Partial Differential Equations
MSO-203-B

T. Muthukumar
tmk@iitk.ac.in

November 14, 2019

Raison d'être

The process of understanding natural phenomena may be viewed in three stages:

1. Modelling the phenomenon as a mathematical equation (algebraic or differential equation) using physical laws such as Newton's law, momentum, conservation laws, balancing forces etc.
2. Solving the equation! This leads to the question of what constitutes as a solution to the equation?
3. Properties of the solution, especially in situations when exact solution is not within our reach.

In this course, we are mostly interested in differential equations in dimension bigger than one!

- Let $\Omega \subset \mathbb{R}$ is an open interval. Then the *derivative* of a function $u : \Omega \to \mathbb{R}$, at $x \in \Omega$, is defined as

$$u'(x) := \lim_{h \to 0} \frac{u(x+h) - u(x)}{h}$$

provided the limit exists.

- Now, let Ω be an open, connected subset of $\mathbb{R}^n$. The *directional derivative* of $u : \Omega \to \mathbb{R}$, at $x \in \Omega$ and in the direction of a given vector $\xi \in \mathbb{R}^n$, is defined as

$$\partial_\xi u(x) := \lim_{h \to 0} \frac{u(x+h\xi) - u(x)}{h},$$

provided the limit exists.

- Let $e_i := (0, 0, \ldots, 1, 0, \ldots, 0)$, where 1 is in the i-th place, denote the standard basis vectors of $\mathbb{R}^n$.
- The i-th *partial derivative* of u at x is the directional derivative of u, at $x \in \Omega$ and along the direction e_i, and is denoted as $u_{x_i}(x)$ or $\frac{\partial u}{\partial x_i}(x)$.
- The *gradient vector* of u is $\nabla u(x) := (u_{x_1}(x), u_{x_2}(x), \ldots, u_{x_n}(x))$.
- The directional derivative along a vector $\xi \in \mathbb{R}^n$ satisfies the identity

$$\partial_\xi u(x) = \nabla u(x) \cdot \xi.$$

- The *divergence* of a vector function $\mathbf{u} = (u_1, \ldots, u_n)$, denoted as $\operatorname{div}(\mathbf{u})$, is defined as $\operatorname{div}(\mathbf{u}) := \nabla \cdot \mathbf{u}$.

Multi-Index Notations

- Note that a k-degree polynomial in one variable is written as $\sum_{1 \leq i \leq k} a_i x^i$.
- How does one denote a k-degree polynomial in n variable (higher dimensions)?
- A k-degree polynomial in n-variables can be concisely written as $\sum_{|\alpha| \leq k} a_\alpha x^\alpha$ where
- the *multi-index* $\alpha = (\alpha_1, \ldots, \alpha_n)$ is a n-tuple where α_i, for each $1 \leq i \leq n$, is a non-negative integer,
- $|\alpha| := \alpha_1 + \ldots + \alpha_n$,
- and, for any $x \in \mathbb{R}^n$, $x^\alpha = x_1^{\alpha_1} \ldots x_n^{\alpha_n}$.

Multi-Index Notations

- The partial differential operator of *order* α is denoted as

$$\partial^\alpha = \frac{\partial^{\alpha_1}}{\partial {x_1}^{\alpha_1}} \ldots \frac{\partial^{\alpha_n}}{\partial {x_n}^{\alpha_n}} = \frac{\partial^{|\alpha|}}{\partial {x_1}^{\alpha_1} \ldots \partial {x_n}^{\alpha_n}}.$$

- If $|\alpha| = 0$, then $\partial^\alpha f = f$.
- For each $k \in \mathbb{N}$, $D^k u(x) := \{\partial^\alpha u(x) \mid |\alpha| = k\}$.
- The case $k = 1$ is the *gradient* vector,

$$\begin{aligned}
\nabla u(x) &:= D^1 u(x) \\
&= \left(\partial^{(1,0,\ldots,0)} u(x), \partial^{(0,1,0,\ldots,0)} u(x), \ldots, \partial^{(0,0,\ldots,0,1)} u(x)\right) \\
&= \left(\frac{\partial u(x)}{\partial x_1}, \frac{\partial u(x)}{\partial x_2}, \ldots, \frac{\partial u(x)}{\partial x_n}\right).
\end{aligned}$$

Multi-Index Notations

- The case $k = 2$ is the *Hessian* matrix

$$D^2 u(x) = \begin{pmatrix} \frac{\partial^2 u(x)}{\partial x_1^2} & \dots & \frac{\partial^2 u(x)}{\partial x_1 \partial x_n} \\ \frac{\partial^2 u(x)}{\partial x_2 \partial x_1} & \dots & \frac{\partial^2 u(x)}{\partial x_2 \partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 u(x)}{\partial x_n \partial x_1} & \dots & \frac{\partial^2 u(x)}{\partial x_n^2} \end{pmatrix}_{n \times n}.$$

- The *Laplace* operator, denoted as Δ, is defined as the trace of the Hessian operator, i.e., $\Delta := \sum_{i=1}^n \frac{\partial^2}{\partial x_i^2}$. Note that $\Delta = \nabla \cdot \nabla$.

Partial Differential map

Example

Let $u(x, y) : \mathbb{R}^2 \to \mathbb{R}$ be defined as $u(x, y) = ax^2 + by^2$. Then

$$\nabla u = (u_x, u_y) = (2ax, 2by)$$

and

$$D^2 u = \begin{pmatrix} u_{xx} & u_{yx} \\ u_{xy} & u_{yy} \end{pmatrix} = \begin{pmatrix} 2a & 0 \\ 0 & 2b \end{pmatrix}.$$

Observe that $\nabla u : \mathbb{R}^2 \to \mathbb{R}^2$ and $D^2 u : \mathbb{R}^2 \to \mathbb{R}^4 = \mathbb{R}^{2^2}$.

- More generally, for a k-times differentiable function u, the n^k-*tensor* $D^k u(x) := \{\partial^\alpha u(x) \mid |\alpha| = k\}$ may be viewed as a map $D^k u : \mathbb{R}^n \to \mathbb{R}^{n^k}$.

Partial Differential Equation

Definition

Let Ω be an open, connected subset of $\mathbb{R}^n$. A k-th *order* partial differential equation of an *unknown* function $u : \Omega \to \mathbb{R}$ is of the form

$$F\left(D^k u(x), D^{k-1}u(x), \dots Du(x), u(x), x\right) = 0, \tag{1.1}$$

for each $x \in \Omega$, where $F : \mathbb{R}^{n^k} \times \mathbb{R}^{n^{k-1}} \times \dots \times \mathbb{R}^n \times \mathbb{R} \times \Omega \to \mathbb{R}$ is a given map such that F depends, at least, on one k-th partial derivative u and is independent of $(k+j)$-th partial derivatives of u for all $j \in \mathbb{N}$.

- For instance, a first order PDE is represented as $F(Du(x), u(x), x) = 0$ and a second order PDE is $F(D^2u(x), Du(x), u(x), x) = 0$.
- A first order PDE with three variable unknown function $u(x, y, z)$ is written as $F(u_x, u_y, u_z, u, x, y, z) = 0$ with F depending, at least, on one of u_x, u_y and u_z.

Classification of PDE in terms of Linearity

The level of difficulty in solving a PDE may depend on its order k and linearity of F.

Definition

A k-th order PDE is *linear* if F in (1.1) has the form

$$Fu := Lu - f(x)$$

where $Lu(x) := \sum_{|\alpha| \le k} a_\alpha(x)\partial^\alpha u(x)$ for given functions f and a_α's. In addition, if $f \equiv 0$ then the PDE is linear and *homogeneous*.

It is called linear because L is linear in u for all derivatives , i.e., $L(\lambda u_1 + \mu u_2) = \lambda L(u_1) + \mu L(u_2)$ for $\lambda, \mu \in \mathbb{R}$.

Classification of PDE in terms of Linearity

Example

1. $xu_y - yu_x = u$ is linear and homogeneous.
2. $xu_x + yu_y = x^2 + y^2$ is linear.
3. $u_{tt} - c^2 u_{xx} = f(x,t)$ is linear.
4. $y^2 u_{xx} + xu_{yy} = 0$ is linear and homogeneous.

Classification of PDE in terms of Linearity

Definition

A k-th order PDE is *semilinear* if F is linear only in the highest (k-th) order, i.e., F has the form

$$\sum_{|\alpha|=k} a_\alpha(x)\partial^\alpha u(x) + f(D^{k-1}u(x), \ldots, Du(x), u(x), x) = 0.$$

Example

1. $u_x + u_y - u^2 = 0$ is semilinear.
2. $u_t + uu_x + u_{xxx} = 0$ is semilinear.
3. $u_{tt}^2 + u_{xxxx} = 0$ is semilinear.

Classification of PDE in terms of Linearity

Definition

A k-th order PDE is *quasilinear* if F has the form

$$\sum_{|\alpha|=k} a_\alpha(D^{k-1}u(x),\ldots,u(x),x)\partial^\alpha u + f(D^{k-1}u(x),\ldots,u(x),x) = 0,$$

i.e., the coefficient of its highest (k-th) order derivative depends on u and its derivative only upto the previous $(k-1)$-th orders.

Example

(i) $u_x + uu_y - u^2 = 0$ is quasilinear.

(ii) $uu_x + u_y = 2$ is quasilinear.

Classification of PDE in terms of Linearity

Definition

A k-th order PDE is *fully nonlinear* if it depends nonlinearly on the highest (k-th) order derivatives.

Example

(i) $u_x u_y - u = 0$ is nonlinear.

(ii) $u_x^2 + u_y^2 = 1$ is nonlinear.

Definition

We say $u : \Omega \to \mathbb{R}$ is a *solution* to the PDE (1.1), if $\partial^\alpha u$ exists for all α explicitly present in (1.1) and u satisfies the equation (1.1).

Example

Consider the first order equation $u_x(x, y) = 0$ in $\mathbb{R}^2$.Freezing the y-variable, the PDE can be viewed as an ODE in x-variable. On integrating both sides with respect to x, $u(x, y) = f(y)$ for any arbitrary function $f : \mathbb{R} \to \mathbb{R}$.Therefore, for every choice of $f : \mathbb{R} \to \mathbb{R}$, there is a solution u of the PDE.Note that the solution u is not necessarily in $C^1(\mathbb{R}^2)$, in contrast to the situation in ODE. By choosing a discontinuous function f, one obtains a solution which is discontinuous in the y-direction.Similarly, a solution of $u_y(x, y) = 0$ is $u(x, y) = f(x)$ for any choice of $f : \mathbb{R} \to \mathbb{R}$ (not necessarily continuous).

Example

Consider the first order equation $u_t(x, t) = u(x, t)$ in $\mathbb{R} \times (0, \infty)$ such that $u(x, t) \neq 0$, for all (x, t). Freezing the x-variable, the PDE can be viewed as an ODE in t-variable. Integrating both sides with respect to t we obtain $u(x, t) = f(x)e^t$, for some arbitrary (not necessarily continuous) function $f : \mathbb{R} \to \mathbb{R}$.

Example

Consider the second order PDE $u_{xy}(x,y) = 0$ in $\mathbb{R}^2$. In contrast to the previous two examples, the PDE involves derivatives in both variables. On integrating both sides with respect to x we obtain $u_y(x,y) = F(y)$, for any arbitrary integrable function $F : \mathbb{R} \to \mathbb{R}$. Now, integrating both sides with respect to y, $u(x,y) = f(y) + g(x)$ for an arbitrary $g : \mathbb{R} \to \mathbb{R}$ and a $f \in C^1(\mathbb{R})$. But the u obtained above is not a solution to $u_{yx}(x,y) = 0$ if g is not differentiable. If we assume mixed derivatives to be same we need to assume $f, g \in C^1(\mathbb{R})$ for the solution to exist.

Solution of PDE

Example

Consider the first order equation $u_x(x,y) = u_y(x,y)$ in $\mathbb{R}^2$.On first glance, the PDE does not seem simple to solve. But, by change of coordinates, the PDE can be rewritten in a simpler form. Choose the coordinates $w = x + y$ and $z = x - y$ and, by chain rule, $u_x = u_w + u_z$ and $u_y = u_w - u_z$. In the new coordinates, the PDE becomes $u_z(w,z) = 0$ which is in the form considered in Example 12.Therefore, its solution is $u(w,z) = f(w)$ for any arbitrary $f : \mathbb{R} \to \mathbb{R}$ and, hence, $u(x,y) = f(x+y)$.

Multiple Family of Solutions

Example

Consider the second order PDE $u_t(x,t) = u_{xx}(x,t)$.

1. Note that $u(x,t) = c$ is a solution of the PDE, for any constant $c \in \mathbb{R}$. This is a family of solutions indexed by $c \in \mathbb{R}$.
2. The function $u : \mathbb{R}^2 \to \mathbb{R}$ defined as $u(x,t) = \frac{x^2}{2} + t + c$, for any constant $c \in \mathbb{R}$, is also a family of solutions of the PDE. Because $u_t = 1$, $u_x = x$ and $u_{xx} = 1$.This family is not covered in the first case.
3. The function $u(x,t) = e^{c(x+ct)}$ is also a family of solutions to the PDE, for each $c \in \mathbb{R}$. Because $u_t = c^2 u$, $u_x = cu$ and $u_{xx} = c^2 u$. This family is not covered in the previous two cases.

Recall that the family of solutions of an ODE is indexed by constants. In contrast to ODE, observe that the family of solutions of a PDE is indexed by either functions or constants.

Well-posedness of PDE

- It has been illustrated via examples that a PDE has a family of solutions.
- The choice of one solution from the family of solutions is made by imposing boundary conditions (boundary value problem) or initial conditions (initial value problem).
- If too many initial/boundary conditions are specified, then the PDE may have *no solution*.
- If too few initial/boundary conditions are specified, then the PDE may have *many solutions*.
- Even with 'right amount' of initial/boundary conditions, but at wrong places, the solution may fail to be stable, i.e., may not depend continuously on the initial or boundary data.
- It is, usually, desirable to solve a *well-posed* problem, in the sense of Hadamard.

Well-posedness of PDE

A PDE, along with the boundary condition or initial condition, is said to be well-posedness if the PDE

(a) admits a solution (existence);
(b) the solution is unique (uniqueness);
(c) and the solution depends continuously on the data given (stability).

Any PDE not meeting the above criteria is said to be *ill-posed*. Further, the stability condition means that a small "change" in the data reflects a small "change" in the solution. The *change* is measured using a *metric* or "distance" in the function space of data and solution, respectively.

Cauchy Problem

- A *Cauchy problem* poses the following question: given the knowledge of u on a smooth hypersurface $\Gamma \subset \Omega$, can one find the solution u of the PDE?
- The prescription of u on Γ is said to be the *Cauchy data*.
- What is the minimum desirable Cauchy data in order to solve the Cauchy problem?
- Taking cue for ODE: Recall that the initial value problem corresponding to a k-th order linear ODE admits a unique solution

$$\begin{cases} \sum_{i=0}^{k} a_i y^{(i)}(x) & = 0 \quad \text{in } I \\ y(x_0) & = y_0 \quad \text{for some } x_0 \in I \\ y^{(i)}(x_0) & = y_0^{(i)} \quad \forall i = \{1, 2, \ldots, k-1\} \text{ for some } x_0 \in I \end{cases}$$

where a_i are continuous on I, a closed subinterval of $\mathbb{R}$, and $x_0 \in I$.

Cauchy Problem

This motivates us to define the Cauchy problem as

$$\begin{cases} F\left(D^k u(x), \ldots, Du(x), u(x), x\right) & = 0 & \text{in } \Omega \\ u(x) & = u_0(x) & \text{on } \Gamma \\ \partial_\nu^i u(x) & = u_i(x) & \text{on } \Gamma \\ & & \forall i = \{1, 2, \ldots, k-1\} \end{cases} \tag{1.2}$$

where Ω is an open connected subset (domain) of $\mathbb{R}^n$ and Γ is a hypersurface contained in Ω. Thus, a natural question at this juncture is whether the knowledge of u and all its normal derivative upto order $(k-1)$ on Γ is sufficient to compute all order derivatives of u on Γ.

First Order Quasilinear PDE

Let $f \in C(\mathbb{R}^n)$. The Cauchy problem for the first order quasilinear PDE

$$\begin{cases} \mathbf{a}(x, u(x)) \cdot \nabla u(x) & = f(x, u) & \text{in } \mathbb{R}^n \\ u(x) & = u_0(x) & \text{on } \{x_n = 0\}. \end{cases}$$

- We seek whether all order derivatives of u on $\{x_n = 0\}$ can be computed.
- Without loss of generality, let us compute at $x = 0$, i.e $u(0)$ and $\nabla u(0) := (\nabla_{x'} u(0), \partial_{x_n} u(0))$ where $x = (x', x_n)$ where x' is the $(n-1)$-tuple.
- If the initial condition u_0 is a smooth function then the x' derivative of u is computed to be the x'-derivatve of u_0, i.e. $\nabla_{x'} u(0) = \nabla_{x'} u_0(0)$.
- It only remains to compute $\partial_{x_n} u(0)$.
- Using the PDE, whenever $a_n(0, u_0(0)) \neq 0$, we have

$$\partial_{x_n} u(0) = \frac{-1}{a_n(0, u_0(0))} \left(\mathbf{a}'(0, u_0(0)) \cdot \nabla_{x'} u(0) - f(0, u_0(0))\right).$$

General Hypersurface

- Now, suppose Γ is a general hyperspace given by the equation $\{\phi = 0\}$ for a smooth function $\phi : \mathbb{R}^n \to \mathbb{R}$ in a neighbourhood of the origin with $\nabla\phi \neq 0$.
- Recall that $\nabla\phi$ is normal to Γ. Without loss of generality, we assume $\phi_{x_n}(x_0) \neq 0$.
- Consider the change of coordinate $(x', x_n) \mapsto y := (x', \phi(x))$, then its Jacobian matrix is given by

$$\begin{pmatrix} I_{(n-1)\times(n-1)} & \mathbf{0}_{n-1} \\ \nabla_{x'}\phi & \phi_{x_n} \end{pmatrix}_{n\times n}$$

  and its determinant at x_0 is non-zero because $\phi_{x_n}(x_0) \neq 0$.

General Hypersurface

- The change of coordinates has mapped the hypersurface to the hyperplane $\{y_n = 0\}$. Rewriting the given PDE in the new variable y, we get

$$Lu = \mathbf{a}(x, u(x)) \cdot \nabla\phi\partial_{y_n} u + \text{ terms not involving } \partial_{y_n} u$$

  and the initial conditions are given on the hyperplane $\{y_n = 0\}$.
- Thus, the necessary condition is $\mathbf{a}(x, u(x)) \cdot \nabla\phi \neq 0$.
- Recall that $\nabla\phi$ is the normal to the hypersurface Γ.

Non-characteristic Hypersurface

Definition

For any given vector field $\mathbf{a} : \mathbb{R}^n \to \mathbb{R}^n$ and $f : \mathbb{R}^n \to \mathbb{R}$, let $Lu := \mathbf{a}(x, u) \cdot \nabla u - f(x, u)$ be the first order quasilinear partial differential operator defined in a neighbourhood of $x_0 \in \mathbb{R}^n$ and Γ be a smooth hypersurface containing x_0. Then Γ is non-characteristic at x_0 if

$$\mathbf{a}(x_0, u_0(x_0)) \cdot \nu(x_0) \neq 0$$

where $\nu(x_0)$ is the normal to Γ at x_0. Otherwise, we say Γ is *characteristic* at x_0 with respect to L. If Γ is *(non)characteristic* at each of its point then we say Γ is (non)characteristic.

