Preface

The theory of *homogenization* of partial differential equations is a concept that deals with the study of features that are different at different length scales. For instance, in material science, homogenization deals with the study of the macroscopic behaviour of a composite medium through its microscopic properties. Fiberglass, bone etc. are examples of composite material. The known and unknown quantities in the study of physical processes in a medium with micro-structure depend on a small parameter $\varepsilon = \frac{l}{L}$, where L is the macroscopic scale length of the dimension of a specimen of the medium and l is the characteristic length of the medium configuration. The physical parameters such as conductivity, elasticity etc. are discontinuous and switch rapidly between different values across a small length scale ε. The study of the limit, as $\varepsilon \to 0$, is the aim of the mathematical theory of homogenization. Though the case $\varepsilon \to 0$ has no real physical meaning, it is important as a tool for numerical computations. The origin of the word "homogenization" is related to the question of replacing a heterogeneous medium by a fictitious homogeneous one (the 'homogenized' material) for computational purposes.

Let Ω be an open bounded subset of $\mathbb{R}^n$. The optimal design problem is to find a subset $\Omega_1 \subset \Omega$ such that the "cost" functional

$$J(a) = \int_\Omega F(x, u(x))\, dx$$

attains its minimum, where $u(x)$ is the solution of the Dirichlet problem

$$\begin{cases} -\operatorname{div}(A(x)\nabla u(x)) & = f(x) \quad \text{in } \Omega \\ \qquad\qquad\quad u & = 0 \qquad\quad \text{on } \partial\Omega, \end{cases} \tag{0.0.1}$$

$A(x) = a(x)I$ and

$$a(x) = \begin{cases} \alpha & \text{on } \Omega_1 \\ \beta & \text{on } \Omega \setminus \Omega_1. \end{cases}$$

The functional J is minimized over all subsets of Ω. The constants α, β are the isotropic conductivity of the material Ω_1 and its complement in Ω, respectively. The optimal design problem concerns with finding the optimal mix of the material constituents in Ω such that it optimizes some "property" (say, energy) of the system given by J.

The existence of an optimal mixture of the material constituents was proved in [Che75] under some regularity hypotheses on Ω. However, when no regularity assumptions are made, then depending on the J, there may arise a situation that we have no optimal mixture of the materials (cf. [Mur71, Mur72]). The situation of no optimal solution corresponds to the case where the corresponding conductivity coefficient a is no longer isotropic. This situation corresponds to the case where the material is mixed finely to form a heterogeneous material. Physically, the material could be an alloy formed by two material with conductivity α and β. Though, from microscopic point of view this is still a mixture of two materials, from the macroscopic view it behaves completely different, with new properties, from the original constituents.

Thus, the situation of 'no optimal solution' motivates us to study the Dirichlet problem (0.0.1) for a heterogeneous material Ω. This is a classical second order elliptic boundary value problem and admits a unique solution. However, note that for a heterogeneous material the coefficient $a(x)$ oscillate rapidly. Thus, when we try to compute the solution for (0.0.1), we need to use grid or mesh at a scale much smaller than the scale of the mixture, which may be practically impossible, leading to large errors in our computation. The mathematical theory of homogenization 'averages out' the heterogeneities and studies an 'equivalent' homogeneous fictitious material whose behaviour reflects that of the original material, when the number of fibres is very large.

Homogenization, as a mathematical discipline, took shape only in the last three decades but the physical ideas of homogenization date back at least to [Poi22, Mos50, Max73, Cla79, Ray92]. A very good historical record of works related to homogenization until 1975 can be found in [Bab76] and the references therein.

An abstract theory of homogenization was introduced by S. Spagnolo in a paper of 1967 (cf. [Spa67]) under the name of G-convergence[1] (also cf. [Spa68, GS73, Spa76]) and further generalised as H-convergence by L. Tartar in [Tar77] and developed by F. Murat and L. Tartar (cf. [Mur78b, MT97]).

[1] The terminology denoting the convergence of Green's operators for boundary problems

There is also a variational theory of homogenization, known as Γ-convergence, proposed by Ennio De Giorgi in a sequence of papers (cf. [GS73, Gio75, GF75]). For a thorough introduction to this theory we refer to [Gio84, Att84, DM93, BD98]. The wide spread application and theory of homogenization can also be found in [BLP78, JKO94, Hor97, CD99, CP99].

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Notations

Symbols

$\mathbb{R}^n$ denotes the n-dimensional Euclidean space over $\mathbb{R}$. $\{e_1, \ldots, e_n\}$ is the standard basis of $\mathbb{R}^n$

Ω denotes an open bounded subset of $\mathbb{R}^n$

$\partial\Omega$ denotes the boundary of Ω

$|E|$ is the Lebesgue measure of a subset $E \subset \mathbb{R}^n$. However, for a vector $\xi \in \mathbb{R}^n$, $|\xi| = \sqrt{\sum_{i=1}^n \xi_i^2}$

$M(\alpha, \beta, \Omega)$ denotes, for $0 < \alpha < \beta$, the class of all $n \times n$ matrices, $A = A(x)$, with $L^\infty(\Omega)$ entries such that,

$$\alpha|\xi|^2 \leq A(x)\xi \cdot \xi \text{ and } |A(x)\xi| \leq \beta|\xi| \quad \text{for a.e. } x \quad \forall \xi \in \mathbb{R}^n$$

${}^tA$ denotes the transpose of a matrix A

Function Spaces

$\mathcal{D}(\Omega)$ or $C_c^\infty(\Omega)$ is the class of all infinitely differentiable functions on Ω with compact support

$\mathcal{D}'(\Omega)$ is the topological dual of $\mathcal{D}(\Omega)$, the space of all distributions

$C_{\text{per}}(Y)$ denotes the class of Y-periodic functions in $C(\mathbb{R}^n)$

$H_0^1(\Omega)$ is the closure of $\mathcal{D}(\Omega)$ in $W^{1,2}(\Omega) = (H^1(\Omega))$ and its norm is denoted by $\|.\|_{H_0^1(\Omega)}$

$H^{-1}(\Omega)$ is the dual space of $H_0^1(\Omega)$ and its norm is denoted by $\|\cdot\|_{H^{-1}(\Omega)}$

$L^\infty(\Omega)$ is the space of all essentially bounded measurable functions and its norm is denoted by $\|.\|_{\infty,\Omega}$

$L^p(\Omega)$ is the space of all p-summable measurable functions and its norm is denoted by $\|.\|_{p,\Omega}$ $(1 \leq p < \infty)$

$L^p(\Omega; X)$ denotes the class of all measurable functions $f : \Omega \to X$ such that $\int_\Omega \|f(x)\|_X^p < \infty$, where X is a Banach space

$L^p_{\text{per}}(Y)$ denotes the class of Y-periodic functions in $L^p_{\text{loc}}(\mathbb{R}^n)$ and its norm denoted by $\|.\|_{p,Y}$ $(1 \leq p \leq \infty)$

$W^{m,p}(\Omega)$ is the collection of all $L^p(\Omega)$ functions such that all distributional derivatives upto order m are also in $L^p(\Omega)$ and its norm is denoted by $\|.\|_{m,p,\Omega}$

$W^{m,p}_{\text{per}}(Y)$ denotes the class of Y-periodic functions in $W^{m,p}(\mathbb{R}^n)$ and its norm denoted by $\|\cdot\|_{m,p,Y}$

General Conventions

$\langle \cdot, \cdot \rangle_{X^\star, X}$ denotes the duality pairing between $X^\star$ and X

$\to$ will denote the convergence in the strong topology of the space

$\rightharpoonup$ will denote the convergence in the weak topology of the space

C_0 is a generic positive constant independent of the parameters w.r.t which a limit is taken; will be different in different inequalities

$X^\star$ denotes the topological dual (space of continuous linear functionals) of the space X

Chapter 1

Asymptotic Expansion

We begin the study of homogenization by considering a linear second order elliptic problem in a domain with periodic structures. More precisely, the coefficients of the PDE have rapid periodic oscillation. The aim of this chapter is to develop the two-scale asymptotic expansion. The mathematical justification of the asymptotic expansion will be discussed in subsequent chapters. The periodic framework models the case where the heterogeneities are very small with respect to the size of the domain and are evenly distributed. This is a realistic assumption for large class of applications. Some good references on periodic homogenization are [BLP78, CD99, JKO94].

