^{-1}(\Omega)$ is an isomorphism and the norm of $\mathcal{A}^{-1}$ is not larger than a^{-1} (cf. (1.3)).

Moreover, the weak solution u of (1.2) can also be characterized as the minimizer in $H_0^1(\Omega)$ of the functional

$$J(v) = \frac{1}{2}\int_\Omega A\nabla v.\nabla v\, dx - \langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)},$$

i.e.,

$$J(u) = \min_{v\in H_0^1(\Omega)} J(v).$$

Thus, the problem of studying the asymptotic behaviour of the second order elliptic problem

$$\begin{cases} -\text{div}(A_\varepsilon\nabla u_\varepsilon) &= f \quad \text{in } \Omega \\ \qquad\quad u_\varepsilon &= 0 \quad \text{on } \partial\Omega, \end{cases} \tag{1.4}$$

with $\{A_\varepsilon\} \subset \mathcal{M}$ is equivalent to finding a functional J on $H_0^1(\Omega)$ whose minimum is the solution of the homogenized elliptic equation such that both the minimizers and minima of J_ε converge to the minimizers and minima of J. Thus, we need to study the convergence of functionals such that the minimizers and minima converge.

2 Direct Method of Calculus of Variation

Definition 2.1. *Let X be a topological space. A function $F : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is said to be* lower semicontinuous (lsc) *at a point $x \in X$ if*

$$F(x) = \sup_{U\in N(x)} \inf_{y\in U} F(y).$$

F is lower semicontinuous *on X if F is lower semicontinuous at each point $x \in X$.*

Remark 2.1. Let X be a topological space satisfying first axiom of countability. Then a function $F : X \to \overline{\mathbb{R}}$ is lower semicontinuous at $x \in X$ iff

$$F(x) \leq \liminf_{n\to\infty} F(x_n)$$

for every sequence $\{x_n\}$ converging to $x \in X$.

Exercise 1. Show that if F is lower semicontinuous then the sublevel set $\{F \leq \alpha\} := \{x \in X : F(x) \leq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$.

Definition 2.2. *A function $F : X \to \overline{\mathbb{R}}$ is* coercive *on X if the closure of the sublevel set $\{F \leq \alpha\} := \{x \in X : F(x) \leq \alpha\}$ is compact in X for every $\alpha \in \mathbb{R}$.*

Exercise 2. Show that if F is a coercive functional on X and $G \geq F$, then G is coercive.

Exercise 3. If F is coercive then there is a non-empty compact set K such that

$$\inf_{x\in X} F(x) = \inf_{x\in K} F(x).$$

Theorem 2.1. *Let X be a topological space. Assume that the function $F : X \to \overline{\mathbb{R}}$ is coercive and lower semicontinuous. Then F has a minimizer in X.*

Proof. If F is identically $+\infty$ or $-\infty$, then every point of X is a minimum point for F. If F takes the value $-\infty$, then all those points are minimizers of F. Suppose now that F is not identically $+\infty$ and $F > -\infty$. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n\to\infty} F(x_n) = \inf_{y\in X} F(y) := d.$$

The existence of such a sequence is clear. Without loss of generality, we can assume that $F(x_n) < +\infty$ for all n. Let $\alpha := \sup_n F(x_n) < +\infty$. Moreover, since F is coercive, the sublevel set $\{F \leq \alpha\}$ is compact and hence there is a subsequence $\{x_k\}$ of $\{x_n\}$ which converges to a point $x \in X$. Since F is lsc we obtain

$$d = \inf_{y\in X} F(y) \leq F(x) \leq \liminf_{k\to\infty} F(x_k) = d.$$

Thus, $F(x) = d$ and hence is the minimizer of F in X. which proves our theorem. ☐

Definition 2.3. *A family of functionals $\{F_n\}$ on X is said to be* equi-coercive, *if for every $\alpha \in \mathbb{R}$, there is a compact set K_α of X such that the sublevel sets $\{F_n \leq \alpha\} \subseteq K_\alpha$ for all n.*