- It says that the coefficient vector $\mathbf{a}$ is not a tangent vector to Γ at x_0.
- The non-characteristic condition depends on the initial hypersurface and the coefficients of first order derivatives in the linear case.
- In the quasilinear case, it also depends on the initial data.

Two Dimension

In the two dimension case, the quasilinear Cauchy problem is

$$\begin{cases} a(x,y,u)u_x + b(x,y,u)u_y &= f(x,y,u) \quad \text{in } \Omega \subset \mathbb{R}^2 \\ u &= u_0 \quad \text{on } \Gamma \subset \overline{\Omega} \end{cases} \tag{1.3}$$

If the parametrization of Γ is $\{\gamma_1(r), \gamma_2(r)\} \subset \overline{\Omega} \subset \mathbb{R}^2$ then the *non-characteristic* condition means if Γ is nowhere tangent to $(a(\gamma_1, \gamma_2, u_0), b(\gamma_1, \gamma_2, u_0))$, i.e.

$$(a(\gamma_1, \gamma_2, u_0), b(\gamma_1, \gamma_2, u_0)) \cdot (-\gamma_2', \gamma_1') \neq 0 \quad \text{for all } r.$$

Example

Consider the equation

$$2u_x(x,y) + 3u_y(x,y) = 1 \quad \text{in } \mathbb{R}^2.$$

Let Γ be a straight line $y = mx + c$ in $\mathbb{R}^2$.The equation of Γ is $\phi(x,y) = y - mx - c$. Then, $\nabla\phi = (-m, 1)$. The parametrization of the line is $\Gamma(r) := (r, mr + c)$ for $r \in \mathbb{R}$. Therefore,

$$\begin{aligned}(a(\gamma_1(r),\gamma_2(r)), b(\gamma_1(r),\gamma_2(r))) \cdot (-\gamma_2'(r), \gamma_1'(r)) &= (2,3)\cdot(-m,1) \\ &= 3 - 2m.\end{aligned}$$

Thus, the line is not a non-characteristic for $m = 3/2$, i.e., all lines with slope $3/2$ is not a non-characteristic.

Two Dimension: Second order Quasilinear

Consider the second order quasilinear Cauchy problem in two variables (x,y)

$$\begin{cases} Pu_{xx} + 2Qu_{xy} + Ru_{yy} &= f(x,y,u,u_x,u_y) \quad \text{in } \Omega \subset \mathbb{R}^2 \\ u(x) &= u_0(x) \quad \text{on } \Gamma \\ \partial_\nu u(x) &= u_1(x) \quad \text{on } \Gamma \end{cases} \tag{1.4}$$

where ν is the unit normal vector to the curve Γ, P, Q, R and f may nonlinearly depend on its arguments (x, y, u, u_x, u_y) and u_0, u_1 are known functions on Γ. Also, one of the coefficients P, Q or R is identically non-zero (else the PDE is not of second order). If the curve Γ is parametrised by $s \mapsto (\gamma_1(s), \gamma_2(s))$ then the directional derivative of u at any point on Γ, along the tangent vector, is $u'(s) = u_x\gamma_1'(s) + u_y\gamma_2'(s)$. But $u'(s) = u_0'(s)$ on Γ.

Thus, instead of the normal derivative, one can prescribe the partial derivatives u_x and u_y on Γ and reformulate the Cauchy problem (1.4) as

$$\left\{\begin{aligned} Pu_{xx} + 2Qu_{xy} + Ru_{yy} &= f(x,y,u,u_x,u_y) && \text{in } \Omega\\ u(x,y) &= u_0(x,y) && \text{on } \Gamma\\ u_x(x,y) &= u_{11}(x,y) && \text{on } \Gamma\\ u_y(x,y) &= u_{12}(x,y) && \text{on } \Gamma. \end{aligned}\right. \tag{1.5}$$

satisfying the *compatibility condition* $u_0'(s) = u_{11}\gamma_1'(s) + u_{12}\gamma_2'(s)$. The compatibility condition implies that among u_0, u_{11}, u_{12} only two can be assigned independently, as expected for a second order equation.

By computing the second derivatives of u on Γ and considering u_{xx}, u_{yy} and u_{xy} as unknowns, we have the system of three equations in three unknowns on Γ,

$$\begin{array}{rlll} Pu_{xx} & +2Qu_{xy} & +Ru_{yy} & = f\\ \gamma_1'(s)u_{xx} & +\gamma_2'(s)u_{xy} & & = u_{11}'(s)\\ & \gamma_1'(s)u_{xy} & +\gamma_2'(s)u_{yy} & = u_{12}'(s). \end{array}$$

This system of equation is solvable whenever the determinant of the coefficients are non-zero, i.e.,

$$\begin{vmatrix} P & 2Q & R\\ \gamma_1' & \gamma_2' & 0\\ 0 & \gamma_1' & \gamma_2' \end{vmatrix} \neq 0.$$

Definition

We say a curve $\Gamma \subset \mathbb{R}^2$ is *characteristic* with respect to (1.5) if $P(\gamma_2')^2 - 2Q\gamma_1'\gamma_2' + R(\gamma_1')^2 = 0$ where $(\gamma_1(s), \gamma_2(s))$ is a parametrisation of Γ.

Two Dimension

If $y = y(x)$ is a representation of the curve Γ (locally, if necessary), we have $\gamma_1(s) = s$ and $\gamma_2(s) = y(s)$. Then the characteristic equation reduces as

$$P\left(\frac{dy}{dx}\right)^2 - 2Q\frac{dy}{dx} + R = 0.$$

Therefore, the characteristic curves of (1.5) are given by the graphs whose equation is

$$\frac{dy}{dx} = \frac{Q \pm \sqrt{Q^2 - PR}}{P}.$$

Two Dimension: Types of Characteristics

Thus, we have three situations depending on the sign of the *discriminant* $d(x) := Q^2 - PR$.

Definition

A second order quasilinear PDE in two dimension is of

(a) *hyperbolic* type at x if $d(x) > 0$, has two families of real characteristic curves,

(b) *parabolic* type at x if $d(x) = 0$, has one family of real characteristic curves and

(c) *elliptic* type at x if $d(x) < 0$, has no real characteristic curves.

Example

For a given $c \in \mathbb{R}$, $u_{yy} - c^2 u_{xx} = 0$ is hyperbolic. Since $P = -c^2$, $Q = 0$ and $R = 1$, we have $d = Q^2 - PR = c^2 > 0$.
How to compute the characteristic curves?Recall that the characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{Q \pm \sqrt{Q^2 - PR}}{P} = \frac{\pm\sqrt{c^2}}{-c^2} = \frac{\mp 1}{c}.$$

Thus, $cy \pm x =$ a constant is the equation for the two characteristic curves. Note that the characteristic curves $y = \mp x/c + y_0$ are boundary of two cones in $\mathbb{R}^2$ with vertex at $(0, y_0)$.

Higher Dimension: Second order Quasilinear

- The classification based on characteristics was done in two dimensions for simplicity.
- The classification is valid for any dimension.
- A second order quasilinear PDE is of the form

$$Lu := A(\nabla u, u, x) \cdot D^2 u - f(\nabla u, u, x) \tag{2.1}$$

  where $A : \mathbb{R}^n \to \mathbb{R}^{n^2}$ and $f : \mathbb{R}^n \to \mathbb{R}$ are given and the dot product in LHS is in $\mathbb{R}^{n^2}$.
- Without loss generality, one may assume that A is symmetric.Because if A is not symmetric, one can replace A with its symmetric part $A^s := \frac{1}{2}(A + A^t)$ in L and L remains unchanged because $A \cdot D^2 u = A^s \cdot D^2 u$.

Non-characteristic Hypersurface

Now repeat the arguments that led to the definition of non-characteristic hypersurface for first order PDE, i.e. compute all the second order derivatives on the data curve.

Definition

Let L as given in (2.1) be defined in a neighbourhood of $x_0 \in \mathbb{R}^n$ and Γ be a smooth hypersurface containing x_0. We say Γ is *non-characteristic* at $x_0 \in \Gamma$ with respect L if

$$A\nu \cdot \nu = \sum_{i,j=1}^{n} A_{ij}(\nabla u(x_0), u(x_0), x_0)\nu_i(x_0)\nu_j(x_0) \neq 0.$$

where $\nu(x_0)$ is the normal vector of Γ at x_0. Otherwise, we say Γ is *characteristic* at x_0 with respect to L. Γ is said to be *(non)-characteristic* if it is (non)-characteristic at each of its point.

Classification

The coefficient matrix $A(\nabla u(x), u(x), x)$ being a real symmetric matrix will admit n eigenvalues at each x. For each x, let $P(x)$ and $Z(x)$ denote the number of positive and zero eigenvalues of $A(\nabla u(x), u(x), x)$.

Definition

We say the partial differential operator given in (2.1) is

- *hyperbolic* at $x \in \Omega$, if $Z(x) = 0$ and either $P(x) = 1$ or $P(x) = n - 1$.
- *elliptic*, if $Z(x) = 0$ and either $P(x) = n$ or $P(x) = 0$.
- is *ultra hyperbolic*, if $Z(x) = 0$ and $1 < P(x) < n - 1$.
- is *parabolic* if $Z(x) > 0$.

Examples

Example

The wave equation $u_{tt} - \Delta_x u = f(x,t)$ for $(x,t) \in \mathbb{R}^{n+1}$ is hyperbolic because the $(n+1) \times (n+1)$ second order coefficient matrix is

$$A := \begin{pmatrix} -I & \mathbf{0} \\ \mathbf{0}^t & 1 \end{pmatrix}$$

has no zero eigenvalue and exactly one positive eigenvalue, where I is the $n \times n$ identity matrix.

Example

The heat equation $u_t - \Delta_x u = f(x,t)$ for $(x,t) \in \mathbb{R}^{n+1}$ is parabolic because the $(n+1) \times (n+1)$ second order coefficient matrix is

$$\begin{pmatrix} -I & \mathbf{0} \\ \mathbf{0}^t & 0 \end{pmatrix}$$

has one zero eigenvalue.

Example

The Laplace equation $\Delta u = f(\nabla u, u, x)$ for $x \in \mathbb{R}^n$ is elliptic because $\Delta u = I \cdot D^2 u(x)$ where I is the $n \times n$ identity matrix. The eigen values are all positive.

'Right' Initial data

- The classification based on characteristics tells us the right amount of initial condition that needs to be imposed for a PDE to be well-posed.
- A hyperbolic PDE, which has two real characteristics, requires as many initial condition as the number of characteristics emanating from initial time and as many boundary conditions as the number of characteristics that pass into the spatial boundary.
- For parabolic, which has exactly one real characteristic, we need one boundary condition at each point of the spatial boundary and one initial condition at initial time.
- For elliptic, which has no real characteristic curves, we need one boundary condition at each point of the spatial boundary.

Standard or Canonical Forms

The classification helps us in reducing a given PDE into simple forms. Given a PDE, one can compute the sign of the discriminant and depending on its clasification we can choose a coordinate transformation (w, z) such that

1. For hyperbolic, $a = c = 0$ (first standard form) $u_{xy} = \tilde{f}(x, y, u, u_x, u_y)$ or $b = 0$ and $a = -c$ (second standard form).
2. If we introduce the linear change of variable $X = x + y$ and $Y = x - y$ in the first standard form, we get the *second standard form* of hyperbolic PDE $u_{XX} - u_{YY} = \hat{f}(X, Y, u, u_X, u_Y)$.
3. For parabolic, $c = b = 0$ or $a = b = 0$. We conveniently choose $c = b = 0$ situation so that $a \neq 0$ (so that division by zero is avoided in the equation for characteristic curves) $u_{yy} = \tilde{f}(x, y, u, u_x, u_y)$.
4. For elliptic, $b = 0$ and $a = c$ to obtain the form $u_{xx} + u_{yy} = \tilde{f}(x, y, u, u_x, u_y)$.

Reduction to Standard Form

- Consider the second order semilinear PDE not in standard form and seek a change of coordinates $w = w(x,y)$ and $z = z(x,y)$, with non-vanishing Jacobian, such that the reduced form is the standard form.
- How does one choose such coordinates w and z. Recall the coeeficients a, b and c obtained in the proof of Exercise 5 of the first assignment!
- If $Q^2 - PR > 0$, we have two characteristics. Thus, choose w and z such that $a = c = 0$.
- This implies we have to choose w and z such that

$$\frac{w_x}{w_y} = \frac{-Q \pm \sqrt{Q^2 - PR}}{P} = \frac{z_x}{z_y}.$$

- Therefore, we need to find w such that along the slopes of the characteristic curves,

$$\frac{dy}{dx} = \frac{Q \pm \sqrt{Q^2 - PR}}{P} = \frac{-w_x}{w_y}.$$

- This means that, using the parametrisation (γ_1, γ_2) of the characteristic curves, $w_x\dot{\gamma}_1(s) + w_y\dot{\gamma}_2(s) = 0$ and $\dot{w(s)} = 0$.
- Similarly for z.
- Thus, w and z are chosen such that they are constant on the characteristic curves.
- Note that $w_x z_y - w_y z_x = w_y z_y \left(\frac{2}{P}\sqrt{Q^2 - PR}\right) \neq 0$.

Example

Let us reduce the PDE $u_{xx} - c^2 u_{yy} = 0$ to its canonical form. Note that $P = 1$, $Q = 0$, $R = -c^2$ and $Q^2 - PR = c^2$ and the equation is hyperbolic. The characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{Q \pm \sqrt{Q^2 - PR}}{P} = \pm c.$$

Solving we get $y \mp cx =$ a constant. Thus, $w = y + cx$ and $z = y - cx$. Now writing

$$\begin{aligned}
u_{xx} &= u_{ww} w_x^2 + 2u_{wz} w_x z_x + u_{zz} z_x^2 + u_w w_{xx} + u_z z_{xx} \\
&= c^2 (u_{ww} - 2u_{wz} + u_{zz}) \\
u_{yy} &= u_{ww} w_y^2 + 2u_{wz} w_y z_y + u_{zz} z_y^2 + u_w w_{yy} + u_z z_{yy} \\
&= u_{ww} + 2u_{wz} + u_{zz} \\
-c^2 u_{yy} &= -c^2 (u_{ww} + 2u_{wz} + u_{zz})
\end{aligned}$$

Substituting into the given PDE, we get $0 = 4c^2 u_{wz}$ or $0 = u_{wz}$.

Example

In the parabolic case, $Q^2 - PR = 0$, we have a single characteristic. Let us reduce the PDE $e^{2x} u_{xx} + 2e^{x+y} u_{xy} + e^{2y} u_{yy} = 0$ to its canonical form. Note that $P = e^{2x}$, $Q = e^{x+y}$, $R = e^{2y}$ and $Q^2 - PR = 0$. The PDE is parabolic. The characteristic curves are given by the equation

$$\frac{dy}{dx} = \frac{Q}{P} = \frac{e^y}{e^x}.$$

Solving, we get $e^{-y} - e^{-x} =$ a constant. Thus, $w = e^{-y} - e^{-x}$. Now, we choose z such that the Jacobian $w_x z_y - w_y z_x \neq 0$. For instance, $z = x$ is one such choice.

Example

Then

$$\begin{aligned}
u_x &= e^{-x}u_w + u_z \\
u_y &= -e^{-y}u_w \\
u_{xx} &= e^{-2x}u_{ww} + 2e^{-x}u_{wz} + u_{zz} - e^{-x}u_w \\
u_{yy} &= e^{-2y}u_{ww} + e^{-y}u_w \\
u_{xy} &= -e^{-y}(e^{-x}u_{ww} - u_{wz})
\end{aligned}$$

Substituting into the given PDE, we get

$$e^x e^{-y} u_{zz} = (e^{-y} - e^{-x})u_w$$

Replacing x, y in terms of w, z gives

$$u_{zz} = \frac{w}{1 + we^z} u_w.$$

In the elliptic case, $Q^2 - PR < 0$, we have no real characteristics. We choose w, z to be the real and imaginary part of the solution of the characteristic equation.

Example

Let us reduce the PDE $x^2 u_{xx} + y^2 u_{yy} = 0$ given in the region $\{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ to its canonical form. Note that $P = x^2$, $Q = 0$, $R = y^2$ and $Q^2 - PR = -x^2y^2 < 0$. The PDE is elliptic. Solving the characteristic equation

$$\frac{dy}{dx} = \pm \frac{\imath y}{x}$$

we get $\ln x \pm \imath \ln y = c$. Let $w = \ln x$ and $z = \ln y$. Then

$$\begin{aligned}
u_x &= u_w/x, \quad u_y = u_z/y \\
u_{xx} &= -u_w/x^2 + u_{ww}/x^2 \\
u_{yy} &= -u_z/y^2 + u_{zz}/y^2
\end{aligned}$$

Substituting into the PDE, we get $u_{ww} + u_{zz} = u_w + u_z$.

Solving a First Order Quasilinear

- Consider the quasilinear PDE $\mathbf{a}(x,u)\cdot\nabla u - f(x,u) = 0$ in a domain $\Omega \subset \mathbb{R}^n$.
- Solving for the unknown $u : \Omega \to \mathbb{R}$ is equivalent to determining the surface S in $\mathbb{R}^{n+1}$ given by

$$S = \{(x,z) \in \Omega \times \mathbb{R} \mid u(x) - z = 0\}.$$

- The equation of the surface S is given by $\{\phi(x,z) := u(x) - z = 0\}$.
- The normal vector to S is given by $\nabla_{(x,z)}\phi = (\nabla u(x), -1)$.
- But using the PDE satisfied by u, we know that

$$(\mathbf{a}(x,u(x)), f(x,u(x))) \cdot (\nabla u(x), -1) = 0.$$

- Thus, the data vector field $V(x,z) := (\mathbf{a}(x,z), f(x,z)) \in \mathbb{R}^{n+1}$ is perpendicular to the normal of S at every point of S.

Integral Surface

The coefficient vector V must lie on the tangent plane of S, at each of its point. Hence, we seek a surface S such that the given coefficient field V is tangential to S, at every point of S.

Definition

A smooth curve in $\mathbb{R}^n$ is said to be an *integral curve* w.r.t a given vector field, if the vector field is tangential to the curve at each of its point. Further, a smooth surface in $\mathbb{R}^n$ is said to be an *integral surface* w.r.t a given vector field, if the vector field is tangential to the surface at each of its point.

Thus, finding the unknown u is equivalent to finding the integral surface corresponding to the data vector field $V(x,z)) = (\mathbf{a}(x,z), f(x,z))$. An integral surface w.r.t V is an union[1] of integral curves w.r.t V.

[1]the union is in the sense that every point in the integral surface belongs to exactly one characteristic

If $\Gamma = \{x(s), z(s)\}$ is an integral curve corresponding to V lying on the solution surface S, i.e. V is tangential to Γ at all its points, then the curve is described by the system ODEs,

$$\frac{dx}{ds} = \mathbf{a}(x(s), z(s)) \text{ and } \frac{dz}{ds} = f(x(s), z(s)).$$

The $n+1$ ODEs obtained are called *characteristic equations*. The method of characteristics converts a first order PDE to a system of ODE whose solution describes the integral curves.