1.1 Periodically Oscillating Functions

Let us build tools required to model rapid oscillations of periodic functions. Let us assume that $Y = \Pi_{i=1}^n [0, l_i)$ is a reference cell (or period) in $\mathbb{R}^n$.

Definition 1.1.1. *A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be Y-*periodic *if, for all $i = 1, 2, \ldots, n$, $f(x + ky_i e_i) = f(x)$ for a.e. $x \in \mathbb{R}^n$ and for all $k \in \mathbb{Z}$.*

For simplicity, we shall, henceforth, take the reference cell to be the unit cube of $\mathbb{R}^n$, i.e., $Y = [0, 1]^n$. This is only for simplicity and to avoid carrying the measure of Y, $|Y|$, in our calculations. Note that $\mathbb{R}^n = \cup_{k \in \mathbb{Z}^n}(k + Y)$, is a disjoint union. Any function $f : Y \to \mathbb{R}$, defined a.e. on Y, may be extended a.e. to $\mathbb{R}^n$ as a Y-periodic function. Let $L^p_{\text{per}}(Y)$ denote the set of all $L^p_{\text{loc}}(\mathbb{R}^n)$ which are Y-periodic equipped with the norm of $L^p(Y)$. For any $f \in L^p_{\text{per}}(Y)$ and $\varepsilon > 0$, we may define a new function $f_\varepsilon : \mathbb{R}^n \to \mathbb{R}$ as

$f_\varepsilon(x) = f(x/\varepsilon)$. Observe that f_ε is εY-periodic in $\mathbb{R}^n$ because, for all $k \in \mathbb{Z}$ and $i = 1, 2, \ldots, n$,

$$f_\varepsilon(x + k\varepsilon e_i) = f\left(\frac{x}{\varepsilon} + ke_i\right) = f\left(\frac{x}{\varepsilon}\right) = f_\varepsilon(x).$$

The second equality is due to the Y-periodicity of f. Observe that f_ε on $\mathbb{R}^n$ has increased number of oscillations, if any[1], compared to f.

Example 1.1. Consider the function f on $[0,1]$ defined as

$$f(y) = \begin{cases} 1 & [0, 1/2) \\ -1 & [1/2, 1] \end{cases}$$

extended to all of $\mathbb{R}$. Define, for any $0 < \varepsilon < 1$, the new function $f_\varepsilon(x) = f(x/\varepsilon)$ on $\mathbb{R}$. Note that the number of points of jump discontinuity for f in $[0,1]$ is only one at $y = 1/2$. However, f_ε has more than one point of jump discontinuity in $[0,1]$. For instance, for $\varepsilon = 1/2$, the function f_ε has three points of jump discontinuity, at $x = 1/4, 1/2, 3/4$, in $[0,1]$.

Example 1.2. Consider $f(y) = \sin(2\pi y)$ on $[0,1]$ extended to all of $\mathbb{R}$. Define $f_\varepsilon(x) = \sin(2\pi x/\varepsilon)$ on $\mathbb{R}$. For any $0 < \varepsilon < 1$, we see that the number of oscillations on $[0,1]$ is increased for f_ε. For instance, for $\varepsilon = 1/2$, f_ε has twice the number of oscillations, as that of f, on $[0,1]$.

Theorem 1.1.2. *Let $1 \leq p \leq +\infty$ and $f \in L^p_{per}(Y)$. Then $f_\varepsilon(x) = f(x/\varepsilon)$, for $0 < \varepsilon < 1$, is bounded in any open cell R that contains any translation of Y, i.e.,*

$$\|f_\varepsilon\|^p_{p,R} \leq C_0 \frac{|R|}{|Y|} \|f\|^p_{p,Y}$$

where C_0 depends only on n, the dimension of Euclidean space.

Theorem 1.1.3. *Let $1 \leq p \leq +\infty$, $f \in L^p_{per}(Y)$ and set $f_\varepsilon(x) = f(x/\varepsilon)$, for $0 < \varepsilon < 1$, on $\mathbb{R}^n$. For $1 \leq p < \infty$,*

$$f_\varepsilon \rightharpoonup \frac{1}{|Y|} \int_Y f(y)\, dy \text{ weakly in } L^p(\Omega)$$

for any bounded open subset $\Omega \subset \mathbb{R}^n$. If $p = \infty$, then

$$f_\varepsilon \rightharpoonup \frac{1}{|Y|} \int_Y f(y)\, dy \text{ weak-* in } L^\infty(\Omega).$$

[1] constant functions have no oscillations

1.2 Second Order Elliptic Equation

Let Ω be an open bounded subset of $\mathbb{R}^n$ and let $\partial\Omega$ denote the boundary of Ω. For any given $0 < \alpha < \beta$, let $M(\alpha, \beta, \Omega)$ denote the class of all $n \times n$ matrices, $A = A(x)$, with $L^\infty(\Omega)$ entries such that,

$$\alpha|\xi|^2 \leq A(x)\xi.\xi \leq \beta|\xi|^2 \quad \text{a.e. } x \quad \forall \xi \in \mathbb{R}^n.$$

Recall the following result on variational inequality on a Hilbert space. Refer [KS00] for a complete theory on variational inequality.

Theorem 1.2.1. *Let $a(x, y)$ be a coercive bilinear form on H, $K \subset H$ be a closed and convex subset of H and $f \in H^\star$. Then there exists a unique solution $x \in K$ to*

$$a(x, y - x) \geq \langle f, y - x\rangle, \quad \forall y \in K. \tag{1.2.1}$$

The case $K = H$ in, the above result, is popularly known as Lax-Milgram result. In the case $K = H$ and $z \in H$, by choosing $y = x + z$ and $y = x - z$, by turn, in (1.2.1), we have the equality $a(x, z) = \langle f, z\rangle$ for all $z \in H$ and for every given $f \in H^\star$.