Exercise 4. If $\{F_n\}$ is a family of equi-coercive, then there is a non-empty compact K (independent of n) such that

$$\inf_{x \in X} F_n(x) = \inf_{x \in K} F_n(x).$$

Proposition 2.1. *A family of functions F_n on X is equi-coercive if and only if there exists a lower semicontinuous coercive function $\Psi : X \to \overline{\mathbb{R}}$ such that $F_n \geq \Psi$ on X, for every n.*

Proof. Let $\Psi : X \to \overline{\mathbb{R}}$ be a lower semicontinuous coercive function such that $F_n \geq \Psi$ on X, for every n. Set $K_\alpha := \{\Psi \leq \alpha\}$. K_α is closed and compact because of the lsc and coercivity of Ψ, respectively. Moreover, $\{F_n \leq \alpha\} \subseteq K_\alpha$, for all n. Thus, F_n are equi-coercive.

Conversely, let F_n be equi-coercive. Then, for each $\alpha \in \mathbb{R}$, there is a compact set K_α such that $\{F_n \leq \alpha\} \subseteq K_\alpha$, for all n. We shall now define $\Psi : X \to \overline{\mathbb{R}}$ as

$$\Psi(x) = \begin{cases} +\infty, & \text{if } x \notin K_\alpha, \forall \alpha \in \mathbb{R} \\ \inf\{\alpha \mid x \in K_\beta \text{ for all } \beta > \alpha\}. \end{cases}$$

We now show that $\Psi \leq F_n$ for all n. Let $x \in X$. If $F_n(x) = +\infty$, for all n, then by definition, $\Psi(x) = F_n(x) = +\infty$. Otherwise, let F_k be a subfamily such that $F_k(x) = \beta_k < \infty$. Thus, $x \in K_{\beta_k}$ for all k and hence $\Psi(x) = \inf_k\{\beta_k\} \leq F_n(x)$. Thus, $\Psi(x) \leq F(x)$, for every $x \in X$. It now remains to show that Ψ is lsc and coercive. Note that any $x \in \{\Psi \leq \alpha\}$ implies $x \in K_\beta$ for all $\beta > \alpha$. Therefore, the sublevel

$$\{\Psi \leq \alpha\} = \cap_{\beta > \alpha} K_\beta$$

is an arbitrary intersection compact sets and hence is closed and compact. ☐

Definition 2.4. *Let X be a vector space. We say a function $F : X \to \overline{\mathbb{R}}$ is* convex *if*

$$F(tx + (1-t)y) \leq tF(x) + (1-t)F(y)$$

for every $t \in (0,1)$ *and for every* $x, y \in X$ *such that* $F(x) < +\infty$ *and* $F(y) < +\infty$. *We say a function* $F : X \to \overline{\mathbb{R}}$ *is* strictly convex *if* F *is not identically* $+\infty$ *and*

$$F(tx + (1-t)y) < tF(x) + (1-t)F(y)$$

for every $t \in (0,1)$ *and for every* $x, y \in X$ *such that* $x \neq y$, $F(x) < +\infty$ *and* $F(y) < +\infty$.

Remark 2.2 (Jensen Inequality)**.** Let X be a real vector space and let $f : X \to \mathbb{R}$ be a convex function. Then for any given $x_1, x_2, \ldots, x_n \in X$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \in [0,1]$ such that $\sum_{i=1}^n \lambda_i = 1$, we have

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i). \tag{2.1}$$

Furthermore, if f is strictly convex then equality holds in (2.1) iff $x_1 = x_2 = \ldots = x_n$. In fact, more generally, if X is a Banach space, (E, μ) is a probability measure space, $f : X \to [0, +\infty]$ is a lsc, convex function, then

$$f\left(\int_E g \, d\mu\right) \leq \int_E f \circ g \, d\mu,$$

for all μ-integrable $g : E \to X$.