Example

We shall now compute the solution of the Cauchy problem

$$\begin{cases} u_t + au_x &= 0 \qquad x \in \mathbb{R} \text{ and } t \in (0, \infty) \\ u(x,0) &= u_0(x) \quad x \in \mathbb{R}. \end{cases} \tag{2.2}$$

where $u_0 : \mathbb{R} \to \mathbb{R}$ is a given smooth function. We now check for non-characteristic property of Γ. Note that $\Gamma \equiv \{(x,0)\}$, the x-axis of xt-plane, is the (boundary) curve on which the value of u is given. Thus, $(\Gamma, u_0) = \{(x, 0, u_0(x))\}$ is the known curve on the solution surface of u. The curve Γ is given by the equation $\{t = 0\}$ and, hence, its normal is $(0,1)$. Γ is non-characteristic, because $(a,1)\cdot(0,1) = 1 \neq 0$. The characteristic equations are:

$$\frac{dx(r,s)}{ds} = a, \quad \frac{dt(r,s)}{ds} = 1, \text{ and } \frac{dz(r,s)}{ds} = 0$$

with initial conditions $x(r,0) = r$, $t(r,0) = 0$ and $z(r,0) = u_0(r)$.

Example

Solving the ODE's, we get

$$x(r,s) = as + c_1(r), \quad t(r,s) = s + c_2(r)$$

and $z(r,s) = c_3(r)$ with initial conditions

$$x(r,0) = c_1(r) = r$$

$$t(r,0) = c_2(r) = 0, \text{ and } z(r,0) = c_3(r) = u_0(r).$$

Therefore,

$$x(r,s) = as + r, \quad t(r,s) = s, \text{ and } z(r,s) = u_0(r).$$

We solve for r, s in terms of x, t and set $u(x,t) = z(r(x,t), s(x,t))$.

$$r(x,t) = x - at \text{ and } s(x,t) = t.$$

Therefore, $u(x,t) = z(r,s) = u_0(r) = u_0(x - at)$.

Cauchy Data on Inflow Boundary

- Consider the linear transport equation

$$u_t + au_x = 0, \text{ in } \Omega := (0,\infty) \times (0,\infty)$$

  and the constant $a \in \mathbb{R}$ is given with $u(x,t) = u_0(x - at)$.
- Note that if $a < 0$ then the equation is solvable in the entire domain if the Cauchy data u_0 is prescribed on $(0,\infty) \times \{0\}$.
- However, when $a > 0$, the Cauchy data has to be prescribed on $\{0\} \times (0,\infty) \cup (0,\infty) \times \{0\}$.
- In other words, the Cauchy data has to be prescribed on that part of boundary where the projected characteristic curves are 'inflow' or the region of entry.

- Consider the linear transport equation

$$u_t + a u_x = 0, \text{ in } \Omega := (0, L) \times (0, \infty)$$

where both $L > 0$ and $a \in \mathbb{R}$ are given with solution $u(x, t) = u_0(x - at)$.
- The boundary of Ω is $\Gamma := \{(0, t) \mid t > 0\} \cup \{(0, L) \times \{0\}\} \cup \{(L, t) \mid t > 0\}$.
- If $a > 0$ then the projected characteristics curves inflow in to a subset of Γ and, hence, u_0 should be prescribed on the subset $\{(0, t) \mid t > 0\} \cup \{(0, L) \times \{0\}\}$ of Γ.
- However, if $a < 0$ then it is enough to prescribe u_0 on the the subset $\{(0, L) \times \{0\}\} \cup \{(L, t) \mid t > 0\}$ of Γ.

Burgers' equation

- Consider the quasilinear Cauchy problem called the *Burgers' equation*

$$\begin{cases} u_t(x,t) + u(x,t)u_x(x,t) &= 0 \quad \text{in } \mathbb{R} \times (0, \infty) \\ u(x, 0) &= u_0(x) \quad \text{on } \mathbb{R} \times \{0\}. \end{cases}$$

- We first check for non-characteristic property of Γ. Note that $(\Gamma, u_0) = \{(x, 0, u_0(x))\}$ is the known curve on the solution surface of u. We parametrize the curve Γ with r-variable, i.e., $\Gamma = \{(r, 0)\}$. Γ is non-characteristic, because $(u, 1) \cdot (0, 1) = 1 \neq 0$.
- The characteristic equations are:

$$\frac{dx(r,s)}{ds} = z, \quad \frac{dt(r,s)}{ds} = 1, \text{ and } \frac{dz(r,s)}{ds} = 0$$

with initial conditions,

$$x(r, 0) = r, \quad t(r, 0) = 0, \text{ and } z(r, 0) = u_0(r).$$

- Solving the ODE corresponding to z, we get $z(r,s) = c_3(r)$ with initial conditions $z(r,0) = c_3(r) = u_0(r)$. Thus, $z(r,s) = u_0(r)$.
- Using this in the ODE of x, we get

$$\frac{dx(r,s)}{ds} = u_0(r).$$

Solving the ODE's, we get

$$x(r,s) = u_0(r)s + c_1(r), \quad t(r,s) = s + c_2(r)$$

with initial conditions

$$x(r,0) = c_1(r) = r \text{ and } t(r,0) = c_2(r) = 0.$$

Therefore,

$$x(r,s) = u_0(r)s + r, \text{ and } t(r,s) = s.$$

- Solving r and s, in terms of x, t and z, we get $s = t$ and $r = x - zt$. Therefore, $u(x,t) = u_0(x - tu)$ is the solution in the implicit form.

- Observe that the projected characteristic curves are given by $x = u_0(r)t + r$ passing through $(r,0) \in \Gamma$ and u is constant along these curves.
- Thus, in contrast to the transport equation, the slope of the projected characteristic curve is $\frac{1}{u_0(r)}$ which is not fixed and depends on the initial condition.
- If the Cauchy data u_0 is such that, for $r_1 < r_2$, $u_0(r_1) > u_0(r_2)$ then the characteristic curves passing through r_1 and r_2 will necessarily intersect.
- The slope of line passing through $(r_2,0)$ is bigger than the one passing through $(r_1,0)$. The characteristic curves will necessarily intersect at some point (x_0,t_0).
- This situation leads to a multi-valued solution because

$$u(x_0,t_0) = u(r_2,0) = u_0(r_2) < u_0(r_1) = u(r_1,0) = u(x_0,t_0).$$

- Thus, even if the Cauchy data is a 'smooth' decreasing initial data one may not be able to find a solution for all time t.

Definition

Let L denote a linear differential operator and $I \subset \mathbb{R}$. Then we say $Ly(x) = \lambda y(x)$ on I is an *eigenvalue problem* (EVP) corresponding to L when both λ and $y : I \to \mathbb{R}$ are unknown.

Example

For instance, if $L = \frac{-d^2}{dx^2}$ then its corresponding eigenvalue problem is $-y'' = \lambda y$.

If $\lambda \in \mathbb{R}$ is fixed then one can obtain a general solution. But, in an EVP[2] we need to find all $\lambda \in \mathbb{R}$ for which the given ODE is solvable. Note that $y \equiv 0$ is a *trivial* solution, for all $\lambda \in \mathbb{R}$.

[2] compare an EVP with the notion of diagonalisation of matrices from Linear Algebra

Definition

A $\lambda \in \mathbb{R}$, for which the EVP corresponding to L admits a non-trivial solution y_λ is called an *eigenvalue* of the operator L and y_λ is said to be an *eigen function* corresponding to λ. The set of all eigenvalues of L is called the *spectrum* of L.

Exercise

Show that any second order ODE of the form

$$y'' + P(x)y' + Q(x)y(x) = R(x)$$

can be written in the form

$$\frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y(x) = r(x).$$

(Find p, q and r in terms of P, Q and R).

Given a finite interval $(a, b) \subset \mathbb{R}$, the Sturm-Liouville (S-L) problem is given as

$$\begin{cases} \frac{d}{dx}\left(p(x)\frac{dy}{dx}\right) + q(x)y + \lambda r(x)y &= 0 \quad x \in (a, b) \\ c_1 y(a) + c_2 y'(a) &= 0 \quad c_1^2 + c_2^2 > 0 \\ d_1 y(b) + d_2 y'(b) &= 0 \quad d_1^2 + d_2^2 > 0. \end{cases} \tag{3.1}$$

The function $y(x)$ and λ are unknown quantities. The pair of boundary conditions given above is called *separated*. The boundary conditions corresponds to the end-point a and b, respectively. Note that both c_1 and c_2 cannot be zero simultaneously and, similar condition on d_1 and d_2.

Definition

The Sturm-Liouville problem with separated boundary conditions is said to be *regular* if:

(a) $p, p', q, r : [a, b] \to \mathbb{R}$ are continuous functions

(b) $p(x) > 0$ and $r(x) > 0$ for $x \in [a, b]$.

We say the S-L problem is *singular* if either the interval (a, b) is unbounded or one (or both) of the regularity condition given above fails. We say the S-L problem is *periodic* if $p(a) = p(b)$ and the separated boundary conditions are replaced with the *periodic* boundary condition $y(a) = y(b)$ and $y'(a) = y'(b)$.

Examples of regular S-L problem

Consider the problem $-y''(x) = \lambda y(x)$ in $x \in (0, a)$ with boundary conditions:

Example

(a) $y(0) = y(a) = 0$. We have chosen $c_1 = d_1 = 1$ and $c_2 = d_2 = 0$. Also, $q \equiv 0$ and $p \equiv r \equiv 1$.

(b) $y'(0) = y'(a) = 0$. We have chosen $c_1 = d_1 = 0$ and $c_2 = d_2 = 1$. Also, $q \equiv 0$ and $p \equiv r \equiv 1$.

(c)
$$\begin{cases} y'(0) &= 0 \\ cy(a) + y'(a) &= 0, \end{cases}$$
where $c > 0$ is a constant.

Example

$$\begin{cases} -\left(x^2 y'(x)\right)' &= \lambda y(x) \quad x \in (1, a) \\ y(1) &= 0 \\ y(a) &= 0, \end{cases}$$

where $p(x) = x^2$, $q \equiv 0$ and $r \equiv 1$.

Remark

In a singular Sturm-Liouville problem, the boundary condition at an (or both) end(s) is dropped if p vanishes in (or both) the corresponding end(s). This is because when p vanishes, the equation at that point is no longer second order. Note that dropping a boundary condition corresponding to a end-point is equivalent to taking both constants zero (for instance, $c_1 = c_2 = 0$, in case of left end-point).

Example

Examples of singular S-L problem:

(a) For each $n = 0, 1, 2, \ldots$, consider the Bessel's equation

$$\begin{cases} -(xy'(x))' &= \left(-\frac{n^2}{x} + \lambda x\right) y(x) \quad x \in (0, a) \\ y(a) &= 0, \end{cases}$$

where $p(x) = r(x) = x$, $q(x) = -n^2/x$. This equation is not regular because $p(0) = r(0) = 0$ and q is not continuous in the closed interval $[0, a]$, since $q(x) \to -\infty$ as $x \to 0$. Note that there is no boundary condition corresponding to 0.

(b) The Legendre equation, $-\left[(1 - x^2)y'(x)\right]' = \lambda y(x)$ with $x \in (-1, 1)$ with no boundary condition. Here $p(x) = 1 - x^2$, $q \equiv 0$ and $r \equiv 1$. This equation is not regular because $p(-1) = p(1) = 0$. Note that there are no boundary conditions because p vanishes at both the end-points.

Example

Examples of periodic S-L problem:

$$\begin{cases} -y''(x) &= \lambda y(x) \quad x \in (-\pi, \pi) \\ y(-\pi) &= y(\pi) \\ y'(-\pi) &= y'(\pi). \end{cases}$$

Spectral Results

We shall now state without proof the spectral theorem for regular S-L problem. Our aim, in this course, is to check the validity of the theorem through some examples.

Theorem

For a regular S-L problem, there exists an increasing sequence of eigenvalues $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_k < \ldots$ *with* $\lambda_k \to \infty$*, as* $k \to \infty$*.*

Example

Consider the boundary value problem,

$$\begin{cases} y'' + \lambda y &= 0 \quad x \in (0, a) \\ y(0) = y(a) &= 0. \end{cases}$$

Example

This is a second order ODE with constant coeffcients. Its characteristic equation is $m^2 + \lambda = 0$. Solving for m, we get $m = \pm\sqrt{-\lambda}$.
Note that the λ can be either zero, positive or negative. If $\lambda = 0$, then $y'' = 0$ and the general solution is $y(x) = \alpha x + \beta$, for some constants α and β. Since $y(0) = y(a) = 0$ and $a \neq 0$, we get $\alpha = \beta = 0$. Thus, we have no non-trivial solution corresponding to $\lambda = 0$.
If $\lambda < 0$, then $\omega = -\lambda > 0$. Hence $y(x) = \alpha e^{\sqrt{\omega}x} + \beta e^{-\sqrt{\omega}x}$. Using the boundary condition $y(0) = y(a) = 0$, we get $\alpha = \beta = 0$ and hence we have no non-trivial solution corresponding to negative λ's.
If $\lambda > 0$, then $m = \pm i\sqrt{\lambda}$ and $y(x) = \alpha\cos(\sqrt{\lambda}x) + \beta\sin(\sqrt{\lambda}x)$. Using the boundary condition $y(0) = 0$, we get $\alpha = 0$ and $y(x) = \beta\sin(\sqrt{\lambda}x)$. Using $y(a) = 0$ (and $\beta = 0$ yields trivial solution), we assume $\sin(\sqrt{\lambda}a) = 0$. Thus, $\lambda = (k\pi/a)^2$ for each non-zero $k \in \mathbb{N}$ (since $\lambda > 0$).

Example

Hence, for each $k \in \mathbb{N}$, there is a solution (y_k, λ_k) with

$$y_k(x) = \sin\left(\frac{k\pi x}{a}\right),$$

and $\lambda_k = (k\pi/a)^2$. Notice the following properties of the eigenvalues λ_k and eigen functions y_k

(i) We have discrete set of λ's such that $0 < \lambda_1 < \lambda_2 < \lambda_3 < \ldots$ and $\lambda_k \to \infty$, as $k \to \infty$.
(ii) The eigen functions y_λ corresponding to λ form a subspace of dimension one.

In particular, in the above example, when $a = \pi$ the eigenvalues, for each $k \in \mathbb{N}$, are (y_k, λ_k) where $y_k(x) = \sin(kx)$ and $\lambda_k = k^2$.

Spectral result for Periodic S-L

Theorem

For a periodic S-L problem, there exists an increasing sequence of eigenvalues $0 \leq \lambda_1 < \lambda_2 < \lambda_3 < \ldots < \lambda_k < \ldots$ with $\lambda_k \to \infty$, as $k \to \infty$. Moreover, $W_1 = W_{\lambda_1}$, the eigen space corresponding to the first eigen value is one dimensional.

Example

Consider the boundary value problem,

$$\begin{cases} y'' + \lambda y &= 0 \qquad \text{in } (-\pi, \pi) \\ y(-\pi) &= y(\pi) \\ y'(-\pi) &= y'(\pi). \end{cases}$$

The characteristic equation is $m^2 + \lambda = 0$. Solving for m, we get $m = \pm\sqrt{-\lambda}$. Note that the λ can be either zero, positive or negative.

Example

If $\lambda = 0$, then $y'' = 0$ and the general solution is $y(x) = \alpha x + \beta$, for some constants α and β. Since $y(-\pi) = y(\pi)$, we get $\alpha = 0$. Thus, for $\lambda = 0$, $y \equiv$ a constant is the only non-trivial solution.
If $\lambda < 0$, then $\omega = -\lambda > 0$. Hence $y(x) = \alpha e^{\sqrt{\omega}x} + \beta e^{-\sqrt{\omega}x}$. Using the boundary condition $y(-\pi) = y(\pi)$, we get $\alpha = \beta$ and using the other boundary condition, we get $\alpha = \beta = 0$. Hence we have no non-trivial solution corresponding to negative λ's.
If $\lambda > 0$, then $m = \pm\imath\sqrt{\lambda}$ and $y(x) = \alpha\cos(\sqrt{\lambda}x) + \beta\sin(\sqrt{\lambda}x)$. Using the boundary condition, we get

$$\alpha\cos(-\sqrt{\lambda}\pi) + \beta\sin(-\sqrt{\lambda}\pi) = \alpha\cos(\sqrt{\lambda}\pi) + \beta\sin(\sqrt{\lambda}\pi)$$

and

$$-\alpha\sin(-\sqrt{\lambda}\pi) + \beta\cos(-\sqrt{\lambda}\pi) = -\alpha\sin(\sqrt{\lambda}\pi) + \beta\cos(\sqrt{\lambda}\pi).$$

Thus, $\beta\sin(\sqrt{\lambda}\pi) = \alpha\sin(\sqrt{\lambda}\pi) = 0$.

Example

For a non-trivial solution, we must have $\sin(\sqrt{\lambda}\pi) = 0$. Thus, $\lambda = k^2$ for each non-zero $k \in \mathbb{N}$ (since $\lambda > 0$).
Hence, for each $k \in \mathbb{N} \cup \{0\}$, there is a solution (y_k, λ_k) with

$$y_k(x) = \alpha_k \ \cos kx + \beta_k \sin kx,$$

and $\lambda_k = k^2$.

Singular Sturm-Liouville Problem

Singular S-L, in general, have continuous spectrum. However, the examples we presented viz. Bessel's equation and Legendre equation have a discrete spectrum, similar to the regular S-L problem. Consider the Legendre equation

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \lambda y = 0 \quad \text{for } x \in [-1, 1].$$

Note that, equivalently, we have the form

$$(1-x^2)y'' - 2xy' + \lambda y = 0 \quad \text{for } x \in [-1, 1].$$

The function $p(x) = 1 - x^2$ vanishes at the endpoints $x = \pm 1$.

Definition

A point x_0 is a *singular* point of

$$y''(x) + P(x)y'(x) + Q(x)y(x) = R(x)$$

if either P or Q (or both) diverges as $x \to x_0$. A *singular point* x_0 is said to be *regular* if $(x - x_0)P(x)$ and $(x - x_0)^2 Q(x)$ tends to finite value as $x \to x_0$.

The end points $x = \pm 1$ are *regular singular* point. The coefficients $P(x) = \frac{-2x}{1-x^2}$ and $Q(x) = \frac{\lambda}{1-x^2}$ are analytic at $x = 0$, with radius of convergence 1. We look for power series form of solutions $y(x) = \sum_{k=0}^{\infty} a_k x^k$. Differentiating (twice) the series term by term, substituting in the Legendre equation and equating like powers of x, we get $a_2 = \frac{-\lambda a_0}{2}$, $a_3 = \frac{(2-\lambda)a_1}{6}$ and for $k \geq 2$,

$$a_{k+2} = \frac{(k(k+1) - \lambda)a_k}{(k+2)(k+1)}.$$

Thus, the constants a_0 and a_1 can be fixed arbitrarily and the remaining constants are defined as per the above relation. For instance, if $a_1 = 0$, we get the non-trivial solution of the Legendre equation as

$$y_1 = a_0 + \sum_{k=1}^{\infty} a_{2k} x^{2k}$$

and if $a_0 = 0$, we get the non-trivial solution as

$$y_2 = a_1 x + \sum_{k=1}^{\infty} a_{2k+1} x^{2k+1},$$

provided the series converge. Note from the recurrence relation that if a coefficient is zero at some stage, then every alternate coefficient, subsequently, is zero. Thus, there are two possibilities of convergence here:

(i) the series terminates after finite stage to become a polynomial
(ii) the series does not terminate, but converges.

