Riesz-Fredhölm Theory

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Contents

1 Introduction

The aim of this lecture note is to show the existence and uniqueness of Fredhölm integral operators of second kind, i.e., show the existence of a solution x of $x - Tx = y$ for any given y, in appropriate function spaces.

2 Integral Operators

Let E be a compact subset of $\mathbb{R}^n$ and $C(E)$ denote the space of complex valued continuous functions on E endowed with the uniform norm $\|f\|_\infty = \sup_{x \in E} |f(x)|$. Recall that $C(E)$ is a Banach space with the uniform norm.

Definition 2.1. *Any continuous function $K : E \times E \to \mathbb{C}$ is called a* continuous kernel.

Since K is continuous on a compact set, K is both bounded, i.e., there is a κ such that

$$|K(x,y)| \leq \kappa \quad \forall x, y \in E$$

and uniformly continuous. In particular, for each $\varepsilon > 0$ there is a $\delta > 0$ such that

$$|K(x_1,y) - K(x_2,y)| \leq \varepsilon \quad \forall y \in E$$

whenever $|x_1 - x_2| < \delta$.

Example 2.1 (Fredhölm integral operator). For any $f \in C(E)$, we define

$$T(f)(x) = \int_E K(x,y)f(y)\,dy$$

where $x \in E$ and $K : E \times E \to \mathbb{R}$ is a continuous function. For each $\varepsilon > 0$,

$$|Tf(x_1) - Tf(x_2)| \leq \int_E |K(x_1,y) - K(x_2,y)||f(y)|\,dy \leq \varepsilon \|f\|_\infty |E|$$

whenever $|x_1 - x_2| < \delta$. Thus, $Tf \in C(E)$ and T defines a map from $C(E)$ to $C(E)$.

Example 2.2 (Volterra integral operator). For any $f \in C[a,b]$, we define

$$T(f)(x) = \int_a^x K(x,y)f(y)\,dy$$

where $x \in [a,b]$ and $K : [a,b] \times [a,b] \to \mathbb{R}$ is a continuous function. Note that, for $x_1, x_2 \in [a,b]$,

$$\begin{aligned} Tf(x_1) - Tf(x_2) &= \int_a^{x_1} K(x_1,y)f(y)\,dy - \int_a^{x_2} K(x_2,y)f(y)\,dy \\ &= \int_a^{x_1} [K(x_1,y) - K(x_2,y)]f(y)\,dy \\ &\quad + \int_{x_2}^{x_1} K(x_2,y)f(y)\,dy. \end{aligned}$$

Therefore, for each $\varepsilon > 0$,

$$\begin{aligned}
|Tf(x_1) - Tf(x_2)| &\leq \int_a^{x_1} |K(x_1, y) - K(x_2, y)||f(y)|\, dy \\
&\quad + \int_{x_2}^{x_1} |K(x_2, y)||f(y)|\, dy \\
&\leq \varepsilon\|f\|_\infty (x_1 - a) + \kappa(x_1 - x_2)\|f\|_\infty \\
&\leq \varepsilon\|f\|_\infty (b - a) + \kappa\delta\|f\|_\infty \\
&\leq [(b - a) + \kappa]\varepsilon\|f\|_\infty
\end{aligned}$$

whenever $|x_1 - x_2| < \delta$ and, without loss of generality, we have assumed $\delta \leq \varepsilon$. Thus, $Tf \in C[a, b]$ and T defines a map from $C[a, b]$ to $C[a, b]$.

One can think of Volterra integral operator as a special case of Fredhölm integral operators by considering a $K : [a, b] \times [a, b]$ such that $K(x, y) = 0$ for $y > x$. Geometrically this means, in the square $[a, b] \times [a, b]$, K takes the value zero in the region above $y = x$ line. Thus, Volterra integral operator is a Fredhölm operator for a K which may be discontinuous on the line $y = x$ in the square. In fact, this particular K is a special case of *weakly singular kernel*.

Definition 2.2. *Let $E \subset \mathbb{R}^n$ be a compact subset. A function $K : E \times E \to \mathbb{C}$ is said to be* weakly singular kernel *if it is continuous for all $x, y \in E$ such that $x \neq y$ and there exist positive constants M and $\alpha \in (0, n)$ such that*

$$|K(x, y)| \leq M|x - y|^{\alpha - n} \quad \forall x, y \in E; x \neq y.$$

Theorem 2.3. *Let $E \subset \mathbb{R}^n$ be a compact subset with non-empty interior. Let $K : E \times E \to \mathbb{C}$ be a continuous function. Then the operator $T : C(E) \to C(E)$, defined as,*

$$(Tf)(x) := \int_E K(x, y) f(y)\, dy, \text{ for each } x \in E$$

is a bounded linear operator, i.e., is in $\mathcal{B}(C(E))$ and

$$\|T\| = \sup_{x \in E} \int_E |K(x, y)|\, dy.$$

Proof. The fact that T is linear is obvious. For each $f \in C(E)$ with $\|f\|_\infty \leq 1$ and $x \in E$, we have

$$|(Tf)x| \leq \int_E |K(x,y)|\,dy.$$

Thus,

$$\|T\| = \sup_{\substack{f\in C(E)\\ \|f\|_\infty\leq 1}} \|Tf\|_\infty \leq \sup_{x\in E} \int_E |K(x,y)|\,dy.$$

It now only remains to show the other inequality. Since K is continuous, there is a $x_0 \in E$ such that

$$\int_E |K(x_0,y)|\,dy = \max_{x\in E} \int_E |K(x,y)|\,dy.$$

For each $\varepsilon > 0$, choose $g_\varepsilon \in C(E)$ as

$$g_\varepsilon(y) := \frac{\overline{K(x_0,y)}}{|K(x_0,y)| + \varepsilon}, \quad \text{for } y \in E.$$

Then $\|g_\varepsilon\|_\infty \leq 1$ and

$$\begin{aligned} \|Tg_\varepsilon\|_\infty &\geq |(Tg_\varepsilon)(x_0)| = \int_E \frac{|K(x_0,y)|^2}{|K(x_0,y)|+\varepsilon}\,dy \\ &\geq \int_E \frac{|K(x_0,y)|^2 - \varepsilon^2}{|K(x_0,y)|+\varepsilon}\,dy = \int_E |K(x_0,y)|\,dy - \varepsilon|E|. \end{aligned}$$

Hence

$$\|T\| = \sup_{\substack{f\in C(E)\\ \|f\|_\infty\leq 1}} \|Tf\|_\infty \geq \|Tg_\varepsilon\|_\infty \geq \int_E |K(x_0,y)|\,dy - \varepsilon|E|,$$

and, since ε is arbitrary, we have

$$\|T\| \geq \int_E |K(x_0,y)|\,dy = \max_{x\in E} \int_E |K(x,y)|\,dy.$$

□

For any operator $T : X \to X$, one can define the iterated operators $T^k : X \to X$ as the k-times composition of T with itself.