Proposition 2.2. *Let* X *be a vector space. Let* $F : X \to \overline{\mathbb{R}}$ *be a strictly convex function. Then* F *has at most one minimizer in* X. □

Proof. If x and y are two minimizers of F in X, then

$$F(x) = F(y) = d := \min_{z \in X} F(z) < +\infty.$$

If $x \neq y$, by strict convexity we have

$$F(tx + (1-t)y) < tF(x) + (1-t)F(y) = d, \quad \forall t \in (0,1).$$

This contradicts the fact that d is a minimum of F. Therefore $x = y$. □

Thus, combining Theorem 2.1 and Proposition 2.2, we have that on a topological vector space X, if F is a lower semicontinuous, coercive and strictly convex function, then F has a unique minimizer. We end this section with a definition from convex analysis.

Definition 2.5 (Convex Conjugate)**.** *Let X be a topological vector space and let $X^\star$ be its topological dual. If $F : X \to \mathbb{R}$, its* convex conjugate *$F^\star : X^\star \to \overline{\mathbb{R}}$ is defined as*

$$F^\star(x^\star) = \sup_{x \in X}\{x^\star(x) - F(x)\}.$$

Exercise 5. If F is convex and lower semicontinuous then $F = (F^\star)^\star$.

Exercise 6. Let A be a $n \times n$ symmetric, positive definite matrix and $F : \mathbb{R}^n \to \mathbb{R}$ be defined as

$$F(x) = \frac{1}{2}\langle x, Ax\rangle.$$

Show that

$$F^\star(x^\star) = \frac{1}{2}\langle x^\star, A^{-1}x^\star\rangle.$$

3 Γ-Convergence

The notion of Γ-convergence was introduced by Ennio De Giorgi in a sequence of papers (cf. [5, 3, 4]). An excellent account of this concept is the book of Dal Maso [2] and A. Braides [1].

Definition 3.1. *A function F is said to be the* Γ-limit *of F_n (denoted as $F_n \xrightarrow{\Gamma} F$) w.r.t the topology of X, if $F = F^+ = F^-$, where*

(i)

$$F^-(x) = \sup_{U \in N(x)} \liminf_{n \to \infty} \inf_{y \in U} F_n(y).$$

(ii)

$$F^+(x) = \sup_{U \in N(x)} \limsup_{n \to \infty} \inf_{y \in U} F_n(y).$$

We say F^- is the Γ-lower limit and F^+ is the Γ-upper limit.

Remark 3.1. If X is a topological space satisfying first axiom of countability, the Γ-limit can be characterised as satisfying the following two conditions:

(i) For every $x \in X$ and for every sequence $\{x_n\}$ converging to x in X, we have

$$\liminf_{n \to \infty} F_n(x_n) \geq F(x).$$

(ii) For every $x \in X$, there exists a sequence $\{x_n\}$ converging to x in X (called the Γ-realising sequence) such that

$$\lim_{n\to\infty} F_n(x_n) = F(x).$$

□

Exercise 7. Show that if $F_n \xrightarrow{\Gamma} F$, $G_n \xrightarrow{\Gamma} G$ and $F_n \leq G_n$, for each n, then $F \leq G$.

Exercise 8. Show that if F_n Γ-converges to F, then F is lower semicontinuous.

Exercise 9. Let X be a topological vector space. Show that if $F_n : X \to \overline{\mathbb{R}}$ is convex for each n, then $\Gamma\text{-}\limsup_n F_n$ is convex. Also show that the $\Gamma\text{-}\liminf_n F_n$ is, in general, not convex.

Exercise 10. Compute the Γ-limit of a constant sequence $F_n = F$ on X.