Suppose the series does not terminate, say for instance, in y_1. Then $a_{2k} \neq 0$, for all k. Consider the ratio

$$\lim_{k\to\infty} \left| \frac{a_{2(k+1)}x^{2(k+1)}}{a_{2k}x^{2k}} \right| = \lim_{k\to\infty} \left| \frac{2k(2k+1)x^2}{(2k+2)(2k+1)} \right| = \lim_{k\to\infty} \left| \frac{2kx^2}{(2k+2)} \right| = x^2.$$

The term involving λ tends to zero. Therefore, by ratio test, y_1 converges in $x^2 < 1$ and diverges in $x^2 > 1$. Also, it can be shown that when $x^2 = 1$ the series diverges (beyond the scope of this course).
Since, Legendre equation is a singular S-L problem, we try to find solution y such that y and its derivative y' are continuous in the closed interval $[-1, 1]$. Thus, the only such possible solutions will be terminating series becoming polynomials. Recall that, for $k \geq 2$,

$$a_{k+2} = \frac{(k(k+1) - \lambda)a_k}{(k+2)(k+1)}.$$

Hence, for any $n \geq 2$, if $\lambda = n(n+1)$, then $a_{n+2} = 0$ and hence every alternate term is zero. Also, if $\lambda = 1(1+1) = 2$, then $a_3 = 0$. If $\lambda = 0(0+1) = 0$, then $a_2 = 0$.

Thus, for each $n \in \mathbb{N} \cup \{0\}$, we have $\lambda_n = n(n+1)$ and one of the solution y_1 or y_2 is a polynomial. Thus, for each $n \in \mathbb{N} \cup \{0\}$, we have the eigenvalue $\lambda_n = n(n+1)$ and the Legendre polynomial P_n of degree n which is a solution to the Legendre equation.

$$P_n(x) = \begin{cases} y_1(x) & n \text{ is even} \\ y_2(x) & n \text{ is odd.} \end{cases}$$

EVP of Bessel's Operator

Consider the EVP, for each fixed $n = 0, 1, 2, \ldots$,

$$\begin{cases} -(xy'(x))' &= \left(-\frac{n^2}{x} + \lambda x\right) y(x) \quad x \in (0, a) \\ y(a) &= 0. \end{cases}$$

As before, since this is a singular S-L problem we shall look for solutions y such that y and its derivative y' are continuous in the closed interval $[0, a]$. We shall assume that the eigenvalues are all real[3]! Thus, λ may be zero, positive or negative.
When $\lambda = 0$, the given ODE reduces to the Cauchy-Euler form

$$-\left(xy'(x)\right)' + \frac{n^2}{x} y(x) = 0$$

or equivalently,

$$x^2 y''(x) + xy'(x) - n^2 y(x) = 0.$$

[3] needs proof

The above second order ODE with variable coefficients can be converted to an ODE with constant coefficients by the substitution $x = e^s$ (or $s = \ln x$). Then, by chain rule,

$$y' = \frac{dy}{dx} = \frac{dy}{ds}\frac{ds}{dx} = e^{-s}\frac{dy}{ds}$$

and

$$y'' = e^{-s}\frac{dy'}{ds} = e^{-s}\frac{d}{ds}\left(e^{-s}\frac{dy}{ds}\right) = e^{-2s}\left(\frac{d^2y}{ds^2} - \frac{dy}{ds}\right).$$

Therefore,

$$y''(s) - n^2 y(s) = 0,$$

where y is now a function of the new variable s. For $n = 0$, the general solution is $y(s) = \alpha s + \beta$, for some arbitrary constants. Thus, $y(x) = \alpha \ln x + \beta$. The requirement that both y and y' are continuous on $[0, a]$ forces $\alpha = 0$. Thus, $y(x) = \beta$. But $y(a) = 0$ and hence $\beta = 0$, yielding the trivial solution.

Now, let $n > 0$ be positive integers. Then the general solution is $y(s) = \alpha e^{ns} + \beta e^{-ns}$. Consequently, $y(x) = \alpha x^n + \beta x^{-n}$. Since y and y' has to be continuous on $[0, a]$, $\beta = 0$. Thus, $y(x) = \alpha x^n$. Now, using the boundary condition $y(a) = 0$, we get $\alpha = 0$ yielding the trivial solution. Therefore, $\lambda = 0$ is not an eigenvalue for all $n = 0, 1, 2, \ldots$.
When $\lambda > 0$, the given ODE reduces to

$$x^2 y''(x) + xy'(x) + (\lambda x^2 - n^2)y(x) = 0.$$

Using the change of variable $s^2 = \lambda x^2$, we get $y'(x) = \sqrt{\lambda} y'(s)$ and $y''(x) = \lambda y''(s)$. Then the given ODE is transformed into the Bessel's equation

$$s^2 y''(s) + sy'(s) + (s^2 - n^2)y(s) = 0.$$

Using the power series form of solution, we know that the general solution of the Bessel's equation is $y(s) = \alpha J_n(s) + \beta Y_n(s)$, where J_n and Y_n are the Bessel functions of first and second kind, respectively. Therefore, $y(x) = \alpha J_n(\sqrt{\lambda} x) + \beta Y_n(\sqrt{\lambda} x)$.

The continuity assumptions of y and y' force that $\beta = 0$, because $Y_n(\sqrt{\lambda} x)$ is discontinuous at $x = 0$. Thus, $y(x) = \alpha J_n(\sqrt{\lambda} x)$. Using the boundary condition $y(a) = 0$, we get $J_n(\sqrt{\lambda} a) = 0$.

Theorem

For each non-negative integer n, J_n has infinitely many positive roots.

For each $n \in \mathbb{N} \cup \{0\}$, let z_{nm} be the m-th zero of J_n, $m \in \mathbb{N}$. Hence $\sqrt{\lambda} a = z_{nm}$ and so $\lambda_{nm} = z_{nm}^2 / a^2$ and the corresponding eigen functions are $y_{nm}(x) = J_n(z_{nm} x / a)$.
For $\lambda < 0$, there are no eigen values. Observing this fact is beyond the scope of this course, hence we assume this fact.

Orthogonality of Eigen Functions

Observe that for a regular S-L problem the differential operator can be written as

$$L = \frac{-1}{r(x)}\frac{d}{dx}\left(p(x)\frac{d}{dx}\right) - \frac{q(x)}{r(x)}.$$

Let V denote the set of all solutions of (3.1). Necessarily, $0 \in V$ and $V \subset C^2(a,b)$. We define the *inner product*[4] $\langle\cdot,\cdot\rangle : V \times V \to \mathbb{R}$ on V as,

$$\langle f, g\rangle := \int_a^b r(x)f(x)g(x)\,dx.$$

[4] a generalisation of the usual scalar product of vectors

Definition

We say two functions f and g are *perpendicular* or *orthogonal* with weight r if $\langle f, g\rangle = 0$. We say f is of *unit length* if its norm $\|f\| = \sqrt{\langle f, f\rangle} = 1$.

Theorem

With respect to the inner product defined above in V, the eigen functions corresponding to distinct eigenvalues of the S-L problem are orthogonal.

Let us examine the orthogonality of the eigenvectors computed in the examples earlier.

Example

We computed in Example 43 the eigenvalues and eigenvectors of the regular S-L problem,

$$\begin{cases} y'' + \lambda y &= 0 \quad x \in (0, a) \\ y(0) = y(a) &= 0 \end{cases}$$

to be (y_k, λ_k) where

$$y_k(x) = \sin\left(\frac{k\pi x}{a}\right)$$

and $\lambda_k = (k\pi/a)^2$, for each $k \in \mathbb{N}$. For $m, n \in \mathbb{N}$ such that $m \neq n$, we need to check that y_m and y_n are orthogonal. Since $r \equiv 1$, we consider

$$\langle y_m(x), y_n(x) \rangle = \int_0^a \sin\left(\frac{m\pi x}{a}\right) \sin\left(\frac{n\pi x}{a}\right)\, dx$$

Exercise

Show that, for any $n \geq 0$ and m positive integer,

1.
$$\int_{-\pi}^{\pi} \cos nt \cos mt\, dt = \begin{cases} \pi, & \text{for } m = n \\ 0, & \text{for } m \neq n. \end{cases}$$

2.
$$\int_{-\pi}^{\pi} \sin nt \sin mt\, dt = \begin{cases} \pi, & \text{for } m = n \\ 0, & \text{for } m \neq n. \end{cases}$$

3.
$$\int_{-\pi}^{\pi} \sin nt \cos mt\, dt = 0.$$

Consequently, show that $\frac{\cos kt}{\sqrt{\pi}}$ and $\frac{\sin kt}{\sqrt{\pi}}$ are of unit length.

Example

We computed in Example 47 the eigenvalues and eigenvectors of the periodic S-L problem,

$$\begin{cases} y'' + \lambda y &= 0 \qquad \text{in } (-\pi, \pi) \\ y(-\pi) &= y(\pi) \\ y'(-\pi) &= y'(\pi) \end{cases}$$

to be, for each $k \in \mathbb{N} \cup \{0\}$, (y_k, λ_k) where

$$y_k(x) = \alpha_k \cos kx + \beta_k \sin kx,$$

and $\lambda_k = k^2$. Again $r \equiv 1$ and the orthogonality follows from the exercise above.

Example

The orthogonality of Legendre polynomial and Bessel function must have been discussed in your course on ODE. Recall that the Legendre polynomials has the property

$$\int_{-1}^{1} P_m(x)P_n(x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{2}{2n+1}, & \text{if } m = n \end{cases}$$

and the Bessel functions have the property

$$\int_{0}^{1} xJ_n(z_{ni}x)J_n(z_{nj}x)\,dx = \begin{cases} 0, & \text{if } m \neq n \\ \frac{1}{2}[J_{n+1}(z_{ni})]^2, & \text{if } m = n \end{cases}$$

where z_{ni} is the i-th positive zero of the Bessel function (of order n) J_n.

Eigen Function Expansion

Observe that an eigenvector y_k, for any k, can be normalised (unit norm) in its inner-product by dividing y_k by its norm $\|y_k\|$. Thus, $y_k/\|y_k\|$, for any k, is a unit vector. For instance, in view of Exercise 2, $\frac{\cos kt}{\sqrt{\pi}}$ and $\frac{\sin kt}{\sqrt{\pi}}$ are functions of unit length.

Definition

Any given function $f : (a, b) \to \mathbb{R}$ is said to be have the *eigen function expansion* corresponding to the S-L problem (3.1), if

$$f(x) \approx \sum_{k=0}^{\infty} a_k y_k,$$

for some constants a_k and y_k are the normalised eigenvectors corresponding to (3.1).

We are using the "$\approx$" symbol to highlight the fact that the issue of convergence of the series is ignored.

- If the eigenvectors (or eigen functions) y_k involves only sin or cos terms, as in regular S-L problem (cf. Example 43), then the series is called *Fourier Sine* or *Fourier Cosine* series.
- If the eigen functions y_k involve both sin and cos, as in periodic S-L problem (cf. Example 47), then the series is called *Fourier series*.
- In the case of the eigenfunctions being Legendre polynomial or Bessel function, we call it *Fourier-Legendre* or *Fourier-Bessel* series, respectively.

Periodic Functions

We isolate the properties of the trigonometric functions, viz., sin, cos etc.

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *periodic* of period T, if $T > 0$ is the smallest number such that

$$f(t+T) = f(t) \quad \forall t \in \mathbb{R}.$$

Such functions are also called *T-periodic* functions.

Example

The trigonometric functions $\sin t$ and $\cos t$ are 2π-periodic functions, while $\sin 2t$ and $\cos 2t$ are π-periodic functions.

Given a L-periodic real-valued function g on $\mathbb{R}$, one can always construct a T-periodic function as: $f(t) = g(Lt/T)$. For instance, $f(t) = \sin\left(\frac{2\pi t}{T}\right)$ is a T-periodic function.

$$\sin\left(\frac{2\pi(t+T)}{T}\right) = \sin\left(\frac{2\pi t}{T} + 2\pi\right) = \sin\left(\frac{2\pi t}{T}\right).$$

In fact, for any positive integer k, $\sin\left(\frac{2\pi kt}{T}\right)$ and $\cos\left(\frac{2\pi kt}{T}\right)$ are T-periodic functions.

Exercise

If $f : \mathbb{R} \to \mathbb{R}$ is a T-periodic function, then show that

1. (i) $f(t-T) = f(t)$, for all $t \in \mathbb{R}$.
2. (ii) $f(t+kT) = f(t)$, for all $k \in \mathbb{Z}$.
3. (iii) $g(t) = f(\alpha t + \beta)$ is (T/α)-periodic, where $\alpha > 0$ and $\beta \in \mathbb{R}$.

Exercise

Show that for a T-periodic integrable function $f : \mathbb{R} \to \mathbb{R}$,

$$\int_{\alpha}^{\alpha+T} f(t)\,dt = \int_0^T f(t)\,dt \quad \forall \alpha \in \mathbb{R}.$$

Fourier Coefficients and Fourier Series

Without loss of generality, to simplify our computation, let us assume that f is a 2π-periodic (similar idea will work for any T-periodic function) function on $\mathbb{R}$. Suppose that $f : (-\pi, \pi) \to \mathbb{R}$, extended to all of $\mathbb{R}$ as a 2π-periodic function, is such that the infinite series

$$a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt)$$

converges uniformly (note the uniform convergence hypothesis) to f. Then,

$$f(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt). \tag{4.1}$$

and integrating both sides of (4.1), from $-\pi$ to π, we get

$$\begin{aligned}\int_{-\pi}^{\pi} f(t)\,dt &= \int_{-\pi}^{\pi}\left(a_0+\sum_{k=1}^{\infty}(a_k\cos kt+b_k\sin kt)\right)\,dt\\ &= a_0(2\pi)+\int_{-\pi}^{\pi}\left(\sum_{k=1}^{\infty}(a_k\cos kt+b_k\sin kt)\right)\,dt\end{aligned}$$

Since the series converges *uniformly* to f, the interchange of integral and series is possible. Therefore,

$$\int_{-\pi}^{\pi} f(t)\,dt = a_0(2\pi)+\sum_{k=1}^{\infty}\left(\int_{-\pi}^{\pi}(a_k\cos kt+b_k\sin kt)\,dt\right)$$

From Exercise 2, we know that

$$\int_{-\pi}^{\pi}\sin kt\,dt=\int_{-\pi}^{\pi}\cos kt\,dt=0,\quad \forall k\in\mathbb{N}.$$

Hence,

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt.$$

To find the coefficients a_k, for each fixed $k\in\mathbb{N}$, we multiply both sides of (4.1) by $\cos kt$ and integrate from $-\pi$ to π. Consequently,

$$\begin{aligned}\int_{-\pi}^{\pi} f(t)\cos kt\,dt &= a_0\int_{-\pi}^{\pi}\cos kt\,dt\\ &\quad+\sum_{j=1}^{\infty}\int_{-\pi}^{\pi}(a_j\cos jt\cos kt+b_j\sin jt\cos kt)\,dt\\ &= \int_{-\pi}^{\pi} a_k\cos kt\cos kt\,dt=\pi a_k.\end{aligned}$$

Similar argument, after multiplying by $\sin kt$, gives the formula for b_k. Thus, we have derived , for all $k\in\mathbb{N}$,

$$\begin{aligned}a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\cos kt\,dt\\ b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} f(t)\sin kt\,dt\\ a_0 &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t)\,dt.\end{aligned}$$

These are the formulae for *Fourier coefficients* of a 2π-periodic functions f, in terms of f. Similarly, if f is a T-periodic function extended to $\mathbb{R}$, then its *Fourier series* is

$$f(t) = a_0 + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{2\pi kt}{T}\right) + b_k \sin\left(\frac{2\pi kt}{T}\right) \right],$$

where

$$a_k = \frac{2}{T} \int_0^T f(t) \cos\left(\frac{2\pi kt}{T}\right) dt \tag{4.2a}$$

$$b_k = \frac{2}{T} \int_0^T f(t) \sin\left(\frac{2\pi kt}{T}\right) dt \tag{4.2b}$$

$$a_0 = \frac{1}{T} \int_0^T f(t)\, dt. \tag{4.2c}$$

The above discussion motivates us to give the following definition.

Definition

If $f : \mathbb{R} \to \mathbb{R}$ is any T-periodic integrable function then we define the *Fourier coefficients* of f, a_0, a_k and b_k, for all $k \in \mathbb{N}$, by (4.2) and the *Fourier series* of f is given by

$$f(t) \approx a_0 + \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{2\pi kt}{T}\right) + b_k \sin\left(\frac{2\pi kt}{T}\right) \right]. \tag{4.3}$$

Note the use of "$\approx$" symbol in (4.3). This is because we have the following issues once we have the definition of Fourier series of f, viz.,

(a) Will the Fourier series of f always converge?

(b) If it converges, will it converge to f?

(c) If so, is the convergence point-wise or uniform[5].

[5] because our derivation of formulae for Fourier coefficients assumed uniform convergence of the series

Answering these question, in all generality, is beyond the scope of this course. However, we shall state some results later that will get us in to working mode. We now present some simple examples on computing Fourier coefficients of functions.

Example

Consider the constant function $f \equiv c$ on $(-\pi, \pi)$. Then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} c \, dt = c.$$

For each $k \in \mathbb{N}$,

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} c \cos kt \, dt = 0$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} c \sin kt \, dt = 0.$$

Example

Consider the trigonometric function $f(t) = \sin t$ on $(-\pi, \pi)$. Then

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} \sin x \, dt = 0.$$

For each $k \in \mathbb{N}$,

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin t \cos kt \, dt = 0$$

and

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} \sin t \sin kt \, dt = \begin{cases} 0 & k \neq 1 \\ 1 & k = 1. \end{cases}$$

Similarly, for $f(t) = \cos t$ on $(-\pi, \pi)$, all Fourier coefficients are zero, except $a_1 = 1$.

Example

Consider the function $f(t) = t$ on $(-\pi, \pi)$. Then

$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\,dt = 0.$$

For each $k \in \mathbb{N}$,

$$\begin{aligned} a_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} t\cos kt\,dt \\ &= \frac{1}{k\pi}\left[-\int_{-\pi}^{\pi}\sin kt\,dt + (\pi\sin k\pi - (-\pi)\sin k(-\pi))\right] \end{aligned}$$

and hence $a_k = 0$, for all k.

Example

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin kt\,dt \\ &= \frac{1}{k\pi}\left[\int_{-\pi}^{\pi}\cos kt\,dt - (\pi\cos k\pi - (-\pi)\cos k(-\pi))\right] \\ &= \frac{1}{k\pi}\left[0 - \left(\pi(-1)^k + \pi(-1)^k\right)\right] = \frac{(-1)^{k+1}2}{k} \end{aligned}$$

Therefore, t as a 2π-periodic function defined in $(-\pi, \pi)$ has the Fourier series expansion

$$t \approx 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin kt$$

Example

Let us consider the same function $f(t) = t$, as in previous example, but defined on $(0,\pi)$. Viewing this as π-periodic function, we compute

$$a_0 = \frac{1}{\pi}\int_0^\pi t\,dt = \frac{\pi}{2}.$$

For each $k \in \mathbb{N}$,

$$\begin{aligned}
a_k &= \frac{2}{\pi}\int_0^\pi t\cos 2kt\,dt = \frac{2}{2k\pi}\left[-\int_0^\pi \sin 2kt\,dt + (\pi\sin 2k\pi - 0)\right] \\
&= \frac{1}{k\pi}\left[\frac{1}{2k}(\cos 2k\pi - \cos(0))\right] = 0 \quad \text{and} \\
b_k &= \frac{2}{\pi}\int_0^\pi t\sin 2kt\,dt = \frac{2}{2k\pi}\left[\int_0^\pi \cos 2kt\,dt - (\pi\cos 2k\pi - 0)\right] \\
&= \frac{1}{k\pi}\left[\frac{1}{2k}(\sin 2k\pi - \sin(0)) - \pi\right] = \frac{-1}{k}.
\end{aligned}$$

Therefore, t as a π-periodic function defined on $(0,\pi)$ has the Fourier series expansion

$$t \approx \frac{\pi}{2} - \sum_{k=1}^\infty \frac{1}{k}\sin 2kt$$

while t as a 2π-periodic function defined in $(-\pi,\pi)$ has the Fourier series expansion

$$t \approx 2\sum_{k=1}^\infty \frac{(-1)^{k+1}}{k}\sin kt.$$

Note that difference in Fourier expansion of the same function when the periodicity changes.