The Lax-Milgram result implies the existence and uniqueness of a weak solution, $u \in H^1_0(\Omega)$, to the second order elliptic equation with Dirichlet boundary condition,

$$\left\{\begin{array}{rll} -\text{div}(A(x)\nabla u(x)) & = f(x) & \text{in } \Omega \\ u(x) & = 0 & \text{on } \partial\Omega, \end{array}\right. \tag{1.2.2}$$

where $A \in M(\alpha, \beta, \Omega)$ and $f \in H^{-1}(\Omega)$. In fact, one also has the estimate

$$\|u\|_{H^1_0(\Omega)} \leq \frac{1}{\alpha}\|f\|_{H^{-1}(\Omega)}. \tag{1.2.3}$$

The bounded elliptic operator $\mathcal{A} : H^1_0(\Omega) \to H^{-1}(\Omega)$, defined as $\mathcal{A} = -\text{div}(A(x)\nabla)$, is an isomorphism and the norm of $\mathcal{A}^{-1}$ is not larger than α^{-1} (cf. (1.2.3)). Moreover, the weak solution u of (1.2.2) can also be characterized as the minimizer in $H^1_0(\Omega)$ of the functional $J : H^1_0(\Omega) \to \mathbb{R}$ defined as

$$J(v) = \frac{1}{2}\int_\Omega A\nabla v.\nabla v\,dx - \langle f, v\rangle_{H^{-1}(\Omega),H^1_0(\Omega)},$$

i.e.,

$$J(u) = \min_{v \in H^1_0(\Omega)} J(v).$$

1.3 Periodic Boundary Conditions

Let $Y = [0,1)^n$ be the unit cell of $\mathbb{R}^n$ and let, for each $i, j = 1, 2, \ldots, n$, $a_{ij} : Y \to \mathbb{R}$ and $A(y) = (a_{ij})$. For any given $f : Y \to \mathbb{R}$, extended Y-periodically to $\mathbb{R}^n$, we want to solve the problem

$$\begin{cases} -\operatorname{div}(A(y)\nabla u(y)) & = f(y) \quad \text{in } Y \\ \qquad\qquad\qquad u & \text{is} \qquad Y - \text{periodic.} \end{cases} \tag{1.3.1}$$

The condition u is Y-periodic is equivalent to saying that u takes equal values on opposite faces of Y.

Let us now identify the solution space for (1.3.1). Let $C^\infty_{\text{per}}(Y)$ be the set of all Y-periodic functions in $C^\infty(\mathbb{R}^n)$. Let $H^1_{\text{per}}(Y)$ denote the closure of $C^\infty_{\text{per}}(Y)$ in the H^1-norm. Being a second order equation, in the weak formulation, we expect the weak solution u to be in $H^1_{\text{per}}(Y)$. Note that if u solves (1.3.1) then $u + c$, for any constant c, also solves (1.3.1). Thus, the solution will be unique up to a constant in the space $H^1_{\text{per}}(Y)$. Therefore, we define the quotient space $W_{\text{per}}(Y) = H^1_{\text{per}}(Y)/\mathbb{R}$ as solution space where the solution is unique.

Solving (1.3.1) is to find $u \in W_{\text{per}}(Y)$, for any given $f \in (W_{\text{per}}(Y))^\star$ in the dual of $W_{\text{per}}(Y)$, such that

$$\int_Y A\nabla u \cdot \nabla v \, dx = \langle f, v\rangle_{(W_{\text{per}}(Y))^\star, W_{\text{per}}(Y)} \quad \forall v \in W_{\text{per}}(Y).$$

The requirement that $f \in (W_{\text{per}}(Y))^\star$ is equivalent to saying that

$$\int_Y f(y)\, dy = 0$$

because f defines a linear functional on $W_{\text{per}}(Y)$ and $f(0) = 0$, where $0 \in H^1_{\text{per}}(Y)/\mathbb{R}$. In particular, the equivalence class of 0 is same as the equivalence class 1 and hence

$$\int_Y f(y)\, dy = \langle f, 1\rangle = \langle f, 0\rangle = 0.$$

Theorem 1.3.1. *Let Y be unit open cell and let $a_{ij} \in L^\infty(\Omega)$ such that the matrix $A(y) = (a_{ij}(y))$ is elliptic with ellipticity constant $\alpha > 0$. For any $f \in (W_{per}(Y))^\star$, there is a unique weak solution $u \in W_{per}(Y)$ satisfying*

$$\int_Y A\nabla u \cdot \nabla v \, dx = \langle f, v\rangle_{(W_{per}(Y))^\star, W_{per}(Y)} \quad \forall v \in W_{per}(Y).$$

Note that the solution u we find from above theorem is an equivalence class of functions which are all possible solutions. Any representative element from the equivalence class is a solution. All the elements in the equivalence differ by a constant. Let u be an element from the equivalence class and let c be the constant

$$c = \frac{1}{|Y|}\int_Y u(y)\,dy.$$

Thus, we have $u-c$ is a solution with zero mean value in Y, i.e., $\int_Y u(y)\,dy = 0$. Therefore, rephrasing (1.3.1) as

$$\begin{cases} -\mathrm{div}(A(y)\nabla u(y)) & = f(y) \quad \text{in } Y \\ u & \text{is} \quad\quad\; Y - \text{periodic} \\ \frac{1}{|Y|}\int_Y u(y)\,dy & = 0 \end{cases}$$

we have unique solution u in the solution space

$$V_{\mathrm{per}}(Y) = \left\{ u \in H^1_{\mathrm{per}}(Y) \mid \frac{1}{|Y|}\int_Y u(y)\,dy = 0 \right\}.$$

Remark 1.3.2. By identifying the cell Y with an equivalent Torus, (1.3.1) may be viewed as posed on the Torus. This formulation has the advantage that the equation has no boundary condition because Torus has no boundary.

1.4 Periodic Composite Material

In this section, we mathematically model a periodic composite material. Let $\Omega \subset \mathbb{R}^n$ denote a periodic composite material. For simplicity, let us consider the case of a composite material which is a mixture of two materials. Let $Y = [0,1)^n$ be the reference cell which is a mixture of materials Y_1 and Y_2 such that $\overline{Y}_1 \cup \overline{Y}_2 = \overline{Y}$ and $Y_1 \cap Y_2 = \emptyset$. For each $0 < \varepsilon < 1$, the dilated cell εY can be used to tile $\mathbb{R}^n$, so that, Ω is also tiled using εY modelling the periodic distribution of its constituents, for some ε very small (cf. Fig. 1.1).

Let us consider the second order elliptic problem with Dirichlet boundary condition on Ω, for a given $f \in H^{-1}(\Omega)$,

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) & = f(x) \quad \text{in } \Omega \\ u & = 0 \quad\quad\;\; \text{on } \partial\Omega, \end{cases}$$

y εY

x

Figure 1.1: partition of Ω into ε-cells

where $A(x) = (a_{ij}(x))$ is a $n \times n$ matrix of measurable $L^\infty(\Omega)$ functions such that

$$\alpha|\xi|^2 \leq A(x)\xi.\xi \quad \forall \xi \in \mathbb{R}^n.$$

We shall now observe that when Ω is a periodic composite material, the functions a_{ij}, for all $i, j = 1, 2, \ldots, n$, are rapidly oscillating periodic functions. For each $i, j \in \{1, 2, \ldots, n\}$, we are given measurable $L^\infty(Y)$ functions $a_{ij} : Y \to \mathbb{R}$ with different values on Y_1 and Y_2 and

$$\alpha|\xi|^2 \leq \sum_{i,j=1}^{n} a_{ij}(y)\xi_i\xi_j \text{ a.e. } \; y \in Y \quad \forall \xi \in \mathbb{R}^n.$$