Theorem 3.1. *Let X be a topological space and F_n be a family functions on X.*

1. *If U is an open subset of X, then*

$$\inf_{x\in U} F^+(x) \geq \limsup_n \inf_{x\in U} F_n(x).$$

2. *If K is a compact subset of X, then*

$$\inf_{x\in K} F^-(x) \leq \liminf_n \inf_{x\in K} F_n(x).$$

Proof. 1. Let $x \in U$. Then, from the definition of Γ-upper limit which says $F(x)$ is sup over all neighbourhoods of x, we have

$$F^+(x) \geq \limsup_{n\to\infty} \inf_{y\in U} F_n(y).$$

Therefore,

$$\inf_{x\in U} F^+(x) \geq \limsup_{n\to\infty} \inf_{y\in U} F_n(y).$$

2. Since F^- is lsc and by the compactness of K, F^- attains its minimum on K (cf. Theorem 2.1). Set $d := \liminf_n \inf_{x\in K} F_n(x)$ and let x_n be a sequence (extracting subsequence, if necessary) in K such that $\lim_n F_n(x_n) = d$. Thus, there is a subsequence x_k which converges to

some $x \in K$. Therefore, for every neighbourhood U of x, $\inf_{y\in U} F_k(y) \leq F_k(x_k)$ for infinitely many k. Now, taking $\liminf$ both sides,

$$\liminf_k \inf_{y\in U} F_k(y) \leq \liminf_k F_k(x_k) = d$$

and taking supremum over all neighbourhoods U of x, we still have

$$F^-(x) = \sup_U \liminf_k \inf_{y\in U} F_k(y) \leq d.$$

Now, since $x \in K$, $\inf_{x\in K} F^-(x) \leq d$.

□

Theorem 3.2 (Fundamental Theorem of Γ-convergence)**.** *Let X be a topological space. Let $\{F_n\}$ be a equi-coercive family of functions and let F_n Γ-converges to F in X, then*

(i) F is coercive.

(ii) $\lim_{n\to\infty} d_n = d$, where $d_n = \inf_{x\in X} F_n(x)$ and $d = \inf_{x\in X} F(x)$. That is, the minima converges.

(iii) The minimizers of F_n converge to a minimizer of F.

Proof. Since $\{F_n\}$ are equi-coercive, by Proposition 2.1, there is a lsc, coercive function Ψ on X such that $F_n \geq \Psi$. Now, by Exercise 7, $F \geq \Psi$ and by Exercise 2 F is coercive.

Now, by putting $U = X$ in Theorem 3.1, we get $d \geq \limsup_n d_n$. We now need to show that $d \leq \liminf_n d_n$. If F_n are all not identically $+\infty$, then $\liminf_n d_n < +\infty$. Set $\liminf_n d_n = \alpha$. By the equi-coercivity of F_n, there is a compact set K_α such that $\{F_n \leq \alpha\} \subseteq K_\alpha$, for all n. Consider,

$$\begin{aligned} d \leq \inf_{y\in K_\alpha} F(y) &\leq \liminf_n \inf_{y\in K_\alpha} F_n(y) \\ &= \liminf_n \inf_{y\in X} F_n(y) \\ &= \liminf_n d_n. \end{aligned}$$

Thus, $\limsup_n d_n \leq d \leq \liminf_n d_n$ and hence, $\lim_n d_n = d$.

Since F is coercive and lsc (Γ-limit is always lsc), then by Theorem 2.1, F attains its minimum. Let x_n^* be a minimizer of F_n, then since F_n are equi-coercive x_n^* belong to a compact set K of X and hence converges up to a subsequence. Let $x_n^* \to x^*$ in X. We need to show that $F(x^*) = d$. By Γ-lower limit,

$$F(x^*) \leq \liminf_n F_n(x_n^*) = \liminf_n d_n = d.$$

But, $d \leq F(x^*)$. Hence $d = F(x^*)$. $\square$

Theorem 3.3 (Compactness)**.** *If X is a topological space satisfying second axiom of countability then any sequence of functionals $F_n : X \to \overline{\mathbb{R}}$ has a Γ-convergent subsequence.*