Theorem (Riemann-Lebesgue Lemma)

Let f be a continuous function in $[-\pi, \pi]$. Show that the Fourier coefficients of f converges to zero, i.e.,

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0.$$

Observe that $|a_k|$ and $|b_k|$ are bounded sequences, since

$$\{|a_k|, |b_k|\} \leq \int_{-\pi}^{\pi} |f(t)|\, dt < +\infty.$$

We need to show that these bounded sequences, in fact, converges to zero. Now set $x = t - \pi/k$ and hence

$$\begin{aligned} b_k &= \int_{-\pi}^{\pi} f(t) \sin kt\, dt = \int_{-\pi-\pi/k}^{\pi-\pi/k} f(x+\pi/k) \sin(kx+\pi)\, dx \\ &= -\int_{-\pi-\pi/k}^{\pi-\pi/k} f(x+\pi/k) \sin kx\, dx. \end{aligned}$$

Therefore, after reassigning x as t,

$$\begin{aligned} 2b_k &= \int_{-\pi}^{\pi} f(t) \sin kt\, dt - \int_{-\pi-\pi/k}^{\pi-\pi/k} f(t+\pi/k) \sin kt\, dt \\ &= -\int_{-\pi-\pi/k}^{-\pi} f(t+\pi/k) \sin kt\, dt \\ &\quad + \int_{-\pi}^{\pi-\pi/k} (f(t) - f(t+\pi/k)) \sin kt\, dt + \int_{\pi-\pi/k}^{\pi} f(t) \sin kt\, dt \\ &= I_1 + I_2 + I_3. \end{aligned}$$

Thus, $|2b_k| \leq |I_1| + |I_2| + |I_3|$. Consider

$$\begin{aligned} |I_3| &= \left| \int_{\pi - \pi/k}^{\pi} f(t) \sin kt \, dt \right| \\ &\leq \int_{\pi - \pi/k}^{\pi} |f(t)| \, dt \\ &\leq \left(\max_{t \in [-\pi, \pi]} |f(t)| \right) \frac{\pi}{k} = \frac{M\pi}{k}. \end{aligned}$$

Similar estimate is also true for I_1. Let us consider,

$$\begin{aligned} |I_2| &= \left| \int_{-\pi}^{\pi - \pi/k} (f(t) - f(t + \pi/k)) \sin kt \, dt \right| \\ &\leq \left(\max_{t \in [-\pi, \pi - \pi/k]} |f(t) - f(t + \pi/k)| \right) \left(2\pi - \frac{\pi}{k} \right) \end{aligned}$$

By the uniform continuity of f on $[-\pi, \pi]$, the maximum will tend to zero as $k \to \infty$. Hence $|b_k| \to 0$. Exactly, similar arguments hold for a_k.

Piecewise Smooth Functions

Definition

A function $f : [a, b] \to \mathbb{R}$ is said to be *piecewise continuously differentiable* if it has a continuous derivative f' in (a, b), except at finitely many points in the interval $[a, b]$ and at each these finite points, the right-hand and left-hand limit for both f and f' exist.

Example

Consider $f : [-1, 1] \to \mathbb{R}$ defined as $f(t) = |t|$ is continuous.It is not differentiable at 0, but it is piecewise continuously differentiable.

Example

Consider the function $f : [-1, 1] \to \mathbb{R}$ defined as

$$f(t) = \begin{cases} -1, & \text{for } -1 < t < 0, \\ 1, & \text{for } 0 < t < 1, \\ 0, & \text{for } t = 0, 1, -1. \end{cases}$$

It is not continuous, but is piecewise continuous. It is also piecewise continuously differentiable.

Exercise (Riemann-Lebesgue Lemma)

Let f be a piecewise continuous function in $[-\pi, \pi]$ such that

$$\int_{-\pi}^{\pi} |f(t)|\, dt < +\infty.$$

Show that the Fourier coefficients of f converges to zero, i.e.,

$$\lim_{k\to\infty} a_k = \lim_{k\to\infty} b_k = 0.$$

Theorem

If f is a T-periodic piecewise continuously differentiable function, then the Fourier series of f converges to $f(t)$, for every t at which f is smooth. Further, at a non-smooth point t_0, the Fourier series of f will converge to the average of the right and left limits of f at t_0.

Corollary

If $f : \mathbb{R} \to \mathbb{R}$ is a continuously differentiable (derivative f' exists and is continuous) T-periodic function, then the Fourier series of f converges to $f(t)$, for every $t \in \mathbb{R}$.

Example

For a given constant $c \neq 0$, consider the piecewise function

$$f(t) = \begin{cases} 0 & \text{if } t \in (-\pi, 0) \\ c & \text{if } t \in (0, \pi). \end{cases}$$

Then,

$$a_0 = \frac{1}{2\pi} \int_0^\pi c \, dt = \frac{c}{2}.$$

For each $k \in \mathbb{N}$,

$$a_k = \frac{1}{\pi} \int_0^\pi c \cos kt \, dt = 0$$

and

$$b_k = \frac{1}{\pi} \int_0^\pi c \sin kt \, dt = \frac{c}{\pi} \left[\frac{1}{k} (-\cos k\pi + \cos(0)) \right] = \frac{c(1 + (-1)^{k+1})}{k\pi}.$$

Example

Therefore,

$$f(t) \approx \frac{c}{2} + \sum_{k=1}^{\infty} \frac{c(1+(-1)^{k+1})}{k\pi} \sin kt.$$

The point $t_0 = 0$ is a non-smooth point of the function f. Note that the right limit of f at $t_0 = 0$ is c and the left limit of f at $t_0 = 0$ is 0. Note that the Fourier series of f ay $t_0 = 0$ converges to $c/2$, the average of c and 0.

Complex Fourier Coefficients

The Fourier series of a 2π-periodic function $f : \mathbb{R} \to \mathbb{R}$ as given in (4.1), can be recast in complex number notation using the formulae

$$\cos t = \frac{e^{\imath t} + e^{-\imath t}}{2}, \quad \sin t = \frac{e^{\imath t} - e^{-\imath t}}{2i}.$$

Note that we can rewrite the Fourier series expansion of f as

$$f(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} (a_k \cos kt + b_k \sin kt)$$

with a factor 2 in denominator of a_0 and make the formulae of the Fourier coefficient having uniform factor.

Thus,

$$\begin{aligned}
f(t) &= \frac{a_0}{2} + \sum_{k=1}^{\infty}(a_k \cos kt + b_k \sin kt) \\
&= \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[\frac{a_k}{2}\left(e^{\imath kt} + e^{-\imath kt}\right) - \frac{\imath b_k}{2}\left(e^{\imath kt} - e^{-\imath kt}\right)\right] \\
f(t) &= \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(\left[\frac{a_k - \imath b_k}{2}\right] e^{\imath kt} + \left[\frac{a_k + \imath b_k}{2}\right] e^{-\imath kt}\right) \\
&= c_0 + \sum_{k=1}^{\infty}\left(c_k e^{\imath kt} + c_{-k} e^{-\imath kt}\right) \\
&= \sum_{k=-\infty}^{\infty} c_k e^{\imath kt}.
\end{aligned}$$

Exercise

Given a 2π-periodic function f such that

$$f(t) = \sum_{k=-\infty}^{\infty} c_k e^{\imath kt}, \tag{4.4}$$

where the convergence is uniform. Show that, for all $k \in \mathbb{Z}$,

$$c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(t) e^{-\imath kt}\, dt.$$

Orthogonality

Consider, for $k \in \mathbb{N}$, the following elements in V

$$e_0(t) = \frac{1}{\sqrt{2\pi}}, \quad e_k(t) = \frac{\cos kt}{\sqrt{\pi}} \text{ and } f_k(t) = \frac{\sin kt}{\sqrt{\pi}}.$$

Example

e_0, e_k and f_k are all of unit length. $\langle e_0, e_k \rangle = 0$ and $\langle e_0, f_k \rangle = 0$. Also, $\langle e_m, e_n \rangle = 0$ and $\langle f_m, f_n \rangle = 0$, for $m \neq n$. Further, $\langle e_m, f_n \rangle = 0$ for all m, n. Check and compare these properties with the standard basis vectors of $\mathbb{R}^n$!

Odd and Even functions

Definition

We say a function $f : \mathbb{R} \to \mathbb{R}$ is *odd* if $f(-t) = -f(t)$ and *even* if $f(-t) = f(t)$.

Example

All constant functions are even functions. For all $k \in \mathbb{N}$, $\sin kt$ are odd functions and $\cos kt$ are even functions.

Exercise

Any odd function is always orthogonal to an even function.

The Fourier series of an odd or even functions will contain only sine or cosine parts, respectively. The reason being that, if f is odd

$$\langle f, 1\rangle = 0 \text{ and } \quad \langle f, \cos kt\rangle = 0$$

and hence $a_0 = 0$ and $a_k = 0$, for all k. If f is even

$$\langle f, \sin kt\rangle = 0$$

and $b_k = 0$, for all k.

Fourier Sine-Cosine Series

Let $f : (0, T) \to \mathbb{R}$ be a piecewise smooth function. To compute the Fourier Sine series of f, we extend f, as an odd function f_o, to $(-T, T)$

$$f_o(t) = \begin{cases} f(t), & \text{for } t \in (0, T) \\ -f(-t) & , \text{for } t \in (-T, 0). \end{cases}$$

Note that f_o is a $2T$-periodic function and is an odd function. Since f_o is odd, the cosine coefficients a_k and the constant term a_0 vanishes in Fourier series expansion of f_o. The restriction of the Fourier series of f_o to f in the interval $(0, T)$ gives the *Fourier sine series* of f. We derive the formulae for Fourier sine coefficient of f.

$$f(t) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{\pi kt}{T}\right) \quad \text{where} \tag{4.5}$$

$$\begin{aligned}
b_k &= \frac{1}{T}\left\langle f_o, \sin\left(\frac{\pi kt}{T}\right)\right\rangle = \frac{1}{T}\int_{-T}^{T} f_o(t)\sin\left(\frac{\pi kt}{T}\right)\,dt \\
&= \frac{1}{T}\left[\int_{-T}^{0} -f(-t)\sin\left(\frac{\pi kt}{T}\right)\,dt + \int_{0}^{T} f(t)\sin\left(\frac{\pi kt}{T}\right)\,dt\right] \\
&= \frac{1}{T}\left[\int_{T}^{0} -f(t)\sin\left(\frac{\pi kt}{T}\right)\,dt + \int_{0}^{T} f(t)\sin\left(\frac{\pi kt}{T}\right)\,dt\right] \\
&= \frac{2}{T}\int_{0}^{T} f(t)\sin\left(\frac{\pi kt}{T}\right)\,dt.
\end{aligned}$$

Example

Let us consider the function $f(t) = t$ on $(0, \pi)$. To compute the Fourier sine series of f, we extend f to $(-\pi, \pi)$ as an odd function $f_o(t) = t$ on $(-\pi, \pi)$. For each $k \in \mathbb{N}$,

$$\begin{aligned}
b_k &= \frac{2}{\pi}\int_0^{\pi} t\sin kt\,dt = \frac{2}{k\pi}\left[\int_0^{\pi}\cos kt\,dt - (\pi\cos k\pi - 0)\right] \\
&= \frac{2}{k\pi}\left[\frac{1}{k}(\sin k\pi - \sin(0)) + \pi(-1)^{k+1}\right] = \frac{(-1)^{k+1}2}{k}.
\end{aligned}$$

Therefore, the Fourier sine series expansion of $f(t) = t$ on $(0, \pi)$ is

$$t \approx 2\sum_{k=1}^{\infty}\frac{(-1)^{k+1}}{k}\sin kt$$

Compare the result with Example 63.

For computing the Fourier cosine series of f, we extend f as an even function to $(-T, T)$,

$$f_e(t) = \begin{cases} f(t), & \text{for } t \in (0, T) \\ f(-t) & , \text{for } t \in (-T, 0). \end{cases}$$

The function f_e is a $2T$-periodic function extended to all of $\mathbb{R}$. The Fourier series of f_e has no sine coefficients, $b_k = 0$ for all k. The restriction of the Fourier series of f_e to f in the interval $(0, T)$ gives the *Fourier cosine series* of f. We derive the formulae for Fourier cosine coefficient of f.

$$f(t) = a_0 + \sum_{k=1}^{\infty} a_k \cos\left(\frac{\pi k t}{T}\right) \tag{4.6}$$

where

$$a_k = \frac{2}{T} \int_0^T f(t) \cos\left(\frac{\pi k t}{T}\right) dt$$

and

$$a_0 = \frac{1}{T} \int_0^T f(t)\, dt.$$

Example

Consider the function $f(t) = t$ on $(0, \pi)$. To compute the Fourier cosine series of f, we extend f to $(-\pi, \pi)$ as an even function $f_e(t) = |t|$ on $(-\pi, \pi)$. Then, $a_0 = \frac{1}{\pi}\int_0^\pi t\,dt = \frac{\pi}{2}$. For each $k \in \mathbb{N}$,

$$\begin{aligned} a_k &= \frac{2}{\pi}\int_0^\pi t\cos kt\,dt = \frac{2}{k\pi}\left[-\int_0^\pi \sin kt\,dt + (\pi\sin k\pi - 0)\right] \\ &= \frac{2}{k\pi}\left[\frac{1}{k}(\cos k\pi - \cos(0))\right] = \frac{2[(-1)^k - 1]}{k^2\pi}. \end{aligned}$$

Therefore, the Fourier cosine series expansion of $f(t) = t$ on $(0, \pi)$ is

$$t \approx \frac{\pi}{2} + 2\sum_{k=1}^{\infty}\frac{(-1)^k - 1}{k^2\pi}\cos kt.$$

Compare the result with the Fourier series of the function $f(t) = |t|$ on $(-\pi, \pi)$.

Fourier Transform and Integral

- We have introduced the notion of Fourier series for *periodic* functions.
- The periodicity was assumed due to the periodicity of sin and cos functions.
- Can we generalise the notion of Fourier series of f, to *non-periodic* functions?
- The answer is a "yes"!
- Note that the periodicity of f is captured by the integer k appearing in the arguments of sin and cos.
- To generalise the notion of Fourier series to non-periodic functions, we shall replace k, a positive integer, with a real number ξ.
- Note that when we replace k with ξ, the sequences a_k, b_k become functions of ξ, $a(\xi)$ and $b(\xi)$ and the discrete sum is replaced by an integral form over $\mathbb{R}$.

Definition

If $f : \mathbb{R} \to \mathbb{R}$ is a piecewise continuous function which vanishes outside a finite interval, then its *Fourier integral* is defined as

$$f(t) = \int_0^\infty \left(a(\xi)\cos\xi t + b(\xi)\sin\xi t\right)\, d\xi,$$

where

$$\begin{aligned} a(\xi) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\cos\xi t\, dt \\ b(\xi) &= \frac{1}{\pi}\int_{-\infty}^{\infty} f(t)\sin\xi t\, dt. \end{aligned}$$

Travelling Wave Technique

- Consider the wave equation $u_{tt} = c^2 u_{xx}$ on $\mathbb{R} \times (0, \infty)$, describing the vibration of an infinite string.
- The equation is hyperbolic and has the two family of characteristics $x \pm ct$= a constant.
- Introduce the new coordinates $w = x + ct$, $z = x - ct$ and set $u(w, z) = u(x, t)$. Thus, we have the following relations, using chain rule:

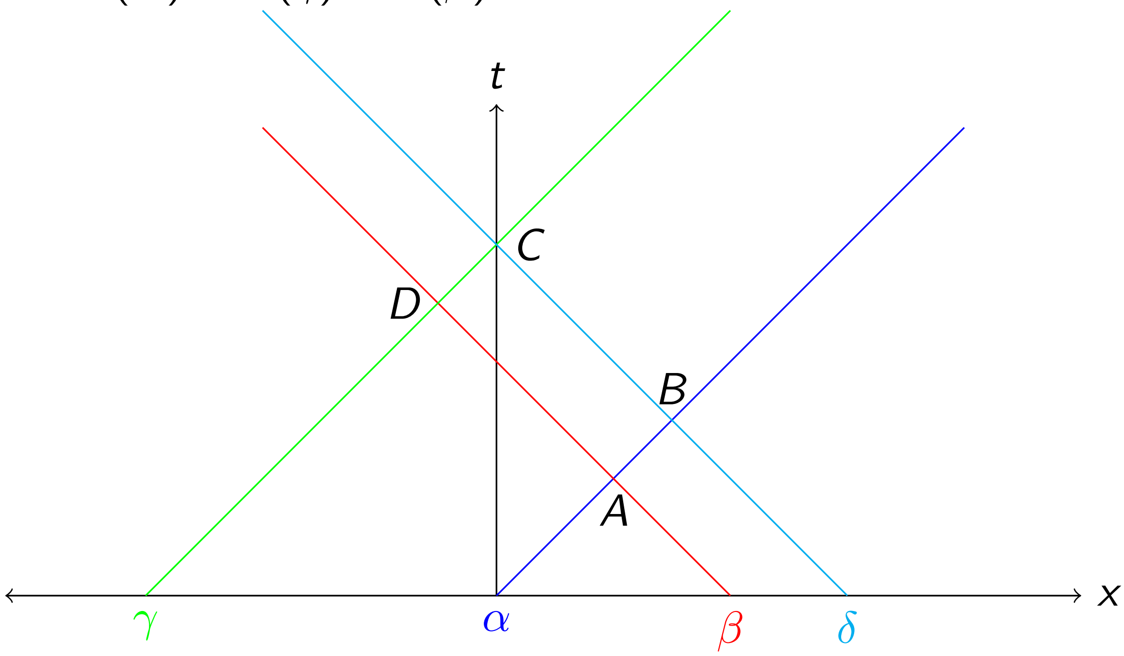
$$\begin{aligned} u_x &= u_w w_x + u_z z_x = u_w + u_z \\ u_t &= u_w w_t + u_z z_t = c(u_w - u_z) \\ u_{xx} &= u_{ww} + 2u_{zw} + u_{zz} \\ u_{tt} &= c^2(u_{ww} - 2u_{zw} + u_{zz}) \end{aligned}$$

- In the new coordinates, the wave equation satisfies $u_{wz} = 0$.

- Integrating twice, we have $u(w,z) = F(w) + G(z)$, for some arbitrary functions F and G. Thus, $u(x,t) = F(x+ct) + G(x-ct)$ is a *general solution* of the wave equation.
- Consider the case where G is chosen to be zero function. Then $u(x,t) = F(x+ct)$ solves the wave equation. At $t = 0$, the solution is simply the graph of F and at $t = t_0$ the solution is the graph of F with origin translated to the left by ct_0.
- Similarly, choosing $F = 0$, we have $u(x,t) = G(x-ct)$ also solves wave equation and at time t is the translation to the right of the graph of G by ct.
- This motivates the name "travelling waves" and "wave equation". The graph of F or G is shifted to left or right, respectively, with a speed of c.