The condition given above is called the *ellipticity* condition. We extend a_{ij} to all of $\mathbb{R}^n$, and for each $0 < \varepsilon < 1$, we define the function $a^\varepsilon_{ij} : \mathbb{R}^n \to \mathbb{R}$ as

$$a^\varepsilon_{ij}(x) = a_{ij}\left(\frac{x}{\varepsilon}\right) \qquad \text{a.e. } x \in \mathbb{R}^n$$

and the $n \times n$ matrix $A_\varepsilon(x) = (a^\varepsilon_{ij}(x))$ is in $M(\alpha, \beta, \Omega)$. Thus, the Dirichlet problem for the composite material Ω is given as, for a given $f \in H^{-1}(\Omega)$,

$$\left\{ \begin{array}{rll} -\text{div}(A_\varepsilon(x)\nabla u_\varepsilon(x)) & = f(x) & \text{in } \Omega \\ u_\varepsilon & = 0 & \text{on } \partial\Omega. \end{array} \right. \tag{1.4.1}$$

By Lax-Milgram result, there exists a unique solution $u_\varepsilon \in H_0^1(\Omega)$ such that

$$\int_\Omega A_\varepsilon(x)\nabla u_\varepsilon(x)\cdot\nabla v(x)\,dx = \langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega)$$

and $\|u_\varepsilon\|_{H_0^1(\Omega)} \leq 1/\alpha\|f\|_{H^{-1}(\Omega)}$. Computing the solution, numerically, is stable if the size of the grid is chosen smaller than ε. But for composite materials, ε is very small and choosing grid smaller than ε leads to impossible computable situation. Therefore, we study the limiting case, as $\varepsilon \to 0$, of the Dirichlet problem (1.4.1).

1.5 Asymptotic Expansion in Two Scales

Note that in the periodic set-up any $x \in \Omega$ has two reprsentations. One is the macroscale representation x and the other is that x is in some translation of the εY cell having the form $x = \varepsilon y$ for some $y \in Y$. Thus, any $x \in \Omega$ may take two representations each in Ω and Y as x and $y = \frac{x}{\varepsilon}$, respectively. Thus, the behaviour of a periodic composite material Ω, as given in (1.4.1), involves two scales, viz., the "macroscopic or slow" scale $x \in \Omega$ and the "microscopic or fast" scale $y = x/\varepsilon \in Y$. We intend to find $u_\varepsilon(x)$ that solves (1.4.1). Thus, our model suggests that u_ε depends on both the slow variable x and fast variable $y = x/\varepsilon$, viewed as independent variables. This suggests us to seek $u_\varepsilon(x)$, with $x \in \Omega$, in the form

$$u_\varepsilon(x) = u_0\left(x, \frac{x}{\varepsilon}\right) + \varepsilon u_1\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^2 u_2\left(x, \frac{x}{\varepsilon}\right) + \dots, \tag{1.5.1}$$

where $u_i(x, y)$ are functions which are Y-periodic in the y-variable. The partial derivative of a function $\phi_\varepsilon(x) := \phi(x, x/\varepsilon)$ is given as

$$\frac{\partial\phi_\varepsilon(x)}{\partial x_i} = \frac{\partial\phi(x,y)}{\partial x_i} = \frac{\partial\phi(x,y)}{\partial x_i} + \frac{\partial y_i}{\partial x_i}\frac{\partial\phi(x,y)}{\partial y_i} = \frac{\partial\phi(x,y)}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial\phi(x,y)}{\partial y_i}.$$

The second order operator of the equation (1.4.1) can be rewritten as

$$\begin{aligned}
-\mathcal{A}_\varepsilon &= -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}^{\varepsilon}(x)\frac{\partial}{\partial x_j}\right) \\
&= -\sum_{i,j=1}^{n} \left(\frac{\partial}{\partial x_i} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_i}\right)\left[a_{ij}(y)\left(\frac{\partial}{\partial x_j} + \frac{1}{\varepsilon}\frac{\partial}{\partial y_j}\right)\right] \\
&= -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right) - \varepsilon^{-2}\sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right) \\
&\quad - \varepsilon^{-1}\left[\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial}{\partial y_j}\right) + \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial}{\partial x_j}\right)\right] \\
&= \mathcal{A}_0 + \varepsilon^{-2}\mathcal{A}_2 + \varepsilon^{-1}\mathcal{A}_1.
\end{aligned}$$

Substituting in (1.4.1), we get

$$(\varepsilon^{-2}\mathcal{A}_2 + \varepsilon^{-1}\mathcal{A}_1 + \mathcal{A}_0)\left[u_0(x,y) + \varepsilon u_1(x,y) + \varepsilon^2 u_2(x,y) + \ldots\right] = f(x).$$

Now, equating like powers of ε both side, we get

$$\begin{aligned}
\mathcal{A}_2 u_0 &= 0 && (1.5.2)\\
\mathcal{A}_2 u_1 + \mathcal{A}_1 u_0 &= 0 && (1.5.3)\\
\mathcal{A}_2 u_2 + \mathcal{A}_1 u_1 + \mathcal{A}_0 u_0 &= f(x) && (1.5.4)\\
\mathcal{A}_2 u_{m+2} + \mathcal{A}_1 u_{m+1} + \mathcal{A}_0 u_m &= 0 \quad \forall m \geq 1. && (1.5.5)
\end{aligned}$$

We first solve for $u_0(\cdot, y)$ in Y using the first equation

$$\left\{\begin{array}{rll} \mathcal{A}_2 u_0(\cdot,y) & = 0 & \text{in } Y \\ u_0(\cdot,y) & \text{is} & Y\text{-periodic in } y. \end{array}\right.$$

Since $0 \in (W_{\text{per}}(Y))^\star$, by Theorem 1.3.1, we have $u_0(\cdot,y) \in H^1_{\text{per}}(Y)$ which are unique up to constant. Thus, $0 \in W_{\text{per}}(Y)$ is a solution. Therefore, $u_0(\cdot,y)$ is independent of y and hence $u_0(x,y) = u(x)$, a function of x.

We now proceed to find u_1 using the problem

$$\left\{\begin{array}{rll} \mathcal{A}_2 u_1(x,y) & = -\mathcal{A}_1 u(x) & \text{in } Y \\ u_1(\cdot,y) & \text{is} & Y\text{-periodic in } y \\ \int_Y u_1(x,y)\,dy & = 0. & \end{array}\right.$$

We first need to check that $-\mathcal{A}_1 u \in (W_{\text{per}}(Y))^\star$ or, equivalently,

$$\int_Y \mathcal{A}_1 u \, dy = 0.$$

We simplify the RHS using the fact that u is a function of x,

$$\begin{aligned}
\mathcal{A}_1 u(x) &= -\sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial u(x)}{\partial x_j}\right) = -\sum_{i,j=1}^{n} \frac{\partial a_{ij}(y)}{\partial y_i}\frac{\partial u(x)}{\partial x_j} \\
&= -\sum_{j=1}^{n}\left(\sum_{i=1}^{n} \frac{\partial a_{ij}(y)}{\partial y_i}\right)\frac{\partial u(x)}{\partial x_j}.
\end{aligned}$$