Theorem 2.4 (Neumann Series)**.** *Let X be a Banach space, $T \in B(X)$ with $\|T\| < 1$ (contraction) and $I : X \to X$ be the identity operator. Then the map $I - T : X \to X$ is invertible given by*

$$(I - T)^{-1} = \sum_{k=0}^{\infty} T^k$$

and the inverse is bounded

$$\|(I - T)^{-1}\| \leq \frac{1}{1 - \|T\|}.$$

Proof. Since X is a Banach space, by Theorem B.12, $\mathcal{B}(X)$ is a Banach space. Thus, owing to Theorem B.4, to show the convergence of the series $\sum_{k=0}^{\infty} T^k$ it is enough to show that it is absolutely convergent. Note that

$$\sum_{k=0}^{\infty} \|T^k\| = \lim_{m\to\infty} \sum_{k=0}^{m} \|T^k\| \leq \lim_{m\to\infty} \sum_{k=0}^{m} \|T\|^k = \sum_{k=0}^{\infty} \|T\|^k.$$

Since $\|T\| < 1$, the geometric series converges to $(1 - \|T\|)^{-1}$ and hence the series $\sum_{k=0}^{\infty} T^k$ is absolutely convergent and, thus, convergence in $\mathcal{B}(X)$. Let $S : X \to X$ be the limit of the series, i.e.,

$$S := \sum_{k=0}^{\infty} T^k.$$

Note that $\|S\| \leq (1 - \|T\|)^{-1}$, hence, is a bounded linear operator on X, i.e., $S \in \mathcal{B}(X)$. It only remains to show that S is the inverse of $I - T$. Note that

$$\begin{aligned}(I - T)S &= (I - T) \lim_{m\to\infty} \sum_{k=0}^{m} T^k = \lim_{m\to\infty} (I - T)(I + T + \ldots + T^m) \\ &= \lim_{m\to\infty} (I - T^{m+1}) = I.\end{aligned}$$

The last equality is due to the fact that the sequence T^{m+1} converges to 0 in $\mathcal{B}(X)$ because $\|T^{m+1}\| \leq \|T\|^{m+1}$ and, since $\|T\| < 1$, $\lim_{m\to\infty} \|T\|^{m+1} \to 0$. Similarly,

$$\begin{aligned}S(I - T) &= \lim_{m\to\infty} \sum_{k=0}^{m} T^k(I - T) = \lim_{m\to\infty} (I + T + \ldots + T^m)(I - T) \\ &= \lim_{m\to\infty} (I - T^{m+1}) = I.\end{aligned}$$

□

If $x \in X$ is the solution of $(I - T)x = y$, for any given $y \in X$, then x can be computed as

$$x = (I - T)^{-1}y = \sum_{k=0}^{\infty} T^k y = \lim_{m\to\infty} \sum_{k=0}^{m} T^k y = \lim_{m\to\infty} x_m.$$

Note that $x_{m+1} = Tx_m + y$, for $m \geq 0$.

Theorem 2.5. *Let X be a Banach space, $T \in B(X)$ with $\|T\| < 1$ (contraction) and $I : X \to X$ be the identity operator. For any given $y \in X$ and arbitrary $x_0 \in X$ the sequence*

$$x_{m+1} := Tx_m + y \quad \text{for } m = 0, 1, 2, \ldots$$

converges to a unique solution $x \in X$ of $(I - T)x = y$.

Proof. Note that $x_1 = Tx_0 + y$, $x_2 = Tx_1 + y = T^2x_0 + (I + T)y$. Thus, by induction, for $m = 1, 2, \ldots,$

$$x_m = T^m x_0 + \sum_{k=0}^{m-1} T^k y.$$

Hence,

$$\lim_{m\to\infty} x_m = \sum_{k=0}^{\infty} T^k y = (I - T)^{-1}y.$$

□

Corollary 2.6. *Let K be a continuous kernel satisfying*

$$\max_{x\in E} \int_E |K(x, y)|\, dy < 1.$$

Then, for each $g \in C(E)$, the integral equation of the second kind

$$f(x) - \int_E K(x, y)f(y)\, dy = g(x) \quad x \in E$$

has a unique solution $f \in C(E)$. Further, for any $f_0 \in C(E)$, the sequence

$$f_{m+1}(x) := \int_E K(x, y)f_m(y)\, dy + g(x), m = 0, 1, 2, \ldots$$

converges uniformly to $f \in C(E)$.

3 Compact Operators

Theorem 3.1. *A subset of a normed space is compact iff it is sequentially compact.*

Definition 3.2. *A subset E of a normed space X is said to be* relatively compact *if $\overline{E}$, the closure of E, is compact in X.*

Theorem 3.3. *Any bounded and finite dimensional subset of a normed space is relatively compact.*

Definition 3.4. *Let X and Y be normed spaces. A linear operator $T : X \to Y$ is said to be* compact *if $T(E)$ is relatively compact in Y, for every bounded subset $E \subset X$.*

Let $\mathcal{K}(X, Y)$ be the space of all compact linear maps from X to Y. To verify compactness of T, it is enough to check the relative compactness of $T(B)$ in Y for the closed unit ball $B \subset X$.

Theorem 3.5. *A linear operator $T : X \to Y$ is compact iff for any bounded sequence $\{x_n\} \subset X$, the sequence $\{Tx_n\} \subset Y$ admits a convergent subsequence.*

Proposition 3.6. *The set $\mathcal{K}(X, Y)$ is a subspace of $\mathcal{B}(X, Y)$.*

Proof. Any compact linear operator is bounded because any relatively compact set is bounded. Thus, $\mathcal{K}(X, Y) \subset \mathcal{B}(X, Y)$. Let $S, T \in \mathcal{K}(X, Y)$. It is easy to check that, for every bounded subset E of X and $\alpha, \beta \in \mathbb{C}$, $(\alpha S + \beta T)(E)$ is relatively compact in Y, since $S(E)$ and $T(E)$ are both relatively compact in Y. □

Theorem 3.7. *Let X, Y and Z be normed spaces, $S \in \mathcal{B}(X, Y)$ and $T \in \mathcal{B}(Y, Z)$. If either one of S or T is compact, then the composition $T \circ S : X \to Z$ is compact.*

Proof. Let $\{x_n\}$ be a bounded sequence in X. Suppose S is compact, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $Sx_{n_k} \to y$ in Y, as $k \to \infty$. Since T is continuous, $T(Sx_{n_k}) \to Ty$ in Z, as $k \to \infty$. Thus, $T \circ S$ is compact.

On the other hand, since S is bounded, Sx_n is bounded sequence in Y. If T is compact, then there is a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $T(Sx_{n_k}) \to z$ in Z, as $k \to \infty$. Thus, $T \circ S$ is compact. □

Theorem 3.8. *If $T \in \mathcal{B}(X, Y)$ such that $T(X)$ is finite dimensional then T is compact.*

Proof. Let $E \subset X$ be bounded. Since T is bounded, $T(E)$ is bounded in Y. But $T(E)$ is a bounded subset of the finite dimensional space $T(X)$ and, hence, $T(E)$ is relatively compact in Y. □

Theorem 3.9. *Let $E \subset \mathbb{R}^n$ be a compact subset with non-empty interior and $K : E \times E \to \mathbb{C}$ is a continuous kernel. Then the bounded linear operator $T : C(E) \to C(E)$, defined as,*

$$(Tf)(x) := \int_E K(x, y) f(y)\, dy, \text{ for each } x \in E$$

is compact.

Proof. Let $B \subset C(E)$ be a bounded subset, i.e., there is a $M > 0$ such that $\|f\|_\infty \leq M$ for all $f \in B$. Thus, for all $f \in B$, we have

$$|Tf(x)| \leq \kappa M |E|$$

for all $x \in E$ and $f \in B$. Thus, $T(B)$ is bounded in $C(E)$. By the uniform continuity of K, for each $\varepsilon > 0$,

$$|Tf(x_1) - Tf(x_2)| \leq \int_E |K(x_1, y) - K(x_2, y)||f(y)|\, dy \leq \varepsilon \|f\|_\infty |E| \leq \varepsilon M |E|$$

for all $x_1, x_2 \in E$ such that $|x_1 - x_2| < \delta$ and for all $f \in B$. Therefore, $T(B)$ is equicontinuous. By Arzelá-Ascoli theorem (cf. Appendix A) $T(B)$ closure is compact and, hence, T is compact. □

Theorem 3.10. *Let $E \subset \mathbb{R}^n$ be a compact subset with non-empty interior and $K : E \times E \to \mathbb{C}$ be a weakly singular kernel. Then the bounded linear operator $T : C(E) \to C(E)$, defined as,*

$$(Tf)(x) := \int_E K(x, y) f(y)\, dy, \text{ for each } x \in E$$

is compact.

Definition 3.11. *Let V and W be vector spaces over the same field. For any $T \in \mathcal{L}(V, W)$, the* kernel *of T, denoted as $N(T)$, is defined as*

$$N(T) = \{x \in V \mid Tx = 0\}.$$

The kernel of T is also referred to as null space *of T and the dimension of kernel of T is called* nullity *of T. Also, the* range *of T, denoted as $R(T)$, is defined as*

$$R(T) = \{Tx \mid x \in V\}.$$

The dimension of range of T is called the rank *of T.*

Exercise 3.1. If $T \in \mathcal{L}(V, W)$, then $N(T)$ is a subspace of V and $R(T)$ is a subspace of W.