Rectangular Property

For any four points A, B, C and D that form a rectangle bounded by the characteristic curves in $\mathbb{R} \times \mathbb{R}^+$ we have $u(A) + u(C) = u(B) + u(D)$ because $u(A) = F(\alpha) + G(\beta)$, $u(C) = F(\gamma) + G(\delta)$, $u(B) = F(\alpha) + G(\delta)$ and $u(D) = F(\gamma) + G(\beta)$.



D'Alembert's Formula

Theorem

Given $g \in C^2(\mathbb{R})$ and $h \in C^1(\mathbb{R})$, there is a unique C^2 solution u of the Cauchy initial value problem (IVP) of the wave equation,

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) & = 0 & \text{in } \mathbb{R} \times (0,\infty) \\ u(x,0) & = g(x) & \text{in } \mathbb{R} \\ u_t(x,0) & = h(x) & \text{in } \mathbb{R}, \end{cases} \tag{5.1}$$

which is given by the d'Alembert's formula

$$u(x,t) = \frac{1}{2}\left(g(x+ct) + g(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\,dy. \tag{5.2}$$

Proof

The general solution is $u(x,t) = F(x+ct) + G(x-ct)$ with $F, G \in C^2(\mathbb{R})$. Using the initial position we get $F(x) + G(x) = g(x)$. Thus, g should be $C^2(\mathbb{R})$. Now, $u_t(x,t) = c\left(F'(w) - G'(z)\right)$ and putting $t = 0$, we get $F'(x) - G'(x) = \frac{1}{c}h(x)$. Thus, h should be $C^1(\mathbb{R})$. Now solving for F' and G', we get $2F'(x) = g'(x) + h(x)/c$. Similarly, $2G'(x) = g'(x) - h(x)/c$. Integrating both these equations, we get

$$F(x) = \frac{1}{2}\left(g(x) + \frac{1}{c}\int_0^x h(y)\,dy\right) + c_1$$

and

$$G(x) = \frac{1}{2}\left(g(x) - \frac{1}{c}\int_0^x h(y)\,dy\right) + c_2.$$

Since $F(x) + G(x) = g(x)$, we get $c_1 + c_2 = 0$. Therefore, the solution to the wave equation is given by (5.2).

Aliter

Let us derive the d'Alembert's formula in an alternate way. Note that the wave equation can be factored as

$$\left(\frac{\partial}{\partial t} + c\frac{\partial}{\partial x}\right)\left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u = u_{tt} - c^2 u_{xx} = 0.$$

We set $v(x,t) = \left(\frac{\partial}{\partial t} - c\frac{\partial}{\partial x}\right) u(x,t)$ and hence

$$v_t(x,t) + cv_x(x,t) = 0 \quad \text{in } \mathbb{R} \times (0,\infty).$$

Notice that the above first order PDE obtained is in the form of homogeneous linear transport equation, which we have already solved. Hence, for some smooth function ϕ,

$$v(x,t) = \phi(x - ct)$$

and $\phi(x) := v(x,0)$.

Aliter

Using v in the original equation, we get the inhomogeneous transport equation $u_t(x,t) - cu_x(x,t) = \phi(x - ct)$.Recall the formula for inhomogenoeus transport equation

$$u(x,t) = g(x - at) + \int_0^t \phi(x - a(t-s), s)\, ds.$$

Since $u(x,0) = g(x)$ and $a = -c$, in our case the solution reduces to,

$$\begin{aligned} u(x,t) &= g(x+ct) + \int_0^t \phi(x + c(t-s) - cs)\, ds \\ &= g(x+ct) + \int_0^t \phi(x + ct - 2cs)\, ds \\ &= g(x+ct) + \frac{-1}{2c}\int_{x+ct}^{x-ct} \phi(y)\, dy \\ &= g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \phi(y)\, dy. \end{aligned}$$

Aliter

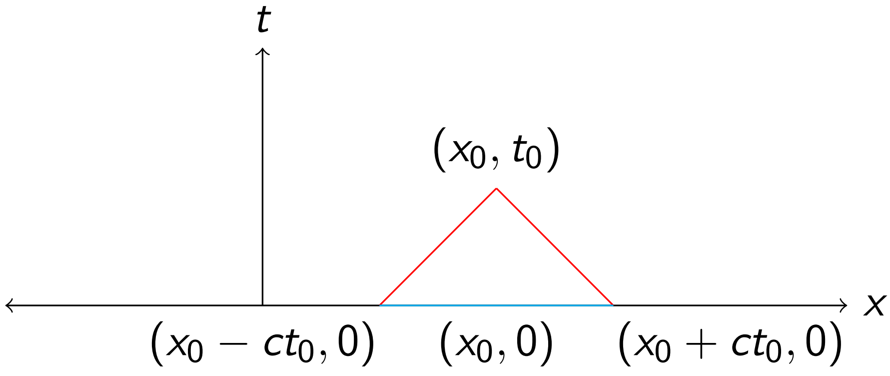
But $\phi(x) = v(x,0) = u_t(x,0) - cu_x(x,0) = h(x) - cg'(x)$ and substituting this in the formula for u, we get

$$\begin{aligned}
u(x,t) &= g(x+ct) + \frac{1}{2c}\int_{x-ct}^{x+ct} \left(h(y) - cg'(y)\right)\, dy \\
&= g(x+ct) + \frac{1}{2}\left(g(x-ct) - g(x+ct)\right) \\
&\quad + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\, dy \\
&= \frac{1}{2}\left(g(x-ct) + g(x+ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(y)\, dy
\end{aligned}$$

A useful observation from the d'Alembert's formula is that the regularity of u is same as the regularity of its initial value g.

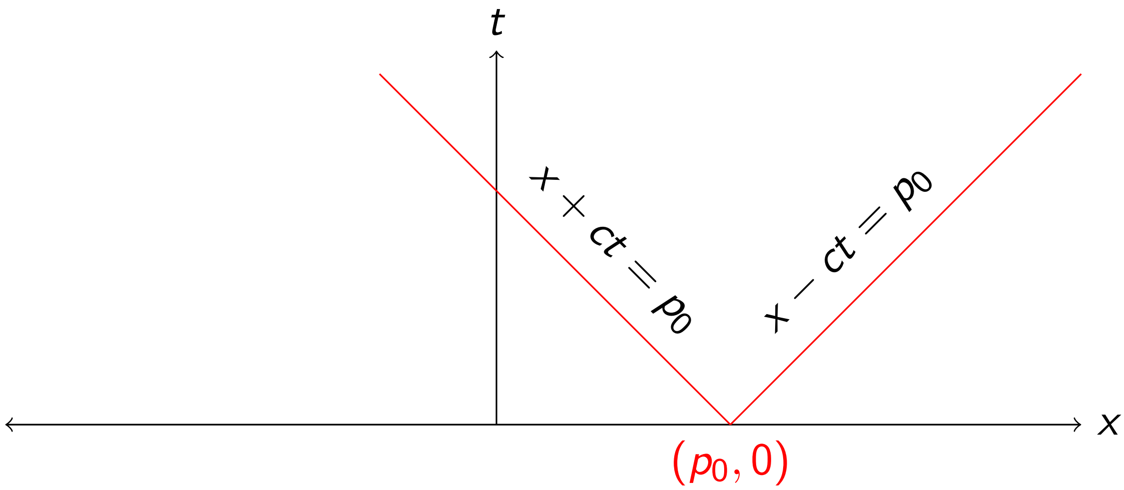
Domain of Dependence

- Note that the value $u(x_0, t_0)$ depends only on the interval $[x_0 - ct_0, x_0 + ct_0]$ because g takes values only on the end-points of this interval
- and h takes values between this interval.
- The closed interval $[x_0 - ct_0, x_0 + ct_0]$ is called the *domain of dependence* of $u(x_0, t_0)$.
- The domain of dependence of (x_0, t_0) is marked on the x-axis by the characteristic curves passing through (x_0, t_0).



Range of Influence

- Given a point $(p_0, 0)$ on the x-axis, what is the region in the $x - t$-plane where the solution u depends on the value of $g(p_0, 0)$ and $h(p_0, 0)$?
- This region turns out to be a cone with vertex at $(p_0, 0)$
- and is called the *range of influence*.
- The range of influence is the region bounded by the characteristic curves passing through $(p_0, 0)$.



Finite Speed of Propagation

- If the initial data g and h are "supported" in the interval $B_{x_0}(R)$
- then the solution u at (x, t) is supported in the region $B_{x_0}(R + ct)$.
- This phenomenon is called the *finite speed of propagation*.

Standing Waves: Separation of Variable

- The *method of separation of variables* was introduced by d'Alembert (1747) and Euler (1748) for the wave equation.
- This technique was also employed by Laplace (1782) and Legendre (1782) while studying the Laplace equation and also by Fourier while studying the heat equation.
- Recall the set-up of the vibrating string given by the equation $u_{tt} = u_{xx}$, we have normalised the constant c.
- Initially at time t, let us say the string has the shape of the graph of v, i.e., $u(x,0) = v(x)$.
- The snapshot of the vibrating string at each time are called the "standing waves".
- The shape of the string at time t_0 can be thought of as some factor (depending on time) of v.
- This observation motivates the idea of "separation of variable", i.e., $u(x,t) = v(x)w(t)$, where $w(t)$ is the factor depending on time, which scales v at time t to fit with the shape of $u(x,t)$.

Finite length Vibrating String

Solve the Cauchy problem

$$\begin{cases} u_{tt}(x,t) - c^2 u_{xx}(x,t) & = 0 \quad \text{in } (0,L) \times (0,\infty) \\ u(x,0) & = g(x) \quad \text{in } [0,L] \\ u_t(x,0) & = h(x) \quad \text{in } [0,L] \\ u(0,t) & = \phi(t) \quad \text{in } (0,\infty) \\ u(L,t) & = \psi(t) \quad \text{in } (0,\infty), \end{cases} \tag{5.3}$$

where ϕ, ψ, g, h satisfies the compatibility condition

$$g(0) = \phi(0), c^2 g''(0) = \phi''(0), h(0) = \phi'(0)$$

and

$$g(L) = \psi(0), c^2 g''(L) = \psi''(0), h(L) = \psi'(0).$$

We are given the initial position $u(x,0) = g(x)$ (at time $t = 0$) and initial velocity of the string at time $t = 0$, $u_t(x,0) = h(x)$ where $g, h : [0,L] \to \mathbb{R}$ are such that $g(0) = g(L) = 0$ and $h(0) = h(L)$. The fact that endpoints are fixed is given by the boundary condition $u(0,t) = u(L,t) = 0$, i.e. $\phi = \psi \equiv 0$.

Let us seek solutions $u(x,t)$ whose variables can be separated, i.e. $u(x,t) = v(x)w(t)$. Differentiating and substituting in the wave equation, we get

$$v(x)w''(t) = c^2 v''(x)w(t).$$

Hence

$$\frac{w''(t)}{c^2 w(t)} = \frac{v''(x)}{v(x)}.$$

Since RHS is a function of x and LHS is a function t, they must be equal a constant, say λ. Thus,

$$\frac{v''(x)}{v(x)} = \frac{w''(t)}{c^2 w(t)} = \lambda.$$

Using the boundary condition $u(0,t) = u(L,t) = 0$, we get

$$v(0)w(t) = v(L)w(t) = 0.$$

If $w \equiv 0$, then $u \equiv 0$ and this cannot be a solution to (5.3). Hence, $w \not\equiv 0$ and $v(0) = v(L) = 0$.

Thus, we need to solve the *eigen value problem* for the second order differential operator.

$$\begin{cases} v''(x) & = \lambda v(x), \quad x \in (0,L) \\ v(0) & = v(L) \quad = 0, \end{cases}$$

Note that the λ can be either zero, positive or negative. If $\lambda = 0$, then $v'' = 0$ and the general solution is $v(x) = \alpha x + \beta$, for some constants α and β. Since $v(0) = 0$, we get $\beta = 0$, and $v(L) = 0$ and $L \neq 0$ implies that $\alpha = 0$. Thus, $v \equiv 0$ and hence $u \equiv 0$. But, this cannot be a solution to (5.3).

If $\lambda > 0$, then $v(x) = \alpha e^{\sqrt{\lambda}x} + \beta e^{-\sqrt{\lambda}x}$. Equivalently,

$$v(x) = c_1 \cosh(\sqrt{\lambda}x) + c_2 \sinh(\sqrt{\lambda}x)$$

such that $\alpha = (c_1 + c_2)/2$ and $\beta = (c_1 - c_2)/2$. Using the boundary condition $v(0) = 0$, we get $c_1 = 0$ and hence

$$v(x) = c_2 \sinh(\sqrt{\lambda}x).$$

Now using $v(L) = 0$, we have $c_2 \sinh \sqrt{\lambda}L = 0$. Thus, $c_2 = 0$ and $v(x) = 0$. We have seen this cannot be a solution.

Finally, if $\lambda < 0$, then set $\omega = \sqrt{-\lambda}$. We need to solve the *simple harmonic oscillator* problem

$$\begin{cases} v''(x) + \omega^2 v(x) & = 0 \qquad x \in (0, L) \\ \qquad\quad v(0) & = v(L) \quad = 0. \end{cases}$$

The general solution is

$$v(x) = \alpha \cos(\omega x) + \beta \sin(\omega x).$$

Using $v(0) = 0$, we get $\alpha = 0$ and hence $v(x) = \beta \sin(\omega x)$. Now using $v(L) = 0$, we have $\beta \sin \omega L = 0$. Thus, either $\beta = 0$ or $\sin \omega L = 0$. But $\beta = 0$ does not yield a solution. Hence $\omega L = k\pi$ or $\omega = k\pi/L$, for all non-zero $k \in \mathbb{Z}$. Since $\omega > 0$, we can consider only $k \in \mathbb{N}$. Hence, for each $k \in \mathbb{N}$, there is a solution (v_k, λ_k) for the eigen value problem with

$$v_k(x) = \beta_k \sin\left(\frac{k\pi x}{L}\right),$$

for some constant b_k and $\lambda_k = -(k\pi/L)^2$.

It now remains to solve w for each of these λ_k. For each $k \in \mathbb{N}$, we solve for w_k in the ODE

$$w_k''(t) + (ck\pi/L)^2 w_k(t) = 0.$$

The general solution is

$$w_k(t) = a_k \cos\left(\frac{ck\pi t}{L}\right) + b_k \sin\left(\frac{ck\pi t}{L}\right).$$

For each $k \in \mathbb{N}$, we have

$$u_k(x,t) = \left[a_k \cos\left(\frac{ck\pi t}{L}\right) + b_k \sin\left(\frac{ck\pi t}{L}\right)\right] \sin\left(\frac{k\pi x}{L}\right)$$

for some constants a_k and b_k.

- The situation corresponding to $k = 1$ is called the *fundamental mode*
- and the *frequency* of the fundamental mode is

$$\frac{c\sqrt{-\lambda_1}}{2\pi} = \frac{1}{2\pi}\frac{c\pi}{L} = \frac{c}{2L}.$$

- The frequency of higher modes are integral multiples of the fundamental frequency.
- Note that the frequency of the vibration is related to eigenvalues of the second order differential operator.

The general solution of (5.3), by principle of superposition, is

$$u(x,t) = \sum_{k=1}^{\infty} \left[a_k \cos\left(\frac{ck\pi t}{L}\right) + b_k \sin\left(\frac{ck\pi t}{L}\right)\right] \sin\left(\frac{k\pi x}{L}\right).$$

Note that the solution is expressed as series, which raises the question of convergence of the series. Another concern is whether all solutions of (5.3) have this form. We ignore these two concerns at this moment. Since we know the initial position of the string as the graph of g, we get

$$g(x) = u(x,0) = \sum_{k=1}^{\infty} a_k \sin\left(\frac{k\pi x}{L}\right).$$

This expression is again troubling and rises the question: Can any arbitrary function g be expressed as an infinite sum of trigonometric functions? But we have answered it in the study of "Fourier series".

Therefore, the constants a_k are given as

$$a_k = \frac{2}{L}\int_0^L g(x) \sin\left(\frac{k\pi x}{L}\right).$$

Finally, by differentiating u w.r.t t, we get

$$u_t(x,t) = \sum_{k=1}^{\infty} \frac{ck\pi}{L}\left[b_k \cos\frac{ck\pi t}{L} - a_k \sin\frac{ck\pi t}{L}\right] \sin\left(\frac{k\pi x}{L}\right).$$

Employing similar arguments and using $u_t(x,0) = h(x)$, we get

$$h(x) = u_t(x,0) = \sum_{k=1}^{\infty} \frac{b_k kc\pi}{L} \sin\left(\frac{k\pi x}{L}\right)$$

and hence

$$b_k = \frac{2}{kc\pi}\int_0^L h(x) \sin\left(\frac{k\pi x}{L}\right).$$

Heat Flow on a Bar

Solve the Cauchy Problem:

$$\begin{cases} u_t(x,t) - c^2 u_{xx}(x,t) & = 0 & \text{in } (0,L)\times(0,\infty) \\ u(0,t) = u(L,t) & = 0 & \text{in } (0,\infty) \\ u(x,0) & = g(x) & \text{on } [0,L] \end{cases}$$

where c is a constant, $(0,L)$ denotes a homogeneous rod of length L and the Dirichlet boundary condition $u(0,t) = u(L,t) = 0$ signifies that the rod is insulated along sides and its ends to keep the temperature at zero temperature. The initial temperature of the rod, at time $t = 0$, is given by $u(x,0) = g(x)$, where $g : [0,L] \to \mathbb{R}$ be such that $g(0) = g(L) = 0$.

Separation of Variable Method

We begin with the ansatz that $u(x,t) = v(x)w(t)$ (variable separated). Substituting u in separated form in the equation, we get

$$v(x)w'(t) = c^2 v''(x)w(t)$$

and, hence,

$$\frac{w'(t)}{c^2 w(t)} = \frac{v''(x)}{v(x)}.$$

Since LHS, a function of t, and RHS, a function x, are equal they must be equal to some constant, say λ. Thus,

$$\frac{w'(t)}{c^2 w(t)} = \frac{v''(x)}{v(x)} = \lambda.$$

Therefore, we need to solve two ODE to obtain v and w,

$$w'(t) = \lambda c^2 w(t) \text{ and } v''(x) = \lambda v(x).$$

We first solve the eigenvalue problem involving v. For each $k \in \mathbb{N}$, there is a pair (λ_k, v_k) which solves the eigenvalue problem involving v, where $\lambda_k = -(k\pi)^2/L^2$ and $v_k(x) = \sin\left(\frac{k\pi x}{L}\right)$. For each $k \in \mathbb{N}$, we solve for w_k to get

$$\ln w_k(t) = \lambda_k c^2 t + \ln \alpha,$$

where α is integration constant. Thus, $w_k(t) = \alpha e^{-(kc\pi/L)^2 t}$. Hence,

$$u_k(x,t) = v_k(x) w_k(t) = \beta_k \sin\left(\frac{k\pi x}{L}\right) e^{-(kc\pi/L)^2 t},$$

for some constants β_k, is a solution to the heat equation. By superposition principle, the general solution is

$$u(x,t) = \sum_{k=1}^{\infty} u_k(x,t) = \sum_{k=1}^{\infty} \beta_k \sin\left(\frac{k\pi x}{L}\right) e^{-(kc\pi/L)^2 t}.$$

We now use the initial temperature of the rod, given as $g : [0, L] \to \mathbb{R}$ to compute the constants. Since $u(x,0) = g(x)$,

$$g(x) = u(x,0) = \sum_{k=1}^{\infty} \beta_k \sin\left(\frac{k\pi x}{L}\right).$$

Further, $g(0) = g(L) = 0$. Thus, g admits a Fourier Sine expansion and hence its coefficients β_k are given as

$$\beta_k = \frac{2}{L} \int_0^L g(x) \sin\left(\frac{k\pi x}{L}\right).$$

Circular Wire

Let Ω be a circle (circular wire) of radius one insulated along its sides. Let the initial temperature of the wire, at time $t = 0$, be given by a 2π-periodic function $g : \mathbb{R} \to \mathbb{R}$. Then there is a solution $u(r, \theta)$ of

$$\begin{cases} u_t(\theta, t) - c^2 u_{\theta\theta}(\theta, t) &= 0 \quad \text{in } \mathbb{R} \times (0, \infty) \\ u(\theta + 2\pi, t) &= u(\theta, t) \quad \text{in } \mathbb{R} \times (0, \infty) \\ u(\theta, 0) &= g(\theta) \quad \text{on } \mathbb{R} \times \{t = 0\} \end{cases}$$

where c is a constant.