Consider

$$\begin{aligned}
\int_Y \mathcal{A}_1 u(x)\, dy &= \langle \mathcal{A}_1 u(x), 1\rangle \\
&= -\int_Y \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial u(x)}{\partial x_j}\right)\cdot 1\, dy \\
&= \sum_{i,j=1}^{n} \frac{\partial u(x)}{\partial x_j}\int_Y a_{ij}(y)\frac{\partial 1}{\partial y_i} = 0.
\end{aligned}$$

Since $-\mathcal{A}_1 u \in (W_{\text{per}}(Y))^\star$, by Theorem 1.3.1, we have $u_1(\cdot, y) \in H^1_{\text{per}}(Y)$ which are unique up to constant. Since the operator $\mathcal{A}_2$ depends on y variable and u_{x_j} is independent of y, we are motivated to define, for each $j = 1, 2, \ldots, n$, the auxiliary periodic function χ_j as a solution to the problem

$$\left\{\begin{array}{rll}
\mathcal{A}_2\chi_j(y) & = -\sum_{i=1}^{n} \frac{\partial a_{ij}(y)}{\partial y_i} & \text{in } Y \\
\chi_j(y) & \text{is} & Y\text{-periodic in } y \\
\int_Y \chi_j(y)\, dy & = 0 &
\end{array}\right. \tag{1.5.6}$$

or equivalently,

$$\left\{\begin{array}{rll}
\operatorname{div}(A(y)\nabla(\chi_j - y_j) & = 0 & \text{in } Y \\
\frac{1}{|Y|}\int_Y \chi_j(y)\, dy & = 0 & \\
\chi_j - y_j & & \text{is } Y\text{-periodic.}
\end{array}\right.$$

Substituting this in the equation of u_1, we get

$$\begin{aligned}\mathcal{A}_2 u_1(x,y) + \mathcal{A}_1 u(x) &= \mathcal{A}_2 u_1(x,y) + \sum_{j=1}^{n} \mathcal{A}_2 \chi_j(y) \frac{\partial u(x)}{\partial x_j} \\ &= \mathcal{A}_2 u_1(x,y) + \mathcal{A}_2 \left(\sum_{j=1}^{n} \chi_j(y) \frac{\partial u(x)}{\partial x_j} \right) \\ &= \mathcal{A}_2 \left(u_1(x,y) + \sum_{j=1}^{n} \chi_j(y) \frac{\partial u(x)}{\partial x_j} \right).\end{aligned}$$

Therefore, we have

$$u_1(x,y) = -\sum_{j=1}^{n} \chi_j(y) \frac{\partial u(x)}{\partial x_j} + \tilde{u}(x)$$

for some function $\tilde{u}(x)$. Finally, we solve for u_2 in the problem

$$\left\{ \begin{array}{rl} \mathcal{A}_2 u_2(x,y) & = f(x) - \mathcal{A}_1 u_1(x,y) - \mathcal{A}_0 u(x) \quad \text{in } Y \\ u_2(\cdot, y) & \text{is } Y\text{-periodic in } y \\ \int_Y u_2(x,y)\, dy & = 0. \end{array} \right.$$

For the above equation to be solvable, by Theorem 1.3.1, it is necessary that $f(x) - \mathcal{A}_1 u_1(x,y) - \mathcal{A}_0 u(x) \in (W_{\mathrm{per}}(Y))^\star$ or, equivalently,

$$\int_Y f(x) = \int_Y \left(\mathcal{A}_1 u_1(x,y) + \mathcal{A}_0 u(x) \right) \, dy.$$

To check this, let us first consider

$$
\begin{aligned}
I &:= \int_Y \mathcal{A}_1 u_1(x,y)\,dy \\
&= -\sum_{i,j=1}^{n} \int_Y \left[\frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial u_1}{\partial y_j}\right) + \frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial u_1}{\partial x_j}\right)\right] dy \\
&= -\sum_{i,j=1}^{n} \left[\int_Y \frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial u_1}{\partial y_j}\right) dy + \int_Y a_{ij}(y)\frac{\partial u_1}{\partial x_j}\frac{\partial 1}{\partial y_i}\,dy\right] \\
&= -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left\{\int_Y a_{ij}(y)\left[\frac{\partial}{\partial y_j}\left(\tilde{u}(x) - \sum_{k=1}^{n}\chi_k(y)\frac{\partial u(x)}{\partial x_k}\right)\right] dy\right\} \\
&= \sum_{i,j,k=1}^{n} \frac{\partial}{\partial x_i}\left[\int_Y \left(a_{ij}(y)\frac{\partial \chi_k(y)}{\partial y_j}\right) dy \frac{\partial u(x)}{\partial x_k}\right] \\
&= \sum_{i,j,k=1}^{n} \left[\int_Y \left(a_{ij}(y)\frac{\partial \chi_k(y)}{\partial y_j}\right) dy\right] \frac{\partial^2 u(x)}{\partial x_i \partial x_k}.
\end{aligned}
$$

Next, we consider

$$
\begin{aligned}
\int_Y \mathcal{A}_0 u(x)\,dy &= -\int_Y \sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(y)\frac{\partial u(x)}{\partial x_j}\right) dy \\
&= -\sum_{i,j=1}^{n} \left(\int_Y a_{ij}(y)\,dy\right) \frac{\partial^2 u(x)}{\partial x_i \partial x_j}.
\end{aligned}
$$

For convenience, we rewrite the above relation by replacing the index j with k to get

$$
\int_Y \mathcal{A}_0 u(x)\,dy = -\sum_{i,k=1}^{n} \left(\int_Y a_{ik}(y)\,dy\right) \frac{\partial^2 u(x)}{\partial x_i \partial x_k}.
$$

Therefore, we need to check that

$$
\begin{aligned}
\int_Y f(x)\,dy &= \sum_{i,k=1}^{n} \left[\int_Y \left(\sum_{j=1}^{n} a_{ij}(y)\frac{\partial \chi_k(y)}{\partial y_j} - a_{ik}(y)\right) dy\right] \frac{\partial^2 u(x)}{\partial x_i \partial x_k} \\
f(x) &= -\sum_{i,k=1}^{n} a_{ik}^0 \frac{\partial^2 u(x)}{\partial x_i \partial x_k},
\end{aligned}
$$

where

$$a_{ik}^0 = \frac{1}{|Y|}\int_Y \left(a_{ik}(y) - \sum_{j=1}^n a_{ij}(y)\frac{\partial \chi_k(y)}{\partial y_j} \right) dy. \tag{1.5.7}$$

Proposition 1.5.1. *The $n \times n$ matrix A_0 with the entries (1.5.7) satisfies the ellipticity condition.*

Proof. Let $a_Y : W_{\text{per}}(Y) \times W_{\text{per}}(Y) \to \mathbb{R}$ be a bilinear form defined as

$$a_Y(\phi,\psi) = \int_Y A(y)\nabla_y\phi(y)\cdot\nabla_y\psi(y)\,dy.$$

Now, by the weak formulation of (1.5.6), $\chi_j \in W_{\text{per}}(Y)$ satisfies $a_Y(\chi_j,\psi) = a_Y(y_j,\psi)$ forall $\psi \in W_{\text{per}}(Y)$. Choosing the test function ψ to be χ_i, we get $a_Y(\chi_j - y_j, \chi_i) = 0$. On the other hand,

$$|Y|a_{ij}^0 = \int_Y A(y)\nabla_y(y_j - \chi_j(y))\cdot\nabla_y y_i\,dy = a_Y(y_j - \chi_j, y_i).$$