Theorem 3.12. *Let X be a normed space and $T : X \to X$ be a compact linear operator. Then $N(I - T)$ is finite dimensional and $R(I - T)$ is closed in X.*

Proof. Let B_1 be the closed unit ball in $N(I - T)$, i.e.,

$$B_1 := \{x \in N(I - T) \mid \|x\| \leq 1\}.$$

If $x \in B_1$, then $x = Tx$ and $\|Tx\| \leq 1$. Thus, $B_1 \subset T(B)$, where B is the closed unit ball of X. Since T is a compact operator, B_1 is compact in X and, hence, $N(I - T)$ is finite dimensional (cf. Theorem B.8).

Let $\{y_n\} \subset R(I - T)$ and suppose that $y_n \to y$ in X. Thus, there is a sequence $\{x_n\} \subset X$ such that $y_n = x_n - Tx_n$. Since $N(I - T)$ is finite dimensional, by Theorem B.6, for each n, there is a $z_n \in N(I - T)$ such that $\|x_n - z_n\| = \inf_{z \in N(I-T)} \|x_n - z\|$. Thus, $y_n = (x_n - z_n) - T(x_n - z_n)$.

If the sequence $\{x_n - z_n\}$ is bounded in X, and since T is compact, there is a subsequence $T(x_{n_k} - z_{n_k}) \to u$ in X. Then, $x_{n_k} - z_{n_k} \to y + u$. Hence, $T(y + u) = u$. Set $s := y + u$. Now consider, $(I - T)s = s - Ts = y + u - u = y$. Therefore, $y \in R(I - T)$ and, hence, $R(I - T)$ is closed. It only remains to prove that $\{x_n - z_n\}$ is a bounded sequence in X.

Suppose not, then, for a subsequence, $\|x_{n_k} - z_{n_k}\| \to \infty$ as $k \to \infty$. Set

$$w_{n_k} := \frac{1}{\|x_{n_k} - z_{n_k}\|}(x_{n_k} - z_{n_k})$$

so that $\|w_{n_k}\| = 1$. Further

$$(I-T)w_{n_k} = \frac{1}{\|x_{n_k} - z_{n_k}\|} y_{n_k}$$

and hence the LHS converges to the zero vector, since the denominator in RHS blows up. Since T is compact, there is a subsequence $\{w_{n_{k_l}}\}$ of $\{w_{n_k}\}$ such that $Tw_{n_{k_l}} \to v$ in X. But, since $(I-T)w_{n_{k_l}} \to 0$ in X, we should have $w_{n_{k_l}} \to v$ in X. Thus, $Tw_{n_{k_l}} \to Tv$ and, hence, $(I-T)v = 0$, i.e., $v \in N(I-T)$. On the other hand,

$$d(w_{n_{k_l}}, N(I-T)) = \frac{d(x_{n_{k_l}}, N(I-T)}{\|x_{n_{k_l}} - z_{n_{k_l}}\|} = 1.$$

Thus, $d(v, N(I-T)) = 1$ and $v \in N(I-T)$, which is impossible. Hence, the sequence $\{x_n - z_n\}$ is bounded in X. □

For any normed space X and compact operator $T : X \to X$, one can define the operators $(I-T)^n : X \to X$, for $n \geq 1$. Let us denote $L := I - T$, then $L^n = (I-T)^n = I - T_n$ where

$$T_n := \sum_{k=1}^{n} (-1)^{k-1} \frac{n!}{k!(n-k)!} T^k.$$

Note that T_n is compact and, hence $N(L^n)$ is finite dimensional and $R(L^n)$ is closed in X, for all $n \geq 1$.

Theorem 3.13. *Let X be a normed space, $T \in \mathcal{K}(X)$ and $L := I - T$. Then there is a unique non-negative integer $r \geq 0$, called the* Riesz number *of T such that*

$$\{0\} \subsetneq N(L) \subsetneq N(L^2) \subsetneq \ldots \subsetneq N(L^r) = N(L^{r+1}) = \ldots$$

and

$$X \supsetneq R(L) \supsetneq R(L^2) \supsetneq \ldots \supsetneq R(L^r) = R(L^{r+1}) = \ldots.$$

Proof. If $x \in N(L^n)$, i.e., $L^n x = 0$, then $L^{n+1}x = L(L^n x) = L0 = 0$. Therefore, $\{0\} \subset N(L) \subset N(L^2) \subset \ldots$. Suppose that all the inclusions are proper. Since $N(L^n)$ is finite dimensional, it is closed proper subspace of $N(L^{n+1})$. Therefore, by Riesz lemma, there is a $x_n \in N(L^{n+1})$ such that

$\|x_n\| = 1$ and $d(x_n, N(L^n)) \geq 1/2$. Thus, we have a bounded sequence $\{x_n\} \subset X$ with $d(x_n, N(L^n)) \geq 1/2$. Consider

$$T(x_n - x_m) = (I - L)(x_n - x_m) = x_n - (x_m + Lx_n - Lx_m).$$

Note that, for $n > m$, we have

$$L^n(x_m + Lx_n - Lx_m) = L^{n-m-1}L^{m+1}x_m + L^{n+1}x_n - L^{n-m}L^{m+1}x_m = 0.$$

Therefore, for $n > m$, $\|Tx_n - Tx_m\| \geq 1/2$, which contradicts that T, since there can be no convergent subsequence of $\{Tx_n\}$. Thus, the sequence of inclusions cannot be proper for all. There exists two consecutive null spaces that are equal. Set

$$r := \min\{k : N(L^k) = N(L^{k+1})\}.$$

We claim that $N(L^r) = N(L^{r+1}) = \ldots$. Note that, for some $k \geq r$, we have shown that $N(L^k) = N(L^{k+1})$. Now, consider $x \in N(L^{k+2})$, then $0 = L^{k+2}x = L^{k+1}Lx$. Thus, $Lx \in N(L^{k+1}) = N(L^k)$, $0 = L^kLx = L^{k+1}x$ and $x \in N(L^{k+1})$. Hence, $N(L^{k+1}) = N(L^{k+2})$ and

$$\{0\} \subsetneq N(L) \subsetneq N(L^2) \subsetneq \ldots \subsetneq N(L^r) = N(L^{r+1}) = \ldots.$$

Let $y \in R(L^{n+1})$, then there is a $x \in X$ such that $L^{n+1}x = y$. Thus, $L^n(Lx) = y$ and $y \in R(L^n)$. We assume the inclusions are all proper. Since $R(L^k)$ is closed subspace, by Riesz lemma, there is a $y_n \in R(L^n)$ such that $\|y_n\| = 1$ and $d(y_n, R(L^{n+1})) \geq 1/2$. Thus, we have a bounded sequence $\{y_n\} \subset X$ with $d(y_n, R(L^{n+1})) \geq 1/2$. Consider

$$T(y_n - y_m) = (I - L)(y_n - y_m) = y_n - (y_m + Ly_n - Ly_m).$$

Note that, for $m > n$, we have

$$y_m + Ly_n - Ly_m = L^{n+1}(L^{m-n-1}x_m + x_n - L^{m-n}x_m)$$

and $y_m + Ly_n - Ly_m \in R(L^{n+1})$. Thus, for $n > m$, $\|Ty_n - Ty_m\| \geq 1/2$, which contradicts that T, since there can be no convergent subsequence of $\{Ty_n\}$. Thus, the sequence of inclusions cannot be proper for all. There exists two consecutive range spaces that are equal. Set

$$s := \min\{k : R(L^k) = R(L^{k+1})\}.$$

We claim that $R(L^s) = R(L^{s+1}) = \ldots$. Note that, for some $k \geq s$, we have shown that $R(L^k) = R(L^{k+1})$. Now, consider $y \in R(L^{k+1})$, then $y = L^{k+1}x = L(L^k x)$. Thus, for some $x_0 \in X$, $L^k x = L^{k+1}x_0$ and $y = L(L^k x) = L(L^{k+1}x_0) = L^{k+2}x_0$. Hence $R(L^{k+1}) = R(L^{k+2})$ and

$$X \supsetneq R(L) \supsetneq R(L^2) \supsetneq \ldots \supsetneq R(L^r) = R(L^{r+1}) = \ldots.$$

It only remains to prove that $r = s$. Suppose $r > s$ and let $x \in N(L^r)$. Then $L^{r-1}x \in R(L^{r-1}) = R(L^r)$ and, hence, there is a $y \in X$ such that $L^r y = L^{r-1}x$. Therefore, $L^{r+1}y = L^r x = 0$ and $y \in N(L^{r+1}) = N(L^r)$. This means that $L^{r-1}x = 0$ and $x \in N(L^{r-1})$ which contradicts the minimality of r.