Proof

Note that $u(\theta, t)$ is 2π-periodic in θ-variable, i.e., $u(\theta + 2\pi, t) = u(\theta, t)$ for all $\theta \in \mathbb{R}$ and $t \geq 0$. We begin with ansatz $u(\theta, t) = v(\theta)w(t)$ with variables separated. Substituting for u in the equation, we get

$$\frac{w'(t)}{c^2 w(t)} = \frac{v''(\theta)}{v(\theta)} = \lambda.$$

For each $k \in \mathbb{N} \cup \{0\}$, the pair (λ_k, v_k) is a solution to the eigenvalue problem where $\lambda_k = -k^2$ and

$$v_k(\theta) = a_k \cos(k\theta) + b_k \sin(k\theta).$$

For each $k \in \mathbb{N} \cup \{0\}$, we get $w_k(t) = \alpha e^{-(kc)^2 t}$. For $k = 0$

$$u_0(\theta, t) = a_0/2 \quad \text{(To maintain consistency with Fourier series)}$$

and for each $k \in \mathbb{N}$, we have

$$u_k(\theta, t) = [a_k \cos(k\theta) + b_k \sin(k\theta)]\, e^{-k^2 c^2 t}.$$

Therefore, the general solution is

$$u(\theta, t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos(k\theta) + b_k \sin(k\theta)\right] e^{-k^2c^2t}.$$

We now use the initial temperature on the circle to find the constants. Since $u(\theta, 0) = g(\theta)$,

$$g(\theta) = u(\theta, 0) = \frac{a_0}{2} + \sum_{k=1}^{\infty} \left[a_k \cos(k\theta) + b_k \sin(k\theta)\right].$$

Further, g is 2π-periodic and, hence, admits a Fourier series expansion. Thus,

$$a_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\cos(k\theta)\, d\theta$$

and

$$b_k = \frac{1}{\pi}\int_{-\pi}^{\pi} g(\theta)\sin(k\theta)\, d\theta.$$

Note that as $t \to \infty$ the temperature of the wire approaches a constant $a_0/2$.

Motivation to Duhamel's Principle

The *Duhamel's principle* states that one can obtain a solution of the inhomogeneous IVP for heat from its homogeneous IVP. Recall that

$$\begin{cases} y'(t) + cy(t) &= f(t) \quad \text{for } t > 0 \\ y(0) &= y_0 \end{cases}$$

has the solution $y(t) = y_0 e^{-ct} + \int_0^t e^{-c(t-s)} f(s)\, ds$.In particular, the solution to the homogeneous equation

$$\begin{cases} z'(t) + cz(t) &= 0 \quad \text{for } t > 0 \\ z(0) &= z_0. \end{cases}$$

is $z(t) = z_0 e^{-ct}$. Therefore, for any fixed s, if we choose $z_0 = f(s)$ then the corresponding solution is $z(t; s) = f(s)e^{-ct}$. Then the solution to inhomogeneous problem with zero initial condition is

$$y(t) = \int_0^t z(t-s; s)\, ds.$$

For a given f and $\Omega \subset \mathbb{R}^n$, let $u(x,t)$ be the solution of the *inhomogeneous* heat equation,

$$\begin{cases} u_t(x,t) - c^2\Delta u(x,t) &= f(x,t) \quad \text{in } \Omega \times (0,T) \\ u(x,t) &= 0 \quad \text{in } \partial\Omega \times (0,T) \\ u(x,0) &= 0 \quad \text{in } \Omega. \end{cases} \tag{6.1}$$

As a first step, for each $s \in (0,\infty)$, consider $w(x,t;s)$ as the solution of the *homogeneous* problem (auxiliary)

$$\begin{cases} w^s_t(x,t) - c^2\Delta w^s(x,t) &= 0 \quad \text{in } \Omega \times (s,T) \\ w^s(x,t) &= 0 \quad \text{in } \partial\Omega \times (s,T) \\ w^s(x,s) &= f(x,s) \quad \text{on } \Omega \times \{s\}. \end{cases}$$

Since $t \in (s,T)$, introducing a change of variable $r = t - s$, we have $w^s(x,t) := w(x,t-s)$ as a solution of

$$\begin{cases} w_r(x,r) - c^2\Delta w(x,r) &= 0 \quad \text{in } \Omega \times (0,T-s) \\ w(x,r) &= 0 \quad \text{in } \partial\Omega \times (0,T-s) \\ w(x,0) &= f(x,s) \quad \text{on } \Omega \times \{0\}. \end{cases}$$

Theorem (Duhamel's Principle)

The function $u(x,t)$ defined as

$$u(x,t) := \int_0^t w^s(x,t)\,ds = \int_0^t w(x,t-s)\,ds$$

solves (6.1).

Proof

Suppose w is $C^{2,1}(\mathbb{R}^n \times (0,T))$, we get

$$\begin{aligned} u_t(x,t) &= \frac{\partial}{\partial t}\int_0^t w(x,t-s)\,ds \\ &= \int_0^t w_t(x,t-s)\,ds + w(x,t-t)\frac{d(t)}{dt} \\ &\quad - w(x,t-0)\frac{d(0)}{dt} \\ &= \int_0^t w_t(x,t-s)\,ds + w(x,0) \\ &= \int_0^t w_t(x,t-s)\,ds + f(x,t). \end{aligned}$$

Similarly,

$$\Delta u(x,t) = \int_0^t \Delta w(x,t-s)\,ds.$$

Thus,

$$\begin{aligned} u_t - c^2\Delta u &= f(x,t) + \int_0^t \left(w_t(x,t-s) - c^2\Delta w(x,t-s)\right)\,ds \\ &= f(x,t). \end{aligned}$$

Second Order Elliptic: The Laplacian

- The Laplacian is the trace of the Hessain matrix, $\Delta := \sum_{i=1}^{n} \frac{\partial^2}{\partial x_i^2}$. Note that in one dimension, $\Delta = \frac{d^2}{dx^2}$.
- Our interest is to solve, for any open subset $\Omega \subset \mathbb{R}^n$, $-\Delta u(x) = f(x)$.
- Since a Cauchy problem for elliptic is over-determined, it is reasonable to seek the solution of $-\Delta u(x) = f(x)$ in Ω and one of the following (or mixture) on $\partial\Omega$.
  1. (Dirichlet condition) $u = g$;
  2. (Neumann condition) $\nabla u \cdot \nu = g$, where $\nu(x)$ is the unit outward normal of $x \in \partial\Omega$;
  3. (Robin condition) $\nabla u \cdot \nu + cu = g$ for any $c > 0$.
  4. (Mixed condition) $u = g$ on Γ_1 and $\nabla u \cdot \nu = h$ on Γ_2, where $\Gamma_1 \cup \Gamma_2 = \partial\Omega$ and $\Gamma_1 \cap \Gamma_2 = \emptyset$.

By the linearity of Laplacian, $u = v + w$ where v is a *harmonic function* of

$$\begin{cases} \Delta v(x) & = 0 & \text{in } \Omega \\ \text{one of} & \text{the above} & \text{inhomogeneous boudary condition on } \partial\Omega, \end{cases}$$

and w is a solution of *Poisson equation*

$$\begin{cases} -\Delta w(x) & = f(x) & \text{in } \Omega \\ \text{one of} & \text{the above} & \text{homogeneous boudary condition on } \partial\Omega. \end{cases}$$

Therefore, we shall solve for u by solving for v and w separately.

Definition

Let Ω be an open subset of $\mathbb{R}^n$. A function $u \in C^2(\Omega)$ is said to be *harmonic* on Ω if $\Delta u(x) = 0$ in Ω.

Harmonic Functions

- The *one dimensional* Laplace equation is an ODE ($\frac{d^2}{dx^2}$) and is solvable with solutions $u(x) = ax + b$ for some constants a and b.
- But in higher dimensions solving Laplace equation is not so simple.
- For instance, a *two dimensional* Laplace equation

$$u_{xx} + u_{yy} = 0$$

has the trivial solution, $u(x, y) = ax + by + c$, all one degree polynomials of two variables.
- In addition, xy, $x^2 - y^2$, $x^3 - 3xy^2$, $3x^2y - y^3$, $e^x \sin y$ and $e^x \cos y$ are all solutions to the two variable Laplace equation.
- In $\mathbb{R}^n$, it is trivial to check that all polynomials up to degree one, i.e.

$$\sum_{|\alpha|\leq 1} a_\alpha x^\alpha$$

is a solution to $\Delta u = 0$ in $\mathbb{R}^n$. However, note that $u(x) = \prod_{i=1}^n x_i$ is a solution to $\Delta u = 0$ in $\mathbb{R}^n$.

- Is there a formula to harmonic functions similar to one dimensions?
- Is it possible to identify properties of harmonic functions without knowing its form?
- In two dimension, one associates with a harmonic function $u(x, y)$, a *conjugate* harmonic function, $v(x, y)$ defined as the solution of a first order system of PDE called the *Cauchy-Riemann* equations,

$$u_x(x, y) = v_y(x, y) \quad \text{and } u_y(x, y) = -v_x(x, y).$$

- Harmonic functions and holomorphic functions (differentiable complex functions) are related in the sense that, for any pair (u, v), harmonic and its conjugate, gives a holomorphic function $f(z) = u(x, y) + iv(x, y)$ where $z = x + iy$. Conversely, for any holomorphic function f, its real part and imaginary part are conjugate harmonic functions.

This observation gives us more examples of harmonic functions, for instance,

- since all complex polynomials $f(z) = z^m$ are holomorphic we have (using the polar coordinates) the real part $u(r,\theta) = r^m \cos m\theta$ and the imaginary part $v(r,\theta) = r^m \sin m\theta$ are harmonic functions in $\mathbb{R}^2$ for all $m \in \mathbb{N}$.
- Similarly, since $f(z) = \log z = \ln r + \imath\theta$ is holomorphic in certain region, we have $u(r,\theta) = \ln r$ and $v(r,\theta) = \theta$ are harmonic in $\mathbb{R}^2 \setminus (0,0)$ and $\mathbb{R}^2 \setminus \{\theta = 0\}$, respectively.

Properties of Harmonic Functions

If u is a harmonic function on Ω then, by Gauss divergence theorem

$$\int_{\partial\Omega} \frac{\partial u}{\partial \nu}\, d\sigma = 0.$$

Theorem (Gauss)

Let Ω be an open bounded subset of $\mathbb{R}^n$ with C^1 boundary. If $V = (v_1, \ldots, v_n)$ on Ω is a vector field such that $v_i \in C^1(\overline{\Omega})$, for all $1 \le i \le n$, then

$$\int_{\Omega} \nabla \cdot V\, dx = \int_{\partial\Omega} V \cdot \nu\, d\sigma. \tag{6.2}$$

Theorem (Maximum Principle)

Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $u \in C(\overline{\Omega})$ be harmonic in Ω. Then

$$\max_{y\in\overline{\Omega}} u(y) = \max_{y\in\partial\Omega} u(y).$$

Since $\partial\Omega \subset \overline{\Omega}$, we have $\max_{\partial\Omega} u \leq \max_{\overline{\Omega}} u$. It only remains to prove the other equality. For the given harmonic function u and for a fixed $\varepsilon > 0$, we set $v_\varepsilon(x) = u(x) + \varepsilon|x|^2$, for each $x \in \overline{\Omega}$. For each $x \in \Omega$, $\Delta v_\varepsilon = \Delta u + 2n\varepsilon > 0$. Recall that[6] if a function v attains local maximum at a point $x \in \Omega$, then in each direction its second order partial derivative $v_{x_ix_i}(x) \leq 0$, for all $i = 1, 2, \dots, n$. Therefore $\Delta v(x) \leq 0$. Thus, we argue that v_ε does not attain (even a local) maximum in Ω. But v_ε has to have a maximum in $\overline{\Omega}$, hence it should be attained at some point $x^\star \in \partial\Omega$, on the boundary.

[6] $v \in C^2(a, b)$ has a local maximum at $x \in (a, b)$ then $v'(x) = 0$ and $v''(x) \leq 0$

For all $x \in \overline{\Omega}$,

$$u(x) \leq v_\varepsilon(x) \leq v_\varepsilon(x^\star) = u(x^\star) + \varepsilon|x^\star|^2 \leq \max_{x\in\partial\Omega} u(x) + \varepsilon \max_{x\in\partial\Omega} |x|^2.$$

The above inequality is true for all $\varepsilon > 0$. Thus, $u(x) \leq \max_{x\in\partial\Omega} u(x)$, for all $x \in \overline{\Omega}$. Therefore, $\max_{\overline{\Omega}} u \leq \max_{x\in\partial\Omega} u(x)$ and hence we have equality.

Strong Maximum Principle

In fact harmonic functions satisfy a much stronger maximum principle whose proof is beyond the scope of this course.

Theorem (Strong Maximum Principle)

Let Ω be an open, connected (domain) subset of $\mathbb{R}^n$. Let u be harmonic in Ω and $M := \max_{y \in \overline{\Omega}} u(y)$. Then

$$u(x) < M \quad \forall x \in \Omega$$

or $u \equiv M$ is constant in Ω.

By the strong maximum principle, if Ω is connected and $g \geq 0$ and $g(x) > 0$ for some $x \in \partial\Omega$ then $u(x) > 0$ for all $x \in \Omega$.

Uniqueness of Harmonic Functions

A consequence of the maximum principle is the uniqueness of the harmonic functions.

Theorem (Uniqueness of Harmonic Functions)

Let Ω be an open, bounded subset of $\mathbb{R}^n$. Let $u_1, u_2 \in C^2(\Omega) \cap C(\overline{\Omega})$ be harmonic in Ω such that $u_1 = u_2$ on $\partial\Omega$, then $u_1 = u_2$ in Ω.

Proof.

Note that $u_1 - u_2$ is a harmonic function and hence, by maximum principle, should attain its maximum on $\partial\Omega$. But $u_1 - u_2 = 0$ on $\partial\Omega$. Thus $u_1 - u_2 \leq 0$ in Ω. Now, repeat the argument for $u_2 - u_1$, we get $u_2 - u_1 \leq 0$ in Ω. Thus, we get $u_1 - u_2 = 0$ in Ω. $\square$

Theorem

Let Ω be an open bounded connected subset of $\mathbb{R}^n$ and $g \in C(\partial\Omega)$. Then the Dirichlet problem

$$\begin{cases} \Delta u(x) & = 0 \qquad x \in \Omega \\ u(x) & = g(x) \quad x \in \partial\Omega. \end{cases} \tag{6.3}$$

has atmost one solution $u \in C^2(\Omega) \cap C(\overline{\Omega})$. Moreover, if u_1 and u_2 are solution to the Dirichlet problem corresponding to g_1 and g_2 in $C(\partial\Omega)$, respectively, then

- (a) *(Comparison) $g_1 \geq g_2$ on $\partial\Omega$ and $g_1(x_0) > g_2(x_0)$ for some $x \in \partial\Omega$ implies that $u_1 > u_2$ in Ω.*
- (b) *(Stability) $|u_1(x) - u_2(x)| \leq \max_{y\in\partial\Omega} |g_1(y) - g_2(y)|$ for all $x \in \Omega$.*

The fact that there is atmost one solution to the Dirichlet problem follows from Theorem 85. Let $w = u_1 - u_2$. Then w is harmonic. (a) Note that $w = g_1 - g_2 \geq 0$ on $\partial\Omega$. Since $g_1(x_0) > g_2(x_0)$ for some $x_0 \in \partial\Omega$, then $w(x) > 0$ for all $x \in \partial\Omega$. This proves the comparison result.
(b) Again, by maximum principle, we have

$$\pm w(x) \leq \max_{y\in\partial\Omega} |g_1(y) - g_2(y)| \quad \forall x \in \Omega.$$

This proves the stability result.

Existence of Solutions

- We have shown above that if a solution exists for (6.3) then it is unique (cf. Theorem 85).
- But does it always exist for any given domain Ω?
- In the modern theory, there are three different methods to address the question of existence, viz., Perron's Method, Layer Potential (Integral Equations) and L^2 methods which are beyond the scope of this course.
- The existence of solution depends on the 'smoothness' of the boundary of Ω and these are beyond the scope of this course!

Neumann Boundary Condition

The Neumann problem is stated as follows: Given $f : \Omega \to \mathbb{R}$ and $g : \partial\Omega \to \mathbb{R}$, find $u : \overline{\Omega} \to \mathbb{R}$ such that

$$\begin{cases} -\Delta u &= f \quad \text{in } \Omega \\ \partial_\nu u &= g \quad \text{on } \partial\Omega \end{cases} \tag{6.4}$$

By Gauss divergence theorem, if u is a solution of the Neumann problem then u satisfies, for every connected component ω of Ω,

$$\begin{aligned} \int_\omega \Delta u &= \int_{\partial\omega} \nabla u \cdot \nu \quad \text{(Using GDT)} \\ -\int_\omega f &= \int_{\partial\omega} g. \end{aligned}$$

Thus, for an inhomogeneous Laplace equation with Neumann boundary condition, the given data f, g must necessarily satisfy the compatibility condition. Otherwise, the Neumann problem does not make any sense.

Non-uniqueness for Neumann Problem

If u is any solution of (6.4), then $u + c$ for any constant c is also a solution of (6.4). More generally, for any v such that v is constant on the connected components of Ω, $u + v$ is a solution of (6.4).

Elliptic Equations

Theorem (Laplacian in 2D Rectangle)

Let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid 0 < x < a \text{ and } 0 < y < b\}$ be a rectangle in $\mathbb{R}^2$. Let $g : \partial\Omega \to \mathbb{R}$ which vanishes on three sides of the rectangle, i.e., $g(0, y) = g(x, 0) = g(a, y) = 0$ and $g(x, b) = h(x)$ where h is a continuous function $h(0) = h(a) = 0$. Then there is a unique solution to

$$\begin{cases} \Delta u(x) &= 0 \quad \text{in } \Omega \\ u(x) &= g(x) \quad \text{on } \partial\Omega, \end{cases}$$

on this rectangle with given boundary value g.

Proof

We begin by looking for solution $u(x,y)$ whose variables are separated, i.e., $u(x,y) = v(x)w(y)$. Substituting this form of u in the Laplace equation, we get

$$v''(x)w(y) + v(x)w''(y) = 0.$$

Hence

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)}.$$

Since LHS is function of x and RHS is function y, they must equal a constant, say λ. Thus,

$$\frac{v''(x)}{v(x)} = -\frac{w''(y)}{w(y)} = \lambda.$$

Using the boundary condition on u,
$u(0,y) = g(0,y) = g(a,y) = u(a,y) = 0$, we get
$v(0)w(y) = v(a)w(y) = 0$. If $w \equiv 0$, then $u \equiv 0$ which is not a solution.
Hence, $w \not\equiv 0$ and $v(0) = v(a) = 0$.