Hence, we see that $|Y|a_{ij}^0 = a_Y(y_j - \chi_j, y_i - \chi_i)$. Therefore, for any non-zero vector $\xi \in \mathbb{R}^n$,

$$\begin{aligned}
A_0\xi\cdot\xi &= \sum_{i,j=1}^n a_{ij}^0\xi_j\xi_i \\
&= \sum_{i,j=1}^n \frac{1}{|Y|}\int_Y A(y)\nabla_y\left(\xi_j[y_j - \chi_j(y)]\right)\cdot\nabla_y\left(\xi_i[y_i - \chi_i(y)]\right)\,dy \\
&= \frac{1}{|Y|}\int_Y A(y)\nabla_y\left(\sum_{j=1}^n \xi_j[y_j - \chi_j]\right)\cdot\nabla_y\left(\sum_{i=1}^n \xi_i[y_i - \chi_i]\right)\,dy \\
&\geq \frac{\alpha|\xi|^2}{|Y|}\int_Y |\nabla_y\eta|^2\,dy = \alpha_0|\xi|^2
\end{aligned}$$

where

$$\alpha_0 = \frac{\alpha}{|Y|}\int_Y |\nabla_y\eta|^2\,dy$$

and $\eta(y) := \sum_{i=1}^n \frac{\xi_i}{|\xi|}(y_i - \chi_i(y))$. To show the ellipticity of (a_{ij}^0), it is enough to show that $\alpha_0 > 0$. On the contrary, suppose $\alpha_0 = 0$ then $|\nabla_y\eta(y)| = 0$ for a.e. $y \in Y$. Since Y is connected, η is constant on Y, say C_0. Thus,

$\sum_{i=1}^{n} \xi_i y_i = \sum_{i=1}^{n} \xi_i \chi_i(y) + |\xi| C_0$. Since χ_i is such that its average is zero, we obtain

$$\sum_{i=1}^{n} \frac{1}{|Y|} \int_Y y_i \, dy = |\xi| C_0.$$

This is not possible for non-zero ξ due to the periodicity of χ_i. WHY??????? □

From the result proved abvoe and the fact that $f \in H^{-1}(\Omega)$, the equation

$$-\sum_{i,k=1}^{n} a_{ik}^0 \frac{\partial^2 u(x)}{\partial x_i \partial x_k} = f(x) \tag{1.5.8}$$

is admits a unique solution, by Lax-Milgram result, $u(x) \in H_0^1(\Omega)$. Observe that the form of (1.5.8) is similar to (1.4.1), but without ε dependence. Thus, (1.5.8) is called the *homogenized* form (1.4.1). The homogenized operator (or effective coefficient) $A_0 = (a_{ik}^0)$ is computed by first computing χ_k in the cell Y using (1.5.6) and using (1.5.7) to compute A_0. Note that a_{ik}^0 are all constant and hence the homogenized equation has constant coefficients. But we caution here that this is very specific to the periodic case.

Now that we have checked that $f(x) - \mathcal{A}_1 u_1(x,y) - \mathcal{A}_0 u(x) \in (W_{\text{per}}(Y))^\star$, by Theorem 1.3.1, we have $u_2(\cdot, y) \in H^1_{\text{per}}(Y)$ which are unique up to constant. We wish to solve for u_2 using the equation

$$\mathcal{A}_2 u_2(x,y) = f(x) - \mathcal{A}_1 u_1(x,y) - \mathcal{A}_0 u(x) \text{ in } Y.$$

Simplifying, as before, we get

$$\begin{aligned} \mathcal{A}_1 u_1(x,y) + \mathcal{A}_0 u(x) &= \sum_{i,k=1}^{n} \left[\sum_{j=1}^{n} a_{ij}(y) \frac{\partial \chi_k(y)}{\partial y_j} - a_{ik}(y) \right] \frac{\partial^2 u(x)}{\partial x_i \partial x_k} \\ &\quad - \sum_{i,j=1}^{n} \frac{\partial}{\partial y_i} \left(a_{ij}(y) \frac{\partial u_1}{\partial x_j} \right) \end{aligned}$$

We first compute the term corresponding to u_1,

$$\begin{aligned}
\sum_{i,j=1}^{n} \frac{\partial}{\partial y_i}\left(a_{ij}(y)\frac{\partial u_1}{\partial x_j}\right) &= \sum_{i,j=1}^{n}\left(\frac{\partial a_{ij}(y)}{\partial y_i}\frac{\partial u_1}{\partial x_j} + a_{ij}(y)\frac{\partial^2 u_1}{\partial y_i \partial x_j}\right) \\
&= \sum_{i,j=1}^{n} \frac{\partial a_{ij}(y)}{\partial y_i}\frac{\partial \tilde{u}(x)}{\partial x_j} - \sum_{i,j,k=1}^{n}\frac{\partial a_{ij}(y)}{\partial y_i}\chi_k(y)\frac{\partial^2 u(x)}{\partial x_j \partial x_k} \\
&\quad - \sum_{i,j,k=1}^{n} a_{ij}(y)\frac{\partial \chi_k(y)}{\partial y_i}\frac{\partial^2 u(x)}{\partial x_j \partial x_k} \\
&= \sum_{i,j=1}^{n} \frac{\partial a_{ij}(y)}{\partial y_i}\frac{\partial \tilde{u}(x)}{\partial x_j} - \sum_{i,j,k=1}^{n}\frac{\partial [a_{ij}(y)\chi_k(y)]}{\partial y_i}\frac{\partial^2 u(x)}{\partial x_j \partial x_k}
\end{aligned}$$

Therefore the equation for u_2 becomes, $\mathcal{A}_2 u_2(x,y) =$

$$\begin{aligned}
f(x) &+ \sum_{i,j=1}^{n} \frac{\partial a_{ij}(y)}{\partial y_i}\frac{\partial \tilde{u}(x)}{\partial x_j} \\
&- \sum_{i,k=1}^{n}\left[\sum_{j=1}^{n}\left(a_{ij}(y)\frac{\partial \chi_k(y)}{\partial y_j} + \frac{\partial [a_{ji}(y)\chi_k(y)]}{\partial y_j}\right) - a_{ik}(y)\right]\frac{\partial^2 u(x)}{\partial x_i \partial x_k}.
\end{aligned}$$

Now, using the homogenized equation (1.5.8) for f in the above relation, we get $\mathcal{A}_2 u_2(x,y) =$

$$\begin{aligned}
&- \sum_{i,k=1}^{n}\left[\sum_{j=1}^{n}\left(a_{ij}(y)\frac{\partial \chi_k(y)}{\partial y_j} + \frac{\partial [a_{ji}(y)\chi_k(y)]}{\partial y_j}\right) - a_{ik}(y) + a^0_{ik}\right]\frac{\partial^2 u(x)}{\partial x_i \partial x_k} \\
&+ \sum_{i,j=1}^{n} \frac{\partial a_{ij}(y)}{\partial y_i}\frac{\partial \tilde{u}(x)}{\partial x_j}
\end{aligned}$$