Proof. Let $\{U_k\}_{k\in\mathbb{N}}$ be a countable base for the topology of X. For each k, let $d_k^n = \inf_{y\in U_k} F_n(y)$. Thus, $\{d_k^n\}_n$ is a sequence in $\overline{\mathbb{R}}$ which is compact, hence has a subsequence $\{d_k^m\}_m$ whose limit as $m \to \infty$ exists in $\overline{\mathbb{R}}$. Thus, for each k, we have subsequence $\{d_k^m\}_m$ whose limit as $m \to \infty$ exists in $\overline{\mathbb{R}}$. Choose the diagonal sequence d_k^k whose limit exists n $\overline{\mathbb{R}}$ as $k \to \infty$. In other words, we have chosen a subsequence F_k of F_n such that

$$\lim_{k\to\infty} d_k^k = \lim_{k\to\infty} \inf_{y\in U_k} F_k(y).$$

Now, define $F(x) = \sup_{U\in N(x)} \lim_{k\to\infty} \inf_{y\in U_k} F_k(y)$ and we have by definition F_k Γ-converges to F. $\square$

Example 3.1. Let $A_\varepsilon \xrightharpoonup{H} A_0$ then we wish to show that $J_\varepsilon \xrightharpoonup{\Gamma} J$ in the weak topology of $H_0^1(\Omega)$ where

$$J_\varepsilon(u) = \int_\Omega A_\varepsilon \nabla u.\nabla u \, dx$$

and

$$J(u) = \int_\Omega A_0 \nabla u.\nabla u \, dx.$$

Let $u \in H_0^1(\Omega)$. We need to find a sequence $\{u_\varepsilon\}$ in $H_0^1(\Omega)$ such that u_ε converges to u weakly in $H_0^1(\Omega)$ and $\lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) = J(u)$. Let $u_\varepsilon \in H_0^1(\Omega)$ be the solution of

$$-\text{div}(A_\varepsilon \nabla u_\varepsilon) = -\text{div}(A_0 \nabla u). \tag{3.1}$$

Then, it follows from H-convergence that $u_\varepsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $\int_\Omega A_\varepsilon \nabla u_\varepsilon.\nabla u_\varepsilon \, dx \to \int_\Omega A_0 \nabla u.\nabla u \, dx$. Thus, we have shown the existence of a sequence $\{u_\varepsilon\}$ converging weakly to u in $H_0^1(\Omega)$ such that

$$\lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) = J(u).$$

Now, let $w_\varepsilon \in H_0^1(\Omega)$ be a sequence such that $w_\varepsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Then, the solution u_ε obtained in (3.1) minimizes the functional

$$\frac{1}{2}J_\varepsilon(v) - \int_\Omega A_0 \nabla u.\nabla v \, dx.$$

Hence, in particular, we have

$$\begin{aligned} \frac{1}{2}\int_\Omega A_\varepsilon \nabla w_\varepsilon.\nabla w_\varepsilon \, dx - \int_\Omega A_0 \nabla u.\nabla w_\varepsilon \, dx \quad &\geq \quad \frac{1}{2}\int_\Omega A_\varepsilon \nabla u_\varepsilon.\nabla u_\varepsilon \, dx \\ &\quad - \int_\Omega A_0 \nabla u.\nabla u_\varepsilon \, dx \end{aligned}$$

and taking liminf on both sides of above inequality we have

$$\liminf_{\varepsilon \to 0} J_\varepsilon(w_\varepsilon) \geq J(u).$$

Hence $J_\varepsilon \overset{\Gamma}{\rightharpoonup} J$ in the weak topology of $H_0^1(\Omega)$.

In the above example, we assume the H-convergence of the matrix coefficients to describe the Γ-limit. A general question of interest is the following: If for any sequence of functionals, by compactness, there is a Γ-limit, then under what conditions one can get an integral representation of Γ-limit. In the next section, we describe the situation in one-dimension.