On the other hand, if $r < s$. Let $y \in R(L^{s-1})$. Then, for some $x \in X$, $L^{s-1}x = y$ and $Ly = L^s x$. Consequently, $Ly \in R(L^s) = R(L^{s+1})$. Hence, there is a $x_0 \in X$ such that $L^{s+1}x_0 = Ly$. Therefore,

$$0 = L^{s+1}x_0 - Ly = L^s(Lx_0 - x),$$

i.e., $Lx_0 - x \in N(L^s) = N(L^{s-1})$ and $L^s x_0 = L^{s-1}x = y$. Thus, $y \in R(L^s)$ which contradicts the minimality of s. □

Theorem 3.14. *Let X be a normed space, $T \in \mathcal{K}(X)$ and $L := I - T$. Then, for each $x \in X$, there exists unique $y \in N(L^r)$ and $z \in R(L^r)$ such that $x = y + z$, i.e., $X = N(L^r) \oplus R(L^r)$.*

Proof. Let $x \in N(L^r) \cap R(L^r)$. Then $x = L^r y$ for some $y \in X$ and $L^r x = 0$. Thus, $L^{2r}y = 0$ and $y \in N(L^{2r}) = N(L^r)$. Therefore, $0 = L^r y = x$.

Let $x \in X$ be an arbitrary element. Then $L^r x \in R(L^r) = R(L^{2r})$. Thus, there is a $x_0 \in X$ such that $L^r x = L^{2r}x_0$ and $L^r(x - L^r x_0)$. Define $z := L^r x_0 \in R(L^r)$ and $y := x - z$. Since $L^r y = L^r x - L^r z = L^r x - L^{2r}x_0 = 0$, $y \in N(L^r)$. □

Theorem 3.15. *Let X be a normed space, $T \in \mathcal{K}(X)$ and $L := I - T$. Then L is injective iff L is surjective. If L is injective (and hence bijective), then its inverse $L^{-1} \in \mathcal{B}(X)$.*

Proof. The injectivity of L is equivalent to saying that the Riesz number is $r = 0$, which means L is surjective. The argument is also true viceversa.

If L is injective and suppose L^{-1} is not bounded. Then there is a sequence $\{x_n\} \subset X$ with $\|x_n\| = 1$ such that $\|L^{-1}x_n\| \geq n$, for all $n \in \mathbb{N}$. Define, for each $n \in \mathbb{N}$,

$$y_n := \frac{1}{\|L^{-1}x_n\|}x_n; \quad z_n := \frac{1}{\|L^{-1}x_n\|}L^{-1}x_n.$$

Then $Lz_n = y_n \to 0$ in X, as $n \to \infty$, and $\|z_n\| = 1$ for all n. By the compactness of T, there is a subsequence $\{z_{n_k}\}$ of z_n such that, for some $z \in X$, $Tz_{n_k} \to z$ as $k \to \infty$. Since $Lz_n = y_n \to 0$, we have $z_{n_k} \to z$, as $k \to \infty$. Also, $Lz = 0$ and $z \in N(L)$. By the injectivity of L, $z = 0$ which contradicts the fact that $\|z_n\| = 1$. Thus, L^{-1} must be bounded. □

Corollary 3.16. *Let $T : X \to X$ be a compact linear operator on a normed space X. If the homogeneous equation $x - Tx = 0$ has only the trivial solution $x = 0$, then for each $f \in X$ the inhomogeneous equation $x - Tx = f$ has a unique solution $x \in X$ which depends continuously on f.*

If the homogeneous equation $x - Tx = 0$ has non-trivial solution, then it has $m \in \mathbb{N}$ linearly independent solutions $x_1, x_2, \ldots, x_m$ and the inhomogeneous equation $x - Tx = f$ is either unsolvable or its general solution is of the form

$$x = x_0 + \sum_{i=1}^{m} \alpha_i x_i$$

where $\alpha_i \in \mathbb{C}$ for each i and x_0 is a particular solution of the inhomogeneous equation.

The decomposition $X = N(L^r) \oplus R(L^r)$ induces a projection operator $P : X \to N(L^r)$ that maps $Px := y$, where $x = y + z$.

Proposition 3.17. *The projection operator $P : X \to N(L^r)$ is compact.*

Proof. We first show that P is a bounded linear operator. Suppose not, then there is a sequence $\{x_n\} \subset X$ with $\|x_n\| = 1$ such that $\|Px_n\| \geq n$ for all $n \in \mathbb{N}$. Define, for each $n \in \mathbb{N}$, $y_n := \frac{1}{\|Px_n\|} x_n$. Then $y_n \to 0$, as $n \to \infty$, and $\|Py_n\| = 1$ for all $n \in \mathbb{N}$. Since $N(L^r)$ is finite-dimensional and $\{Py_n\}$ is bounded, by Theorem 3.3, there is a subsequence $\{y_{n_k}\}$ such that $Py_{n_k} \to z$ in $N(L^r)$, as $k \to \infty$.

Also, since $y_{n_k} \to 0$, we have $Py_{n_k} - y_{n_k} \to z$, as $k \to \infty$. Note that $Py_{n_k} - y_{n_k} \in R(L^r)$, by direct decomposition, thus $z \in R(L^r)$ because $R(L^r)$ is closed. Since $z \in N(L^r) \cap R(L^r)$, $z = 0$ and $Py_{n_k} \to 0$ which contradicts $\|Py_{n_k}\| = 1$. Thus, P must be bounded. Moreover, since $P(X) = N(L^r)$ is finite dimensional, by Theorem 3.8, P is compact. □

4 Fredhölm Alternative

Definition 4.1. *Let V and W be complex vector spaces. A mapping $\langle\cdot,\cdot\rangle : V \times W \to \mathbb{C}$ is called a* bilinear form *if*

$$\langle \alpha_1 x_1 + \alpha_2 x_2, y\rangle = \alpha_1\langle x_1, y\rangle + \alpha_2\langle x_2, y\rangle, \langle x, \beta_1 y_1 + \beta_2 y_2\rangle = \beta_1\langle y, x_1\rangle + \beta_2\langle x, y_2\rangle$$

for all $x_1, x_2, x \in X$ and $y_1, y_2, y \in Y$ and $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{C}$. Further, a bilinear form is called nondegenerate *if for every non-zero $x \in X$ there exists a $y \in Y$ such that $\langle x, y\rangle \neq 0$ and, for every non-zero $y \in Y$ there is a $x \in X$ such that $\langle x, y\rangle \neq 0$.*

Definition 4.2. *If two normed spaces X and Y are equipped with a nondegenerate bilinear form, then we call it a* dual system *denoted by $\langle X, Y\rangle$.*

Example 4.1. Let $E \subset \mathbb{R}^n$ be a non-empty compact subset. We define the bilinear form in $\langle C(E), C(E)\rangle$ as

$$\langle f, g\rangle := \int_E f(x)g(x)\,dx.$$

which makes the pair a dual system.

Definition 4.3. *Let $\langle X_1, Y_1\rangle$ and $\langle X_2, Y_2\rangle$ be two dual systems. The operators $S : X_1 \to X_2$ and $T : Y_2 \to Y_1$ are called* adjoint *if $\langle Sx, y\rangle = \langle x, Ty\rangle$ for all $x \in X_1$ and $y \in Y_2$.*

Theorem 4.4. *Let $E \subset \mathbb{R}^n$ be a non-empty compact subset and K be a continuous kernel on $E \times E$. Then the compact integral operators*

$$Sf(x) := \int_E K(x, y)f(y)\,dy \quad x \in E$$

and

$$Tg(x) := \int_E K(y, x)g(y)\,dy \quad x \in E$$

are adjoint in the dual system $\langle C(E), C(E)\rangle$.