As before, we are motivated to define, for each $i,k = 1,2,\ldots,n$, the auxiliary periodic function θ_{ik} as a solution to the problem

$$\left\{\begin{array}{rll}
\mathcal{A}_2\theta_{ik}(y) & = a_{ik}(y) - a^0_{ik} - \sum_{j=1}^{n}\left(a_{ij}(y)\frac{\partial \chi_k(y)}{\partial y_j} + \frac{\partial [a_{ji}(y)\chi_k(y)]}{\partial y_j}\right) & \text{in } Y \\
\theta_{ik}(y) & \text{is } Y\text{-periodic in } y & \\
\int_Y \theta_{ik}(y)\,dy & = 0. &
\end{array}\right. \tag{1.5.9}$$

Substituting the auxiliary problems (1.5.9) and (1.5.6) in the equation of u_2, we get

$$\mathcal{A}_2 u_2(x,y) = \mathcal{A}_2 \left(\sum_{i,k=1}^{n} \theta_{ik} \frac{\partial^2 u(x)}{\partial x_i \partial x_k} - \sum_{j=1}^{n} \chi_j(y) \frac{\partial \tilde{u}(x)}{\partial x_j} \right)$$

Therefore, we have

$$u_2(x,y) = \sum_{i,k=1}^{n} \theta_{ik} \frac{\partial^2 u(x)}{\partial x_i \partial x_k} - \sum_{j=1}^{n} \chi_j(y) \frac{\partial \tilde{u}(x)}{\partial x_j} + \tilde{u}_2(x)$$

for some function $\tilde{u}_2(x)$. This way one can proceed for all values of u_k, using the recurrence relation (1.5.5). Finally, substituting the values of $u_k(x,y)$ in (1.5.1), we have

$$\begin{aligned} u_\varepsilon(x) &= u(x) + \varepsilon \left(-\sum_{j=1}^{n} \chi_j(y) \frac{\partial u(x)}{\partial x_j} + \tilde{u}(x) \right) \\ &\quad + \varepsilon^2 \left(\sum_{i,k=1}^{n} \theta_{ik} \frac{\partial^2 u(x)}{\partial x_i \partial x_k} - \sum_{j=1}^{n} \chi_j(y) \frac{\partial \tilde{u}(x)}{\partial x_j} + \tilde{u}_2(x) \right) + \ldots \end{aligned}$$

The following theorem summarises the result we have obtained above.

Theorem 1.5.2. *Let $f \in H^{-1}(\Omega)$ and u_ε be the solution of (1.4.1). Then there exists $u \in H_0^1(\Omega)$ such that*

$$u_\varepsilon(x) \to u(x) \text{ as } \varepsilon \to 0$$

where $u \in H_0^1(\Omega)$ is the unique solution of (1.5.8) and $A_0 = (a_{ij}^0)$ is a matrix with constant entries and is elliptic. Further, the effective coefficients a_{ij}^0 depend only on the matrix A (we started with), and not on any other data, viz., f and Ω etc.

Mathematically the result obtained is not precise, for we ignored various crucial issues during our computation. For instance, we have conveniently assumed the differentiability of the $L^\infty(Y)$ functions a_{ij} on Y. Also, we have at few places conveniently swapped derivative and integral. In spite of these pitfalls the asymptotic expansion approach gives a fair idea on what is expected to be the homogenized form of (1.4.1).

Example 1.3. Let us understand the above method in the one dimensional case. Let $a : [0,1] \to \mathbb{R}$ be a function such that $0 < \alpha \leq a(x) < \beta$ a.e. in $[0,1]$. Thus, a satisfies the ellipticity condition and is in $L^\infty(Y)$, and is extended periodically to all of $\mathbb{R}$ and $a_\varepsilon(x) = a(x/\varepsilon)$, for $x \in (a,b) \subset \mathbb{R}$. Let $Y = (0,1)$ and the equation (1.4.1) takes the form

$$\begin{cases} -\frac{d}{dx}(a_\varepsilon(x)\frac{du_\varepsilon(x)}{dx}) & = f(x) & \text{in } (a,b) \\ u_\varepsilon(a) & = u_\varepsilon(b) & = 0. \end{cases}$$

We know that the asymptotic expansion of u_ε involves the function u which is a solution to the homogenized equation (1.5.8). Thus, to compute u we need to find the effective coefficient a_0 such that

$$-\frac{d}{dx}\left(a_0(x)\frac{du(x)}{dx}\right) = f(x) \text{ in } (a,b)$$

We already know that a_0 can be computed by finding the function χ that solves (1.5.6), in one dimension which takes the form

$$\begin{cases} -\frac{d}{dy}\left(a(y)\frac{d\chi(y)}{dy}\right) & = -\frac{da(y)}{dy} & \text{in } (0,1) \\ \chi(y) & \text{is} & Y\text{-periodic in } y \\ \int_Y \chi(y)\,dy & = 0. \end{cases}$$

Simplifying the differential equation, we get

$$-\frac{d}{dy}\left(a(y)\left[\frac{d\chi(y)}{dy} - 1\right]\right) = 0$$

and, therefore, $a(y)\chi'(y) = a(y) + c$, for some constant c. This first order differential equation will admit a periodic solution iff

$$\int_Y \left(1 + \frac{c}{a(y)}\right)dy = 1 + c\int_Y \frac{1}{a(y)}\,dy = 0.$$

Note that due to the ellipticity condition, $a(y) > 0$ for all y, and hence there is no division by zero. The effective coefficient is given by formula (1.5.7)

$$a_0 = \int_Y \left(a(y) - a(y)\frac{d\chi(y)}{dy}\right)dy = \int_Y (a(y) - a(y) - c)\,dy = -c.$$

Hence,

$$a_0 = \left(\int_Y \frac{1}{a(y)} \, dy \right)^{-1}.$$

The interesting fact to be observed here is that the effective coefficient a_0 is the inverse of the weak limit in $L^p(\Omega)$ of $1/a_\varepsilon$ rather that the weak limit of a_ε, as one would expect. So, the homogenized coefficient is not always same as taking the averages. $\square$

We caution that for the homogenized equation (1.5.8) to be solvable we need to check that the effective coefficients a^0_{ik} satisfy the ellipticity condition and are bounded. We shall address these questions after we give a precise mathematical formulation of deriving the homogenized equation.

Chapter 2

Two-Scale Convergence

The aim of this chapter is to give a rigorous treatment of the formal asymptotic expansion of homogenization problems with periodic coefficients, discussed in previous chapters. The two-scale method justifies the formal expansion. The notion of *two-scale convergence* was introduced by G. Nguetseng (cf. [Ngu89]) in 1989 and further developed by G. Allaire (cf. [All92, All94, LNW02]).

In Chapter 1, we studied (1.4.1) by introducing a formal two-scale asymptotic expansion for the solution u_ε, i.e.,

$$u_\varepsilon(x) = \sum_{i=0}^{\infty} \varepsilon^i u_i \left(x, \frac{x}{\varepsilon}\right).$$

The above power series expansion was proposed, expecting that the solution u_ε will exhibit a two-scale oscillation, viz., in x and $y = \frac{x}{\varepsilon}$ variables. This is due to the presence of two scales in (1.4.1). The formal two-scale asymptotic expansion method is not rigorous. Two-scale convergence method incorporates the lessons learnt from the two-scale asymptotic expansion and gives rise to a rigorous justification of the homogenization process. In this chapter, we shall consider more general coefficients $A(x, y) = (a_{ij}(x, y))$ defined on $\Omega \times Y$ instead of $A(y)$.