4 Integral Representation (One-Dimension)

For any given $1 < p < \infty$ and $c_1, c_2, c_3 > 0$, let $\mathcal{F} = \mathcal{F}(p, c_1, c_2, c_3)$ be the class of all functionals $F : W^{1,p}(\Omega) \to [0, +\infty)$ such that

$$F(u) = \int_\Omega f(x, \nabla u(x)) \, dx$$

where $f : \Omega \times \mathbb{R}^n \to [0, +\infty)$

H 1. *is a Borel function such that $\xi \mapsto f(x, \xi)$ is convex for all $x \in \Omega$,*

H 2. *and satisfies the* growth conditions *of order p*

$$c_1|\xi|^p - c_2 \leq f(x, \xi) \leq c_3(1 + |\xi|^p), \quad \forall x \in \Omega, \xi \in \mathbb{R}^n.$$

Exercise 11. If f satisfies **H**1 and **H**2, then f satisfies the local Lipschitz condition

$$|f(x,\xi) - f(x,\zeta)| \leq k(1 + |\xi|^{p-1} + |\zeta|^{p-1})|\xi - \zeta| \quad \forall \xi, \zeta \in \mathbb{R}^n.$$

The constant k depends only on c_3 and p.

We take $n = 1$ in the dimension of Euclidean space and set $\Omega = (a,b)$. Observe that any functional in $\mathcal{F}$ is invariant by addition of a constant c, i.e., $F(u+c) = F(u)$. Thus, it is sufficient to characterize in the space

$$X = \{u \in W^{1,p}(\Omega) \mid u(b) = 0\}$$

equipped with L^p norm instead of $W^{1,p}(\Omega)$. Since X is embedded in $L^\infty(a,b)$, $L^1(a,b) \subset X^\star$.

Proposition 4.1. *Let* $X = \{u \in W^{1,p}(\Omega) \mid u(b) = 0\}$ *equipped with* L^p *norm. Let* $F \in \mathcal{F}$ *and consider its integrand* f *as a function on* X*, then* $F^\star : X^\star \to \mathbb{R}$ *is given as*

$$F^\star(\phi) = \int_a^b f^\star\left(x, -\int_a^x \phi(t)\,dt\right) dx, \quad \forall \phi \in L^1(a,b).$$

Proof. Let us assume $f(x,\cdot) \in C^1(\mathbb{R})$ for all $x \in (a,b)$. Due to the growth conditions and continuity of f,

$$f^\star(x,\xi^\star) = \sup_{\xi\in\mathbb{R}}\{\xi^\star \cdot \xi - f(x,\xi)\} = \max_{\xi\in\mathbb{R}}\{\xi^\star \cdot \xi - f(x,\xi)\}.$$

Thus, if ζ is the point at which maximum is attained, then

$$f^\star(x,\zeta^\star) = \zeta^\star \cdot \zeta - f(x,\zeta) \quad \text{if and only if } \zeta^\star - \frac{\partial f}{\partial \zeta}(x,\zeta) = 0. \tag{4.1}$$

Let $\phi \in L^1(a,b)$, define $\Phi \in W^{1,1}(a,b)$ as,

$$\Phi(x) = -\int_a^x \phi(t)\,dt.$$

Note that $\Phi' = -\phi$ and $\Phi(a) = 0$. Thus, the convex conjugate of F is given as

$$\begin{aligned}
F^\star(\phi) &= \sup_{v\in X}\left\{\int_a^b \big(\phi(x)v(x) - f(x,v'(x)\big)\,dx\right\} \\
&= \sup_{v\in X}\left\{\int_a^b \big(\Phi(x)v'(x) - f(x,v'(x)\big)\,dx\right\} \quad \text{(using integration by parts)} \\
&= \max_{v\in X}\left\{\int_a^b \big(\Phi(x)v'(x) - f(x,v'(x)\big)\,dx\right\} \\
&= \int_a^b \big(\Phi(x)u'(x) - f(x,u'(x)\big)\,dx.
\end{aligned}$$