Proof. Note that

$$\begin{aligned}\langle Sf, g\rangle &= \int_E Sf(x)g(x)\,dx = \int_E\int_E K(x, y)f(y)\,dy g(x)\,dx \\ &= \int_E f(y)\int_E K(x, y)g(x)\,dx\,dy = \int_E f(y)Tg(y)\,dy = \langle f, Tg\rangle.\end{aligned}$$

□

The above result is also true for a weakly singular kernel K whose proof involves approximating K by continuous kernels.

Lemma 4.5. *Let $\langle X, Y\rangle$ be a dual system. Then to every set of linearly independent elements $\{x_1, \dots, x_n\} \subset X$, then there exists a set $\{y_1, \dots, y_n\} \subset Y$ such that $\langle x_i, y_j\rangle = \delta_{ij}$ for all i, j. The result also holds true with the roles of X and Y interchanged.*

Proof. The result is true for $n = 1$, by the nondegeneracy of the bilinear form. We shall prove the result by induction. Let us assume the result for $n \geq 1$ and consider the $n+1$ linearly independent $\{x_1, \dots, x_{n+1}\}$. By induction hypothesis, for each $m = 1, 2, \dots, n+1$, the linearly independent set $\{x_1, \dots, x_{m-1}, x_{m+1}, \dots, x_{n+1}\}$ of n elements in X has a set of n elements $\{y^m_1, \dots, y^m_{m-1}, y^m_{m+1}, \dots, y^m_{n+1}\}$ in Y such that $\langle x_i, y^m_j\rangle = \delta_{ij}$ for all i, j except $i, j \neq m$. Since $\{x_1, \dots, x_{n+1}\}$ is linear independent, we have

$$x_m - \sum_{\substack{j=1\\ j\neq m}}^{n+1} \langle x_m, y^m_j\rangle x_j \neq 0.$$

Thus, by nondegeneracy of bilinear form there is a $z_m \in Y$ such that

$$\left\langle x_m - \sum_{\substack{j=1\\ j\neq m}}^{n+1} \langle x_m, y^m_j\rangle x_j, z_m \right\rangle \neq 0.$$

The LHS is same as

$$\alpha_m := \left\langle x_m, z_m - \sum_{\substack{j=1\\ j\neq m}}^{n+1} y^m_j \langle x_j, z_m\rangle \right\rangle.$$

Define

$$y_m := \frac{1}{\alpha_m} \left\{ z_m - \sum_{\substack{j=1\\ j\neq m}}^{n+1} y^m_j \langle x_j, z_m\rangle \right\}.$$

Then $\langle x_m, y_m\rangle = 1$, and for $i \neq m$, we have

$$\langle x_i, y_m\rangle = \frac{1}{\alpha_m} \left\{ \langle x_i, z_m\rangle - \sum_{\substack{j=1\\ j\neq m}}^{n+1} \langle x_i, y^m_j\rangle \langle x_j, z_m\rangle \right\} = 0$$

because $\langle x_i, y_j^m \rangle = \delta_{ij}$. Thus, we obtained $\{y_1, \ldots, y_{n+1}\}$ such that $\langle x_i, y_j \rangle = \delta_{ij}$ for all i, j. □

Theorem 4.6. *Let $\langle X, Y \rangle$ be a dual system and $S : X \to X$, $T : Y \to Y$ be compact adjoint operators. Then*

$$dim(N(I - S)) = dim(N(I - T)) < \infty.$$

Proof. By Theorem 3.12,

$$\dim(N(I - S)) = m; \quad \dim(N(I - T)) = n.$$

We need to show that $m = n$. Suppose that $m < n$. If $m > 0$, we choose a basis $\{x_1, \ldots, x_m\} \subset N(I - S)$ and a basis $\{y_1, \ldots, y_n\} \subset N(I - T)$. By Lemma 4.5, there exists elements $\{a_1, a_2, \ldots a_m\} \subset Y$ and $\{b_1, b_2, \ldots, b_n\} \subset X$ such that $\langle x_i, a_j \rangle = \delta_{ij}$, for $i, j = 1, 2, \ldots, m$, and $\langle b_i, y_j \rangle = \delta_{ij}$ for $i, j = 1, 2, \ldots, n$. Define a linear operator $F : X \to X$ by

$$Fx := \sum_{i=1}^{m} \langle x, a_i \rangle b_i$$

for $m > 0$. If $m = 0$ then $F \equiv 0$ is the zero operator. Note that $F : N[(I - S)^r] \to X$ is bounded by Theorem B.9 and $P : X \to N[(I - S)^r]$ is a compact projection operator by Proposition 3.17. Then, by Theorem 3.7, $FP : X \to X$ is compact. Since linear combination of compact operators are compact, $S - FP$ is compact. Consider

$$\langle x - Sx + FPx, y_j \rangle = \langle x, (I - T)y_j \rangle + \langle FPx, y_j \rangle = \langle FPx, y_j \rangle.$$

Then

$$\langle x - Sx + FPx, y_j \rangle = \begin{cases} \langle Px, a_j \rangle & j = 1, 2, \ldots, m \\ 0 & j = m + 1, \ldots, n. \end{cases}$$

If $x \in N(I - S + FP)$, then by above equation $\langle Px, a_j \rangle = 0$ for all $j = 1, \ldots, m$. Therefore, $FPx = 0$ and, hence, $x \in N(I - S)$. Consequently, $x = \sum_{i=1}^{m} \alpha_i x_i$, i.e., $\alpha_i = \langle x, a_i \rangle$. But $Px = x$ for $x \in N(I - S)$, therefore $\alpha_i = \langle Px, a_i \rangle = 0$ for all $i = 1, \ldots, m$ which implies that $x = 0$. Thus, $I - S + FP$ is injective. Hence the inhomogeneous equation

$$x - Sx + FPx = b_n$$

has a unique solution x. Note that

$$0 = \langle x - Sx + FPx, y_n \rangle = \langle b_n, y_n \rangle = 1$$

is a contradiction. Therefore, $m \geq n$. Arguing similarly by interchanging the roles of S and T, we get $n \geq m$ implying that $m = n$. $\square$

Theorem 4.7. *Let $\langle X, Y \rangle$ be a dual system and $S : X \to X$, $T : Y \to Y$ be compact adjoint operators. Then*

$$R(I - S) = \{x \in X \mid \langle x, y \rangle = 0, \forall y \in N(I - T)\}$$

and

$$R(I - T) = \{y \in Y \mid \langle x, y \rangle = 0, \forall x \in N(I - S)\}.$$

Proof. The case of $\dim(N(I - T)) = 0$ is trivial because, in that case, $\dim(N(I - S)) = 0$ and $R(I - S) = X$ (by Theorem 3.15 and Theorem 4.6). Hence, the result is trivially true. Suppose that the $\dim(N(I - T)) = m > 0$. Let $x \in R(I - S)$, i.e., $x = (I - S)x_0$ for some $x_0 \in X$. Then, for all $y \in N(I - T)$,

$$\langle x, y \rangle = \langle x_0 - Sx_0, y \rangle = \langle x_0, y - Ty \rangle = 0.$$

Conversely, assume that $x \in X$ satisfies $\langle x, y \rangle = 0$ for all $y \in N(I - T)$. From the proof of previous theorem, there is a unique solution $x_0 \in X$ of $(I - S + FP)x_0 = x$. Then

$$\langle Px_0, a_j \rangle = \langle (I - S + FP)x_0, y_j \rangle = \langle x, y_j \rangle = 0 \quad \forall j = 1, 2, \dots, m.$$

Then $FPx_0 = 0$ and thus $(I - S)x_0 = x$ and $x \in R(I - S)$. The argument for $R(I - T)$ is similar. $\square$

The above two theorems together is called the *Fredhölm alternative.*

Corollary 4.8. *Let $E \subset \mathbb{R}^n$ be a non-empty compact subset with non-empty interior and K be a continuous or weakly singular kernel on $E \times E$. Then either the homogeneous integral equations*

$$u(x) - \int_E K(x, y)u(y)\, dy = 0 \quad x \in E \tag{4.1}$$

and

$$v(x) - \int_E K(y,x)v(y)\,dy = 0 \quad x \in E \tag{4.2}$$

only have the trivial solutions $u = 0$ and $v = 0$, and the inhomogeneous integral equations

$$u(x) - \int_E K(x,y)u(y)\,dy = f(x) \quad x \in E$$

and

$$v(x) - \int_E K(y,x)v(y)\,dy = g(x) \quad x \in E$$

have unique solution $u, v \in C(E)$ for given $f, g, \in C(E)$, respectively, or both (4.1) and (4.2) have the same finite number $m \in \mathbb{N}$ of linearly independent solutions and the inhomogeneous integral equations are solvable iff

$$\int_E f(x)v(x)\,dx = 0$$

for all v solving (4.2) and

$$\int_E u(x)g(x)\,dx = 0$$

for all u solving (4.1), respectively.