Thus, we need to solve,

$$\begin{cases} v''(x) & = \lambda v(x), \quad x \in (0,a) \\ v(0) & = v(a) \quad\quad = 0, \end{cases}$$

the *eigen value problem* for the second order differential operator. Note that the λ can be either zero, positive or negative.
If $\lambda = 0$, then $v'' = 0$ and the general solution is $v(x) = \alpha x + \beta$, for some constants α and β. Since $v(0) = 0$, we get $\beta = 0$, and $v(a) = 0$ and $a \neq 0$ implies that $\alpha = 0$. Thus, $v \equiv 0$ and hence $u \equiv 0$. But, this can not be a solution to (6.3).
If $\lambda > 0$, then $v(x) = \alpha e^{\sqrt{\lambda}x} + \beta e^{-\sqrt{\lambda}x}$. Equivalently,

$$v(x) = c_1 \cosh(\sqrt{\lambda}x) + c_2 \sinh(\sqrt{\lambda}x)$$

such that $\alpha = (c_1 + c_2)/2$ and $\beta = (c_1 - c_2)/2$. Using the boundary condition $v(0) = 0$, we get $c_1 = 0$ and hence

$$v(x) = c_2 \sinh(\sqrt{\lambda}x).$$

Now using $v(a) = 0$, we have $c_2 \sinh\sqrt{\lambda}a = 0$. Thus, $c_2 = 0$ and $v(x) = 0$. We have seen this cannot be a solution.

If $\lambda < 0$, then set $\omega = \sqrt{-\lambda}$. We need to solve

$$\begin{cases} v''(x) + \omega^2 v(x) & = 0 \qquad x \in (0, a) \\ v(0) & = v(a) \quad = 0. \end{cases} \tag{6.5}$$

The general solution is

$$v(x) = \alpha \cos(\omega x) + \beta \sin(\omega x).$$

Using the boundary condition $v(0) = 0$, we get $\alpha = 0$ and hence $v(x) = \beta \sin(\omega x)$. Now using $v(a) = 0$, we have $\beta \sin \omega a = 0$. Thus, either $\beta = 0$ or $\sin \omega a = 0$. But $\beta = 0$ does not yield a solution. Hence $\omega a = k\pi$ or $\omega = k\pi/a$, for all non-zero $k \in \mathbb{Z}$. Hence, for each $k \in \mathbb{N}$, there is a solution (v_k, λ_k) for (6.5), with

$$v_k(x) = \beta_k \sin\left(\frac{k\pi x}{a}\right),$$

for some constant β_k and $\lambda_k = -(k\pi/a)^2$.

We now solve w corresponding to each λ_k. For each $k \in \mathbb{N}$, we solve for w_k in the ODE

$$\begin{cases} w_k''(y) & = \left(\frac{k\pi}{a}\right)^2 w_k(y), \quad y \in (0, b) \\ w(0) & = 0. \end{cases}$$

Thus, $w_k(y) = c_k \sinh(k\pi y/a)$. Therefore, for each $k \in \mathbb{N}$,

$$u_k = \delta_k \sin\left(\frac{k\pi x}{a}\right) \sinh\left(\frac{k\pi y}{a}\right)$$

is a solution. The general solution is of the form (principle of superposition) (convergence?)

$$u(x, y) = \sum_{k=1}^{\infty} \delta_k \sin\left(\frac{k\pi x}{a}\right) \sinh\left(\frac{k\pi y}{a}\right).$$

The constant δ_k are obtained by using the boundary condition $u(x, b) = h(x)$ which yields

$$h(x) = u(x, b) = \sum_{k=1}^{\infty} \delta_k \sinh\left(\frac{k\pi b}{a}\right) \sin\left(\frac{k\pi x}{a}\right).$$

Since $h(0) = h(a) = 0$, the function h admits a Fourier Sine series. Thus $\delta_k \sinh\left(\frac{k\pi b}{a}\right)$ is the k-th Fourier sine coefficient of h, i.e.,

$$\delta_k = \left(\sinh\left(\frac{k\pi b}{a}\right)\right)^{-1} \frac{2}{a} \int_0^a h(x) \sin\left(\frac{k\pi x}{a}\right).$$

Laplacian on 2D Disk

Theorem (2D Disk)

Let $\Omega = \{(x, y) \in \mathbb{R}^2 \mid x^2 + y^2 < R^2\}$ be the disk of radius R in $\mathbb{R}^2$. Let $g : \partial\Omega \to \mathbb{R}$ is a continuous function. Then there is a unique solution to

$$\begin{cases} \Delta u(x) &= 0 & \text{in } \Omega \\ u(x) &= g(x) & \text{on } \partial\Omega, \end{cases}$$

on the unit disk with given boundary value g.

The Laplace operator in polar coordinates,

$$\Delta := \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2}$$

where r is the magnitude component $(0 \leq r < \infty)$ and θ is the direction component $(0 \leq \theta < 2\pi)$. The direction component is also called the azimuth angle or polar angle.

This is easily seen by using the relation $x = r\cos\theta$ and $y = r\sin\theta$. Then

$$\frac{\partial x}{\partial r} = \cos\theta,\ \frac{\partial y}{\partial r} = \sin\theta \text{ and } \frac{\partial u}{\partial r} = \cos\theta\frac{\partial u}{\partial x} + \sin\theta\frac{\partial u}{\partial y}.$$

Also,

$$\frac{\partial^2 u}{\partial r^2} = \cos^2\theta\frac{\partial^2 u}{\partial x^2} + \sin^2\theta\frac{\partial^2 u}{\partial y^2} + 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y}.$$

Similarly,

$$\frac{\partial x}{\partial \theta} = -r\sin\theta,\ \frac{\partial y}{\partial \theta} = r\cos\theta,\ \frac{\partial u}{\partial \theta} = r\cos\theta\frac{\partial u}{\partial y} - r\sin\theta\frac{\partial u}{\partial x}$$

and

$$\frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \sin^2\theta\frac{\partial^2 u}{\partial x^2} + \cos^2\theta\frac{\partial^2 u}{\partial y^2} - 2\cos\theta\sin\theta\frac{\partial^2 u}{\partial x\partial y} - \frac{1}{r}\frac{\partial u}{\partial r}.$$

Therefore, $\frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} - \frac{1}{r}\frac{\partial u}{\partial r}$ and, hence,

$$\Delta u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} + \frac{1}{r}\frac{\partial u}{\partial r}.$$

Proof

Then $\partial\Omega$ is the circle of radius R. Then, solving for $u(x,y)$ in the Dirichlet problem is to equivalent to finding $U(r,\theta) : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} & = 0 & \text{in } \Omega \\ U(r,\theta + 2\pi) & = U(r,\theta) & \text{in } \Omega \\ U(R,\theta) & = G(\theta) & \text{on } \partial\Omega \end{cases} \tag{7.1}$$

where $U(r,\theta) = u(r\cos\theta, r\sin\theta)$, $G : [0,2\pi) \to \mathbb{R}$ is $G(\theta) = g(R\cos\theta, R\sin\theta)$. Note that both U and G are 2π periodic w.r.t θ. We will look for solution $U(r,\theta)$ whose variables can be separated, i.e., $U(r,\theta) = v(r)w(\theta)$ with both v and w non-zero.

Substituting it in the polar form of Laplacian, we get

$$\frac{w}{r}\frac{d}{dr}\left(r\frac{dv}{dr}\right)+\frac{v}{r^2}\frac{d^2w}{d\theta^2}=0$$

and hence

$$\frac{-r}{v}\frac{d}{dr}\left(r\frac{dv}{dr}\right)=\frac{1}{w}\left(\frac{d^2w}{d\theta^2}\right).$$

Since LHS is a function of r and RHS is a function of θ, they must equal a constant, say λ. We need to solve the eigen value problem,

$$\begin{cases} w''(\theta)-\lambda w(\theta) & =0 \quad \theta\in\mathbb{R} \\ w(\theta+2\pi) & =w(\theta) \quad \forall\theta. \end{cases}$$

Note that the λ can be either zero, positive or negative. If $\lambda=0$, then $w''=0$ and the general solution is $w(\theta)=\alpha\theta+\beta$, for some constants α and β. Using the periodicity of w,

$$\alpha\theta+\beta=w(\theta)=w(\theta+2\pi)=\alpha\theta+2\alpha\pi+\beta$$

implies that $\alpha=0$. Thus, the pair $\lambda=0$ and $w(\theta)=\beta$ is a solution.

If $\lambda>0$, then

$$w(\theta)=\alpha e^{\sqrt{\lambda}\theta}+\beta e^{-\sqrt{\lambda}\theta}.$$

If either of α and β is non-zero, then $w(\theta)\to\infty$ as $\theta\to\pm\infty$, which contradicts the periodicity of w. Thus, $\alpha=\beta=0$ and $w\equiv 0$, which cannot be a solution. If $\lambda<0$, then set $\omega=\sqrt{-\lambda}$ and the equation becomes

$$\begin{cases} w''(\theta)+\omega^2 w(\theta) & =0 \quad \theta\in\mathbb{R} \\ w(\theta+2\pi) & =w(\theta) \quad \forall\theta \end{cases}$$

Its general solution is

$$w(\theta)=\alpha\cos(\omega\theta)+\beta\sin(\omega\theta).$$

Using the periodicity of w, we get $\omega=k$ where k is an integer. For each $k\in\mathbb{N}$, we have the solution (w_k,λ_k) where

$$\lambda_k=-k^2 \quad \text{and } w_k(\theta)=\alpha_k\cos(k\theta)+\beta_k\sin(k\theta).$$

For the λ_k's, we solve for v_k, for each $k = 0, 1, 2, \ldots$,

$$r\frac{d}{dr}\left(r\frac{dv_k}{dr}\right) = k^2 v_k.$$

For $k = 0$, we get $v_0(r) = \alpha \ln r + \beta$. But $\ln r$ blows up as $r \to 0$, but any solution U and, hence v, on the closed unit disk (compact subset) has to be bounded. Thus, we must have the $\alpha = 0$. Hence $v_0 \equiv \beta$. For $k \in \mathbb{N}$, we need to solve for v_k in

$$r\frac{d}{dr}\left(r\frac{dv_k}{dr}\right) = k^2 v_k.$$

Use the change of variable $r = e^s$. Then $e^s\frac{ds}{dr} = 1$ and $\frac{d}{dr} = \frac{d}{ds}\frac{ds}{dr} = \frac{1}{e^s}\frac{d}{ds}$. Hence $r\frac{d}{dr} = \frac{d}{ds}$. $v_k(e^s) = \alpha e^{ks} + \beta e^{-ks}$. $v_k(r) = \alpha r^k + \beta r^{-k}$. Since r^{-k} blows up as $r \to 0$, we must have $\beta = 0$. Thus, $v_k = \alpha r^k$. Therefore, for each $k = 0, 1, 2, \ldots$,

$$U_k(r, \theta) = a_k r^k \cos(k\theta) + b_k r^k \sin(k\theta).$$

The general solution is

$$U(r, \theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left(a_k r^k \cos(k\theta) + b_k r^k \sin(k\theta)\right).$$

To find the constants, we must use $U(R, \theta) = G(\theta)$. If $G \in C^1[0, 2\pi]$, then G admits Fourier series expansion. Therefore,

$$G(\theta) = \frac{a_0}{2} + \sum_{k=1}^{\infty}\left[R^k a_k \cos(k\theta) + R^k b_k \sin(k\theta)\right]$$

where

$$a_k = \frac{1}{R^k\pi}\int_{-\pi}^{\pi} G(\theta)\cos(k\theta)\, d\theta,$$

$$b_k = \frac{1}{R^k\pi}\int_{-\pi}^{\pi} G(\theta)\sin(k\theta)\, d\theta.$$

Using this in the formula for U and the uniform convergence of Fourier series, we get $U(r,\theta) =$

$$
\begin{aligned}
&= \frac{1}{\pi}\int_{-\pi}^{\pi} G(\eta)\left[\frac{1}{2} + \sum_{k=1}^{\infty}\left(\frac{r}{R}\right)^k (\cos k\eta \cos k\theta + \sin k\eta \sin k\theta)\right] d\eta \\
&= \frac{1}{\pi}\int_{-\pi}^{\pi} G(\eta)\left[\frac{1}{2} + \sum_{k=1}^{\infty}\left(\frac{r}{R}\right)^k \cos k(\eta-\theta)\right] d\eta.
\end{aligned}
$$

Using the relation

$$
\begin{aligned}
\sum_{k=1}^{\infty}\left(\frac{r}{R}\right)^k \cos k(\eta-\theta) &= \mathrm{Re}\left[\sum_{k=1}^{\infty}\left(\frac{r}{R}e^{i(\eta-\theta)}\right)^k\right] = \mathrm{Re}\left[\frac{\frac{r}{R}e^{i(\eta-\theta)}}{1-\frac{r}{R}e^{i(\eta-\theta)}}\right] \\
&= \frac{R^2 - rR\cos(\eta-\theta)}{R^2 + r^2 - 2rR\cos(\eta-\theta)} - 1 \\
&= \frac{rR\cos(\eta-\theta) - r^2}{R^2 + r^2 - 2rR\cos(\eta-\theta)}
\end{aligned}
$$

in $U(r,\theta)$ we get

Poisson Formula

$$U(r,\theta) = \frac{R^2 - r^2}{2\pi}\int_{-\pi}^{\pi} \frac{G(\eta)}{R^2 + r^2 - 2rR\cos(\eta-\theta)} d\eta.$$

Note that the formula derived above for $U(r,\theta)$ can be rewritten in Cartesian coordinates and will have the form

$$u(x) = \frac{R^2 - |x|^2}{2\pi R}\int_{S_R(0)} \frac{g(y)}{|x-y|^2} dy.$$

Any $y \in S_R(0)$ has the representation (R,η) and the integration is over the arc length element, i.e. $dy = Rd\eta$. Further, by law of cosines, $|x-y|^2 = R^2 + r^2 - 2rR\cos(\eta-\theta)$. This is called the *Poisson* formula.

Laplacian on 3D Sphere

Theorem (3D Sphere)

Let $\Omega = \{(x,y,z) \in \mathbb{R}^3 \mid x^2+y^2+z^2 < 1\}$ be the unit sphere in $\mathbb{R}^3$. Let $g : \partial\Omega \to \mathbb{R}$ is a continuous function. Then there is a unique solution to the Dirichlet problem on the unit sphere with given boundary value g.

Given the nature of domain, the Laplace operator in spherical coordinates,

$$\Delta := \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial}{\partial\phi}\right) + \frac{1}{r^2\sin^2\phi}\frac{\partial^2}{\partial\theta^2}.$$

where $r \in [0,\infty)$ is the magnitude component, $\phi \in [0,\pi]$ (zenith angle or inclination) and $\theta \in [0,2\pi)$ (azimuth angle).

Proof

Solving for u is equivalent to finding $U(r,\phi,\theta) : \Omega \to \mathbb{R}$ such that

$$\begin{cases} \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial U}{\partial r}\right) + \frac{1}{r^2\sin\phi}\frac{\partial}{\partial\phi}\left(\sin\phi\frac{\partial U}{\partial\phi}\right) & \\ \qquad\qquad + \frac{1}{r^2\sin^2\phi}\frac{\partial^2 U}{\partial\theta^2} = 0 & \text{in } \Omega \\ \qquad\qquad U(1,\phi,\theta) = G(\phi,\theta) & \text{on } \partial\Omega \end{cases} \tag{7.2}$$

where $U(r,\phi,\theta)$ and $G(\phi,\theta)$ are appropriate spherical coordinate function corresponding to u and g. We will look for solution $U(r,\phi,\theta)$ whose variables can be separated, i.e., $U(r,\phi,\theta) = v(r)w(\phi)z(\theta)$ with v, w and z non-zero. Substituting it in the spherical form of Laplacian, we get

$$\frac{wz}{r^2}\frac{d}{dr}\left(r^2\frac{dv}{dr}\right) + \frac{vz}{r^2\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dw}{d\phi}\right) + \frac{vw}{r^2\sin^2\phi}\frac{d^2z}{d\theta^2} = 0$$

and hence

$$\frac{1}{v}\frac{d}{dr}\left(r^2\frac{dv}{dr}\right) = \frac{-1}{w\sin\phi}\frac{d}{d\phi}\left(\sin\phi\frac{dw}{d\phi}\right) - \frac{1}{z\sin^2\phi}\frac{d^2z}{d\theta^2}.$$

Since LHS is a function of r and RHS is a function of (ϕ, θ), they must equal a constant, say λ. If azimuthal symmetry is present then $z(\theta)$ is constant and hence $\frac{dz}{d\theta} = 0$. We need to solve for w,

$$\sin\phi w''(\phi) + \cos\phi w'(\phi) + \lambda \sin\phi w(\phi) = 0, \quad \phi \in (0, \pi)$$

Set $x = \cos\phi$. Then $\frac{dx}{d\phi} = -\sin\phi$.

$$w'(\phi) = -\sin\phi \frac{dw}{dx} \text{ and } w''(\phi) = \sin^2\phi \frac{d^2w}{dx^2} - \cos\phi \frac{dw}{dx}$$

In the new variable x, we get the Legendre EVP

$$(1-x^2)w''(x) - 2xw'(x) + \lambda w(x) = 0 \quad x \in [-1, 1].$$

We have already seen that this is a singular S-L problem. For each $k \in \mathbb{N} \cup \{0\}$, we have the solution (w_k, λ_k) where

$$\lambda_k = k(k+1) \quad \text{and } w_k(\phi) = P_k(\cos\phi).$$

For the λ_k's, we solve for v_k. For each $k = 0, 1, 2, \ldots$,

$$\frac{d}{dr}\left(r^2 \frac{dv_k}{dr}\right) = k(k+1)v_k.$$

For $k = 0$, we get $v_0(r) = -\alpha/r + \beta$. But $1/r$ blows up as $r \to 0$ and U must be bounded in the closed sphere. Thus, we must have the $\alpha = 0$. Hence $v_0 \equiv \beta$. For $k \in \mathbb{N}$, we need to solve for v_k in

$$\frac{d}{dr}\left(r^2 \frac{dv_k}{dr}\right) = k(k+1)v_k.$$

Use the change of variable $r = e^s$. Then $e^s \frac{ds}{dr} = 1$ and $\frac{d}{dr} = \frac{d}{ds}\frac{ds}{dr} = \frac{1}{e^s}\frac{d}{ds}$. Hence $r\frac{d}{dr} = \frac{d}{ds}$. Solving for m in the quadratic equation $m^2 + m = k(k+1)$. $m_1 = k$ and $m_2 = -k-1$. $v_k(e^s) = \alpha e^{ks} + \beta e^{(-k-1)s}$. $v_k(r) = \alpha r^k + \beta r^{-k-1}$.Since r^{-k-1} blows up as $r \to 0$, we must have $\beta = 0$. Thus, $v_k = \alpha r^k$. Therefore, for each $k = 0, 1, 2, \ldots$,

$$U_k(r, \phi, \theta) = a_k r^k P_k(\cos\phi).$$

The general solution is

$$U(r, \phi, \theta) = \sum_{k=0}^{\infty} a_k r^k P_k(\cos\phi).$$

Since we have azimuthal symmetry, $G(\phi, \theta) = G(\phi)$. To find the constants, we use $U(1, \phi, \theta) = G(\phi)$, hence

$$G(\phi) = \sum_{k=0}^{\infty} a_k P_k(\cos\phi).$$

Using the orthogonality of P_k, we have

$$a_k = \frac{2k+1}{2} \int_0^{\pi} G(\phi) P_k(\cos\phi) \sin\phi \, d\phi.$$