By computing Euler equations, we have $\Phi - \frac{\partial f}{\partial u}(x,u') = c$, for some constant c. But $\Phi(a) = 0$ and $\frac{\partial f}{\partial u}(a,u'(a)) = 0$, implies that $c = 0$ and thus, $\Phi = \frac{\partial f}{\partial u}(x,u')$ a.e. on (a,b). By choosing $\zeta^\star = \Phi(x)$ and $\zeta = u'(x)$ in (4.1), we have

$$\Phi(x) = \frac{\partial f}{\partial u}(x,u'(x)) \quad \text{if and only if } f^\star(x,\Phi(x)) = \Phi(x)u'(x) - f(x,u'(x)).$$

Hence,

$$\begin{aligned}
F^\star(\phi) &= \int_a^b \big(\Phi(x)u'(x) - f(x,u'(x)\big)\,dx \\
&= \int_a^b f^\star(x,\Phi(x)\,dx \\
&= \int_a^b f^\star\left(x, -\int_a^x \phi(t)\,dt\right)\,dx
\end{aligned}$$

Now, for a general f satisfying hypotheses **H**1 and **H**2, we define $f_\varepsilon(x,\xi) = \int_a^b \rho_\varepsilon(x-y)f(y,\xi)\,dy$, where ρ_ε are the sequence of mollifiers. Observe that f_ε are convex in the second variable and, by Jensen's inequality, $f_\varepsilon \geq f$. Also, observe that $\lim_\varepsilon f_\varepsilon^\star(x,\xi^\star) = f^\star(x,\xi^\star)$ for all $x \in (a,b)$ and $\xi^\star \in \mathbb{R}$. We have, for each ε,

$$F_\varepsilon^\star(\phi) = \int_a^b f_\varepsilon^\star\left(x, -\int_a^x \phi(t)\,dt\right)\,dx \quad \forall \phi \in L^1(a,b).$$

Now, by dominated convergence theorem and $F^\star \geq F_\varepsilon^\star$, we get

$$F^\star(\phi) \geq \lim_k F_\varepsilon^\star(\phi) = \int_a^b f^\star\left(x, -\int_a^x \phi(t)\,dt\right)\,dx.$$

Also, by the convex conjugate definition, $f^\star(x,\xi^\star) \geq \xi^\star\xi - f(x,\xi)$ for all $x,\xi,\xi^\star$. Now, choose $\xi^\star = \Phi(x), \xi = v'$, where $v \in X$ and integrate both sides of above inequality,

$$\begin{aligned}\int_a^b f^\star(x,\Phi(x))\,dx &\geq \int_a^b (\Phi(x)v'(x) - f(x,v'(x)))\,dx \\ &= \int_a^b (\phi(x)v(x) - f(x,v'(x)))\,dx.\end{aligned}$$

Taking supremum over $v \in V$, we have $F^\star(\phi) \leq \int_a^b f^\star(x,\Phi(x))\,dx$. □

Proposition 4.2. *Let $g_n : \Omega \times \mathbb{R}^n \to [0,+\infty)$ satisfy hypotheses* **H***1 and* **H***2, for all n. If $g_n(\cdot,\xi)$ weak* converges to $g(\cdot,\xi)$ for all $\xi \in \mathbb{R}$, then $g_n(\cdot,v(\cdot))$ weak* converges to $g(\cdot,v(\cdot))$, for all $v \in C([a,b])$.*

Proof. Let $v \in C([a,b])$ and $\phi \in L^1(a,b)$. Also, let (x_{i-1},x_i) be k number of partitions of (a,b) for $i = 1,2,\ldots,k$ such that $x_0 = a$ and $x_k = b$. Consider,

$$\begin{aligned}\left|\int_a^b (g_n(x,v) - g(x,v))\,\phi\,dx\right| &\leq \sum_{i=1}^k \left|\int_{(x_{i-1},x_i)} (g_n(x,v(x)) - g_n(x,v(x_i)))\,\phi\,dx\right| \\ &\quad + \sum_{i=1}^k \left|\int_{(x_{i-1},x_i)} (g_n(x,v(x_i)) - g(x,v(x_i)))\,\phi\,dx\right| \\ &\quad + \sum_{i=1}^k \left|\int_{(x_{i-1},x_i)} (g(x,v(x_i)) - g(x,v(x)))\,\phi\,dx\right|\end{aligned}$$