Appendices

A Ascoli-Arzelá Result

Definition A.1. *Let X be a topological space. A set $E \subset X$ is said to be totally bounded if, for every given $\varepsilon > 0$, there exists a finite collection of points $\{x_1, x_2, \cdots, x_n\} \subset X$ such that $E \subset \cup_{i=1}^n B_\varepsilon(x_i)$.*

Exercise A.1. If $E \subset X$ is totally bounded then $E^n \subset X^n$ is also totally bounded.

Definition A.2. *A subset $A \subset C(X)$ is said to be* bounded *if there exists a $M \in \mathbb{N}$ such that $\|f\|_\infty \leq M$ for all $f \in A$.*

Definition A.3. *A subset $A \subset C(X)$ is said to be* equicontinuous *at $x_0 \in X$ if, for every given $\varepsilon > 0$, there is an open set U of x_0 such that*

$$|f(x) - f(x_0)| < \varepsilon \quad \forall x \in U; f \in A.$$

A is said to be equicontinuous *if it is equicontinuous at every point of X.*

Theorem A.4. *Let X be a compact topological space and Y be a totally bounded metric space. If a subset $A \subset C(X, Y)$ is equicontinuous then A is totally bounded.*

Proof. Let A be equicontinuous and $\varepsilon > 0$. Then, for each $x \in X$, there is a open set U_x containing x such that

$$|f(y) - f(x)| < \frac{\varepsilon}{3} \quad \forall y \in U_x; f \in A.$$

Since X is compact, there is a finite set of points $\{x_i\}_1^n \subset X$ such that $X = \cup_{i=1}^n U_{x_i}$. Define the subset E_A of Y^n as,

$$E_A := \{(f(x_1), f(x_2), \cdots, f(x_n)) \mid f \in A\}$$

which is endowed with the product metric, i.e.,

$$d(y, z) = \max_{1 \le i \le n} \{|y_i - z_i|\}$$

where $y, z \in Y^n$ are n-tuples. Since Y is totally bounded, Y^n is also totally bounded (cf. Exercise A.1). Thus, E_A is totally bounded and there are m number of n-tuples, $y_j := (f_j(x_1), f_j(x_2), \cdots, f_j(x_n)) \in Y^n$, for each $1 \le j \le m$, such that $E_A \subset \cup_{j=1}^m B_{\varepsilon/3}(k_j)$. For any $f \in A$, there is a j such that $d(y_j, z_f) < \frac{\varepsilon}{3}$ where $z_f = (f(x_1), f(x_2), \cdots, f(x_n))$. In particular, given any $f \in A$, there is a j such that, for all $1 \le i \le n$,

$$|f_j(x_i) - f(x_i)| < \frac{\varepsilon}{3}.$$

Given $f \in A$, fix the j as chosen above.Now, for any given $x \in X$, there is a i such that $x \in U_{x_i}$. For this choice of i, j, we have

$$|f(x) - f_j(x)| \le |f(x) - f(x_i)| + |f(x_i) - f_j(x_i)| + |f_j(x_i) - f_j(x)|$$

The first and third term is smaller that $\varepsilon/3$ by the continuity of f and f_j, respectively, and the second term is smaller than $\varepsilon/3$ by choice of f_j. Hence A is totally bounded, i.e., $A \subset \cup_{j=1}^m B_\varepsilon(f_j)$, equivalently, for any $f \in A$ there is a j such that $\|f - f_j\|_\infty < \varepsilon$. □

Lemma A.5. *Let X be compact topological space. If $A \subset C(X)$ is bounded then there is a compact subset $K \subset \mathbb{R}$ such that $f(x) \in K$ for all $f \in A$ and $x \in X$.*

Proof. Choose an element $g \in A$. Since A is bounded in the uniform topology, there is a M such that $\|f - g\|_\infty < M$ for all $f \in A$. Since X is compact, $g(X)$ is compact. Hence there is a $N > 0$ such that $g(X) \subset [-N, N]$. Then $f(X) \subset [-M - N, M + N]$ for all $f \in A$. Set $K := [-M - N, M + N]$ and we are done. □

Corollary A.6 (other part of Ascoli-Arzela Theorem)**.** *Let X be a compact topological space. If a subset $A \subset C(X)$ is closed, bounded and equicontinuous then A is compact.*

Proof. Since A is bounded, by Lemma above, we have $A \subset C(X, K) \subset C(X)$ for some compact subset $K \subset \mathbb{R}$. Then, by the Theorem above, A is totally bounded. Since A is a closed and totally bounded subset of the metric space $C(X)$, A is compact. □

B Normed Spaces and Bounded Operators

Definition B.1. *Let V and W be real or complex vector spaces. A* linear map *from V to W is a function $T : V \to W$ such that*

$$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y) \quad \forall x, y \in V \text{ and } \forall \alpha, \beta \in \mathbb{R} \text{ or } \mathbb{C}.$$

Observe that a linear map is defined between vector spaces over the same field of scalars.

Exercise B.1. Show that a linear map T satisfies $T(0) = 0$.

Let $\mathcal{L}(V, W)$ be the space of linear maps from V to W.

Definition B.2. *A* normed space *is a pair $(X, \|\cdot\|)$, where X is a vector space over $\mathbb{C}$ or $\mathbb{R}$ and $\|\cdot\| : X \to [0, \infty)$ is a function such that*

(i) $\|x\| = 0$ *iff* $x = 0$,

(ii) $\|\lambda x\| = |\lambda|\|x\|$ *for all $x \in X$ and $\lambda \in \mathbb{F}$ (absolute homogeneity),*

(iii) $\|x + y\| \leq \|x\| + \|y\|$ *for all $x, y \in X$. (sub-additivity or triangle inequality)*

The function $\|\cdot\|$ is called the *norm* of a vector from X. Norm is a generalisation of the notion length of a vector in a Euclidean space.

Exercise B.2. Show that every normed space is a metric space with the metric $d(x, y) = \|x - y\|$.

Exercise B.3. Show that the map $\|\cdot\| : X \to [0, \infty)$ is uniformly continuous on X.

Proof. Observe that $\|x\| = \|x - y + y\| \leq \|x - y\| + \|y\|$. Thus, $\|x\| - \|y\| \leq \|x - y\|$. Similarly, $\|y\| \leq \|y - x\| + \|x\|$. Thus, $|\|x\| - \|y\|| \leq \|x - y\|$. □

Exercise B.4. The operations addition (+) and scalar multiplication are continuous from $X \times X$ and $X \times \mathbb{C}$ to X, respectively.