The second term converges to zero, by hypothesis, and by uniform local Lipschitz continuity (cf. Exercise 11 of g_n and g, we have the result. □

Lemma 4.1. *Let $g_n : \Omega \times \mathbb{R}^n \to [0,+\infty)$ satisfy hypotheses* **H***1 and* **H***2, for all n. Then, there exists a subsequence of $\{g_n\}$ and a $g : (a,b) \times \mathbb{R} \to [0,+\infty)$ such that $g_n(\cdot,\xi)$ weak* converges to $g(\cdot,\xi)$ for all $\xi \in \mathbb{R}$.*

Theorem 4.1. *Let $\{F_n\}$ be a sequence in $\mathcal{F}$ with integrand f_n and $F \in \mathcal{F}$ with integrand f. Then the following statements are equivalent:*

1. *$F_n(\cdot,I)$ Γ-converges to $F(\cdot,I)$ in $W^{1,p}(I)$, for all open intervals I of (a,b).*

2. $f_n^\star(\cdot,\xi^\star)$ *weak* converges to* $f^\star(\cdot,\xi^\star)$, *for all* $\xi^\star \in \mathbb{R}$.

The proof of above lemma and theorem are being skipped and can be found in [1].

Example 4.1. Let $0 < \alpha \leq a_\varepsilon(x) \leq \beta < +\infty$ and $g \in L^2(a,b)$. Let $F_\varepsilon : H_0^1(a,b) \to \mathbb{R}$ be defined as

$$F_\varepsilon(u) = \int_a^b \left\{ \frac{1}{2} a_\varepsilon(x)|u'|^2 - gu \right\} dx.$$

The Euler-Lagrange equations yields that the minimizers u_ε,

$$\begin{cases} -\frac{d}{dx}\left(a_\varepsilon(x)\frac{du_\varepsilon}{dx}\right) = g \text{ in } (a,b) \\ u_\varepsilon(a) = u_\varepsilon(b) = 0. \end{cases}$$

Now, set $f_\varepsilon(x,\xi) := a_\varepsilon(x)|\xi|^2$. Then, $f_\varepsilon^\star(x,\xi^\star) = \frac{\xi^2}{4a_\varepsilon(x)}$. But, for each $\xi^\star \in \mathbb{R}^n$, $f_\varepsilon^\star(\cdot,\xi^\star)$ converges weak* in $L^\infty(a,b)$ to $f^\star(\cdot,\xi^\star)$, where $f^\star(x,\xi^\star) = \frac{\xi^2}{4b(x)}$ and

$$\frac{1}{a_\varepsilon(x)} \rightharpoonup \frac{1}{b(x)}.$$

□

References

[1] A. Braides, *Γ-Convergence for Beginners*, vol. 22 of Oxford Lecture series in Mathematics and its Applications, Oxford University Press, 2002.

[2] G. Dal Maso, *An Introduction to Γ-Convergence*, vol. 8 of Progress in Nonlinear Differential Equations and Their Applications, Birkhäuser Boston, 1993.

[3] E. De Giorgi, *Sulla convergenza di alcune successioni di integrali del tipo dell'area*, Rendi Condi di Mat., 8 (1975), pp. 277–294.

[4] E. De Giorgi and T. Franzoni, *Su un tipo di convergenza variazionale*, Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur., 58 (1975), pp. 842–850.

[5] E. De Giorgi and S. Spagnolo, *Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine*, Boll. Un. Mat. It., 8 (1973), pp. 391–411.

[6] D. Kinderlehrer and G. Stampacchia, *An Introduction to Variational Inequalities and their applications*, vol. 31 of Classics in Applied Mathematics, SIAM, 2000.