Exercise B.5. Show that for a Cauchy sequence $\{x_n\}$ in X, we have

$$\|x_m - x_n\| < \frac{1}{2^n} \quad \forall m \geq n.$$

Definition B.3 (Infinite Series)**.** *An infinite series in a normed space X, say $\sum_{i=1}^{\infty} x_i = x_1 + x_2 + \ldots$, is said to be* convergent *if the sequence s_n is convergent, where $s_n = \sum_{i=1}^{n} x_i$ is the sequence of partial sums. An infinite series is said to be* absolutely convergent *if the series $\sum_{i=1}^{\infty} \|x_i\|$ is convergent.*

Theorem B.4. *A normed space X is a Banach space iff every absolutely convergent series in X is convergent.*

Proof. Let X be Banach space and let $x = \sum_{i=1}^{\infty} x_i$ be an absolutely convergence series. Let $y_n = \sum_{i=1}^{n} x_i$ be the partial sum. It is enough to show that $\{y_n\}$ is Cauchy in X. Given $\varepsilon > 0$, there exists a N_0 such that $\sum_{i=N_0}^{\infty} \|x_i\| < \varepsilon$. We choose $m, n \geq N_0$ and, without loss of generality, fix $N_0 \leq m < n$. Then

$$\|y_n - y_m\| = \left\| \sum_{i=m+1}^{n} x_i \right\| \leq \sum_{i=m+1}^{n} \|x_i\| \leq \sum_{i=N_0}^{\infty} \|x_i\| < \varepsilon.$$

Thus, $\{y_n\}$ is a Cauchy sequence in X and hence converges. Hence, the given absolutely convergent series converges.

Conversely, let every absolutely convergent series in X converge. We need to show that every Cauchy sequence in X converges. Let $\{x_n\}$ be a Cauchy sequence in X. Therefore, by Exercise B.5,

$$\|x_m - x_n\| < \frac{1}{2^n} \quad \forall m \geq n.$$

Now, let us construct a series in X using the given Cauchy sequence. Set $x_0 = 0$ and define $y_k = x_k - x_{k-1}$ for all $i \geq 1$. Then, observe that $\sum_{k=1}^{n} y_k = x_n$. Therefore, the n^{th} partial sum of the series $\sum_{k=1}^{\infty} y_k$ is x_n. Observe that $\|y_k\| < 1/2^{k-1}$. Thus, by comparison test, the series absolutely convergent and hence, by hypothesis, converges. Therefore its sequence of partial sums $\{x_n\}$ converges. Therefore X is Banach since $\{x_n\}$ was a arbitrary sequence in X. □

Theorem B.5. *Let $\{x_1, x_2, \ldots, x_n\}$ be a linearly independent set of vectors in a normed space X. Then there is a constant $c > 0$ such that for every choice of $\lambda_1, \lambda_2, \ldots, \lambda_n$ we have*

$$\left\| \sum_{i=1}^{n} \lambda_i x_i \right\| \geq c \left(\sum_{i=1}^{n} |\lambda_i| \right). \quad \text{(c independent of the scalars)} \tag{B.1}$$

Proof. Set $s = \sum_{i=1}^{n} |\lambda_i|$. If $s = 0$ then $\lambda_i = 0$, for each $i = 1, 2, \ldots, n$. Thus, (B.1) holds trivially, for any $c > 0$. Suppose that $s > 0$. Then, observe that proving (B.1) is equivalent to showing the existence of a constant $c > 0$ such that for all scalars α_i of the satisfying $\sum_{i=1}^{n} |\alpha_i| = 1$, we have

$$\left\| \sum_{i=1}^{n} \alpha_i x_i \right\| \geq c.$$

The equivalence is obtained by dividing s on both sides of (B.1) and setting $\alpha_i = \frac{\lambda_i}{s}$. Suppose our claim is false, then for every $m \in \mathbb{N}$, there is a set of scalars $\{\alpha_i^m\}_1^n$ such that $\sum_{i=1}^{n} |\alpha_i^m| = 1$ and

$$\|y_m\| = \left\| \sum_{i=1}^{n} \alpha_i^m x_i \right\| < \frac{1}{m}.$$

Thus, $\|y_m\| \to 0$ as $m \to \infty$. Since $\sum_{i=1}^{n} |\alpha_i^m| = 1$, for each i, $|\alpha_i^m| \leq 1$. Fixing $i = 1$, we observe that the sequence $\{\alpha_i^m\}_m$ is bounded in $\mathbb{R}$. By invoking Bolzano-Weierstrass theorem, $\{\alpha_1^m\}_m$ has a convergent subsequence $\{\gamma_1^m\}$ that converges to α_1. Let $y_m^1 = \gamma_1^m x_1 + \sum_{i=2}^{n} \alpha_i^m x_i$ which is a subsequence of y_m. Repeating the argument for y_m^1, we get a subsequence $y_m^2 = \sum_{i=1}^{2} \gamma_i^m x_i + \sum_{i=3}^{n} \alpha_i^m x_i$ with α_2 being the limit of the subsequence of $\{\alpha_2^m\}$. Thus, repeating the procedure n times, we have a subsequence $\{y_m^n\}_m$ of y_m which is given by

$$y_m^n = \sum_{i=1}^{n} \gamma_i^m x_i$$

where, for each i, $\gamma_i^m \to \alpha_i$ and $\sum_{i=1}^n |\gamma_i^m| = 1$. Thus, letting $m \to \infty$, we have

$$y_m^n \to y = \sum_{i=1}^n \alpha_i x_i$$

and $\sum_{i=1}^n |\alpha_i| = 1$. Thus, $\alpha_i \neq 0$ for some i. Since the set $\{x_1, x_2, \ldots, x_n\}$ is linearly independent $y \neq 0$. By Exercise B.3, if $y_m^n \overset{m\to\infty}{\longrightarrow} y$, then $\|y_m^n\| \overset{m\to\infty}{\longrightarrow} \|y\|$. But $\|y_m\| \to 0$, hence the subsequence $\|y_m^n\| \to 0$. Therefore $\|y\| = 0$ implies $y = 0$ which is a contradiction. □

Theorem B.6. *Let Y be finite dimensional subspace of a normed space X. Then, for any $x \in X$, there is a $y \in Y$ such that*

$$\|x - y\| = \inf_{z \in Y} \|x - z\|.$$

Lemma B.7 (Riesz Lemma)**.** *Let Y be a proper closed subspace of a normed space X. Then, for every $0 < \varepsilon < 1$, there is a point $x_\varepsilon \in X$ such that $\|x_\varepsilon\| = 1$ and*

$$\varepsilon \leq d(x_\varepsilon, Y) \leq 1,$$

where $d(x, Y) = \inf_{y \in Y} \|x - y\|$.

Proof. Since $Y \neq X$, choose $x \in X$ such that $x \notin Y$. Since Y is closed $d(x, Y) > 0$. Now, for any $0 < \varepsilon < 1$, there is a $y_0 \in Y$ such that

$$d(x, Y) \leq \|x - y_0\| \leq \frac{d(x, Y)}{\varepsilon}.$$

The above inequality can be rewritten as

$$\varepsilon \leq \frac{d(x, Y)}{\|x - y_0\|} \leq 1. \tag{B.2}$$

Set $x_\varepsilon = \frac{x - y_0}{\|x - y_0\|}$. Observe that

$$\begin{aligned} \|x_\varepsilon - y\| &= \frac{1}{\|x - y_0\|}(x - y_0 - \|x - y_0\| y) \\ &= \frac{1}{\|x - y_0\|}\|x - y_1\| \quad (\text{ where } y_1 = y_0 - \|x - y_0\| y \in Y). \end{aligned}$$

Therefore, $d(x_\varepsilon, Y) = \frac{1}{\|x - y_0\|} d(x, Y)$ and by (B.2), we have our claim. □

Theorem B.8. *If a normed space X is such that the unit ball $B(X)$ is compact, then X is finite dimensional.*

Proof. Let us suppose that X is infinite dimensional. Let $x_1 \in X$ such that $\|x_1\| = 1$. The $X_1 = [x_1]$ is a one dimensional subspace of X. Since X is infinite dimensional, $[x_1]$ is a proper subspace of X. By Riesz lemma, there is a $x_2 \in X$ with $\|x_2\| = 1$ such that $\|x_2 - x_1\| \geq 1/2$. Now, $X_2 = [x_1, x_2]$ is a two-dimensional proper subspace of X. Therefore, again by Riesz lemma, there is a $x_3 \in X$ with $\|x_3\| = 1$ such that $\|x_3 - x\| \geq 1/2$ for all $x \in X_2$. In particular, $\|x_3 - x_1\| \geq 1/2$ and $\|x_3 - x_2\| \geq 1/2$. Arguing further in a similar way, we obtain a sequence $\{x_n\}$ in $B(X)$ such that $\|x_m - x_n\| \geq 1/2$ for all $m \neq n$. Thus, we have obtained a bounded sequence in $B(X)$ which cannot converge for any subsequence, which contradicts the hypothesis that $B(X)$ is compact. Therefore $\dim(X) = \infty$. □

Theorem B.9. *Let X and Y be normed spaces. If X is finite dimensional, then every linear map from X to Y is continuous.*

Proof. Let X be finite dimensional and and let $T \in \mathcal{L}(X, Y)$. If $T = 0$, then the result is trivial, since $\mathcal{L}(X, Y) = \{0\}$. Let $X \neq 0$ and $\{e_1, e_2, \ldots, e_m\}$ be a basis for X. Let $x_n \to x$ in X. For some scalars, since $x_n = \sum_{i=1}^m \lambda_i^n e_i$ and $x = \sum_{i=1}^m \lambda_i e_i$, we have $\lambda_i^n \to \lambda_i$ for all $1 \leq i \leq m$ (by Theorem B.5). Consider,

$$\begin{aligned} Tx_n &= \sum_{i=1}^m \lambda_i^m T e_i \quad \text{(by linearity of } T\text{)} \\ &\longrightarrow \sum_{i=1}^m \lambda_i T e_i = Tx. \end{aligned}$$

The convergence is valid by the continuity of addition and scalar multiplication (cf. Exercise B.4). Thus T is continuous. □

Definition B.10. *For any given normed spaces X and Y, a linear map $T \in \mathcal{L}(X, Y)$ is said to be* bounded *if there is a constant $c > 0$ such that $\|Tx\| \leq c\|x\|, \quad \forall x \in X$.*

Basically, bounded operators map bounded sets in X to bounded sets in Y. Let $\mathcal{B}(X, Y)$ be the space of bounded linear maps from X to Y.

Any map in $\mathcal{L}(X, Y) \setminus \mathcal{B}(X, Y)$ is said to be *unbounded* linear map. We shall now prove an interesting result which says that $\mathcal{B}(X, Y) = \mathcal{C}(X, Y)$, the space of continuous linear maps from X to Y. In other words, by looking at bounded linear maps we are, in fact, looking at continuous linear maps. Such a result is possible only because of the underlying linear structure of the space. The following theorem proves these remarks rigorously.

Theorem B.11. *Let X and Y be normed spaces and let $T \in \mathcal{L}(X, Y)$. Then the following are equivalent:*

(i) $T \in \mathcal{C}(X, Y)$.

(ii) T *is continuous at some point* $x_0 \in X$.

(iii) $T \in \mathcal{B}(X, Y)$.

Proof. The above equivalence are true if $T = 0$. Hence, henceforth, we assume $T \neq 0$.

(i) implies (ii) is trivial from the definition of continuity.

Let us now assume (ii). Then, for any $\varepsilon > 0$, there is a $\delta > 0$ such that $\|Tx - Tx_0\| \leq \varepsilon$ whenever $\|x - x_0\| \leq \delta$. Set $B_\delta(x_0) = \{x \in X \mid \|x - x_0\| \leq \delta$. For any non-zero $x \in X$, $x_0 + \frac{\delta}{\|x\|}x \in B_\delta(x_0)$. Therefore $\frac{\delta}{\|x\|}\|Tx\| \leq \varepsilon$ and hence $\|Tx\| \leq \frac{\varepsilon}{\delta}\|x\|$. Thus, T is bounded.

We shall now assume T is bounded and prove (i). There is a $c > 0$ such that $\|Tx\| \leq c\|x\|$. Therefore, $\|Tx - Ty\| \leq c\|x - y\|$ for all $x, y \in X$. Thus, for any given $\varepsilon > 0$, $\|Tx - Ty\| \leq \varepsilon$ whenever $\|x - y\| \leq \varepsilon/N$. □

We shall now introduce a norm in $\mathcal{B}(X, Y)$ to make it a normed space. Observe that, in the definition of bounded linear map, we seek the existence of a constant $c > 0$. The question is what is the smallest such constant $c > 0$? Note that

$$\frac{\|Tx\|}{\|x\|} \leq c \quad \forall x \in X \text{ and } x \neq 0.$$

Thus, the smallest such constant would be $\sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} \leq c$.

Exercise B.6. For any $T \in \mathcal{B}(X, Y)$

$$\sup_{\substack{x \in X \\ x \neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x \in X \\ \|x\| = 1}} \|Tx\| = \sup_{\substack{x \in X \\ \|x\| \leq 1}} \|Tx\|.$$

Proof. Observe that for any $x \in X$, the vector $z = \frac{x}{\|x\|} \in X$ such that $\|z\| = 1$. Thus, by the linearity of T,

$$\sup_{\substack{x\in X\\x\neq 0}} \frac{\|Tx\|}{\|x\|} = \sup_{\substack{x\in X\\x\neq 0}} \left\| T\left(\frac{x}{\|x\|}\right)\right\| = \sup_{\substack{z\in X\\\|z\|=1}} \|Tz\|.$$

□

Exercise B.7. Show that the function $\|\cdot\| : \mathcal{B}(X,Y) \to [0,\infty)$ defined as

$$\|T\| = \sup_{\substack{x\in X\\x\neq 0}} \frac{\|Tx\|}{\|x\|}, \quad \forall T \in \mathcal{B}(X,Y)$$

is a norm on $\mathcal{B}(X,Y)$. Thus, $\mathcal{B}(X,Y)$ is a normed space.

Exercise B.8. Show that if $T \in \mathcal{B}(X,Y)$ and $S \in \mathcal{B}(Y,Z)$, then the composition $S \circ T = ST$ is in $\mathcal{B}(X,Z)$ and $\|ST\| \leq \|S\|\|T\|$. In particular, show that $\mathcal{B}(X)$ is an algebra under composition of operators.

Theorem B.12. *If Y is complete then $\mathcal{B}(X,Y)$ is a complete normed space. In particular, $X^\star$ is a Banach space.*

Proof. Let $\{T_n\}$ be a Cauchy sequence in $\mathcal{B}(X,Y)$. Then $\{T_n x\}$ is a Cauchy sequence in Y, for all $x \in X$. Since Y is complete, there is a $y \in Y$ such that $T_n x \to y$. Set $Tx = y$. It now remains to show that $T \in \mathcal{B}(X,Y)$ and $T_n \to T$ in $\mathcal{B}(X,Y)$. For any given $x_1, x_2 \in X$ and scalars $\lambda_1, \lambda_2 \in \mathbb{R}$, we have

$$\begin{aligned} T(\lambda_1 x_1 + \lambda_2 x_2) &= \lim_{n\to\infty} T_n(\lambda_1 x_1 + \lambda_2 x_2) \\ &= \lim_{n\to\infty} \lambda_1 T_n x_1 + \lim_{n\to\infty} \lambda_2 T_n x_2 \\ &= \lambda_1 T x_1 + \lambda_2 T x_2. \end{aligned}$$

Thus, $T \in \mathcal{L}(X,Y)$. Since T_n is Cauchy, for every $\varepsilon > 0$, there is a $N_0 \in \mathbb{N}$ such that $\|T_m - T_n\| < \varepsilon$ for all $m, n \geq N_0$. Therefore, for all $x \in X$, we have,

$$\|T_n x - T_m x\| \leq \|T_n - T_m\|\|x\| < \varepsilon\|x\| \quad \forall m, n \geq N_0.$$

Now, by letting $m \to \infty$, we have $\|T_n x - Tx\| \leq \varepsilon\|x\|$ for all $x \in X$ and $n \geq N_0$. Thus, $\|T_n - T\| \to 0$. Also, since

$$\|Tx\| = \|Tx - T_n x\| + \|T_n x\| \leq (\varepsilon + \|T_n\|)\|x\|,$$

we have $T \in \mathcal{B}(X,Y)$. □