2.1 Vector-valued Function Spaces

Let X be a Banach space. Let $\mathcal{D}'(\Omega; X)$ denote the class of all linear continuous X-valued functions on $\mathcal{D}(\Omega)$. For $1 \leq p \leq \infty$, let $L^p(\Omega; X)$ denote the class of all measurable functions $f : \Omega \to X$ such that $\int_\Omega \|f(x)\|_X^p < \infty$.

Theorem 2.1.1. *For $1 \le p \le \infty$, the space $L^p(\Omega; X)$ is a Banach space w.r.t the norm*

$$\|f\|_{p,\Omega,X} = \left(\int_\Omega \|f(x)\|_X^p \, dx \right)^{1/p}.$$

Further, for $1 < p < \infty$, if X is reflexive then $L^p(\Omega; X)$ is reflexive. Also, for $1 \le p < \infty$, if X is separable then $L^p(\Omega; X)$ is separable.

Theorem 2.1.2 (Pettis' theorem (cf. [Yos95]))**.** *Let X be a separable Banach space X. A function $f : \Omega \to X$ is measurable if and only if the real-valued functions $x \mapsto \langle G, f(x) \rangle$ is measurable for every $G \in X^\star$, the dual of X.*

If X is chosen to be the Banach space $C_{\text{per}}(Y)$, the set of all Y-periodic functions in $C(\mathbb{R}^n)$, then $f \in L^p(\Omega; C_{\text{per}}(Y))$ implies that, for each $x \in \Omega$, $f(x) : \mathbb{R}^n \to \mathbb{R}$ is a Y-periodic function. Thus, f can be seen as a real-valued two variable function on $\Omega \times \mathbb{R}^n$.

Theorem 2.1.3 (cf. See [All92, LNW02])**.** *A function $f \in L^1(\Omega; C_{per}(Y))$ if and only if there exists a zero measure subset $E \subset \Omega$ such that:*

(a) for any $x \in \Omega \setminus E$, the function $y \mapsto f(x, y)$ is continuous and Y-periodic;

(b) for any $y \in Y$, the function $x \mapsto f(x, y)$ is measurable;

(c) the map $x \mapsto \sup_{y \in Y} |f(x, y)|$ is in $L^1(\Omega)$, i.e.,

$$\int_\Omega \sup_{y \in Y} |f(x, y)| dx < \infty.$$

Proof. Suppose $f \in L^1(\Omega; C_{\text{per}}(Y))$, then (a) and (c) are obvious from definitions and it only remains to prove (b). Since the map $f : \Omega \to C_{\text{per}}(Y)$ is measurable, by Pettis' theorem, $x \mapsto \langle G, f(x) \rangle$ is measurable for every $G \in [C_{\text{per}}(Y)]^\star$. In particular, for any fixed $y \in Y$, choose G to be the Dirac measure δ_y at y, i.e., $\langle \delta_y, g \rangle = \int_Y g(t) \, d\delta_y = g(y)$, for all $g \in C_{\text{per}}(Y)$. Thus,

$$x \mapsto \langle \delta_y, f(x) \rangle = f(x, y)$$

is measurable.

Conversely, if (a), (b) and (c) are satisfied then it only remains to prove that the map $f : \Omega \to C_{\text{per}}(Y)$ is measurable. By Pettis' theorem, it is enough to prove that $x \mapsto \langle G, f(x)\rangle$ is measurable, for every $G \in [C_{\text{per}}(Y)]^\star$. Without loss of generality, assume G to be a positive functional because any G can be split into positive and negative functionals. By Riesz representation, there is a unique positive measure μ_G such that $\langle G, g\rangle = \int_Y g\, d\mu_G$, for all $g \in C_{\text{per}}(Y)$. We now approximate G by a sequence of functionals G_k which are finite linear combination of Dirac measures. Let $\{Y_i\}_1^k$ be a partition of Y into k-disjoint cubes of side length $\frac{1}{k}$ and $\lambda_i := \mu_G(Y_i)$. Let $y_i \in Y_i$, χ_{Y_i} be the characteristic function of Y_i in Y, extended periodically to $\mathbb{R}^n$ and $\delta_i := \delta_{y_i}$ be the Dirac measure at y_i. Define G_k as,

$$\langle G_k, g\rangle := \sum_{i=1}^{k} \int_Y g(y)\, d(\lambda_i \delta_i) = \sum_{i=1}^{k} \lambda_i g(y_i).$$

Note that $x \mapsto \langle G_k, f(x)\rangle = \sum_{i=1}^k \lambda_i f(x, y_i)$ is measurable because it is sum of measurable functions. We now claim that

$$\lim_{k\to\infty} \langle G_k, f(x)\rangle = \langle G, f(x)\rangle$$

which will imply that $x \mapsto \langle G, f(x)\rangle$ is measurable. Note that

$$\begin{aligned}
\langle G_k, f(x)\rangle &= \sum_{i=1}^{k} \lambda_i f(x, y_i) = \sum_{i=1}^{k} f(x, y_i)\mu_G(Y_i) \\
&= \int_Y \left(\sum_{i=1}^{k} f(x, y_i)\chi_{Y_i}(y) \right) d\mu_G(y).
\end{aligned}$$

For each $x \in \Omega$, let $A_k(x)$ be the class of all simple functions of the form $s_x(y) = \sum_{i=1}^k \alpha_i(x)\chi_{Y_i}(y)$ and $s_x(y) \le f(x, y)$, for all $y \in Y$. Also, let $S(x)$ be the class of all simple functions satisfying $s_x(y) \le f(x, y)$, for all $y \in Y$.

Then

$$
\begin{aligned}
\lim_{k\to\infty}\langle G_k, f(x)\rangle &= \lim_{k\to\infty}\int_Y \left(\sum_{i=1}^{k} f(x,y_i)\chi_{Y_i}(y)\right) d\mu_G(y) \\
&\le \lim_{k\to\infty}\left[\sup_{s_x\in A_k(x)}\int_Y s_x(y)\,d\mu_G(y)\right] \\
&\le \sup_{s_x\in S(x)}\int_Y s_x(y)\,d\mu_G(y) \\
&\le \int_Y f(x,y)\,d\mu_G(y) \\
&= \langle G, f(x)\rangle.
\end{aligned}
$$

By taking $-f$ instead of f in the above argument and by linearity of duality, we obtain the reverse inequality and, hence, the equality. □

For $1 \le p < \infty$, let $X_p(\Omega; Y)$ generically denote one of the following spaces: $L^p(\Omega; C_{\text{per}}(Y))$, $L^p_{\text{per}}(Y; C(\Omega))$, $C(\bar{\Omega}; C_{\text{per}}(Y))$. The demand of smoothness in one of the variables is mandatory. The space $X_p(\Omega; Y)$ is a separable Banach space and $X_p(\Omega; Y)$ is dense in $L^p(\Omega \times Y)$.

Theorem 2.1.4. *Let $1 \le p < \infty$. For any $\phi \in L^p(\Omega; C_{per}(Y))$, the functions $\phi^\varepsilon(x) := \phi\left(x, \frac{x}{\varepsilon}\right)$ are measurable in x and satisfies:*

(a)

$$\|\phi^\varepsilon\|_{p,\Omega} \le \|\phi\|_{L^p(\Omega; C_{per}(Y))} \tag{2.1.1}$$

and, hence, $\phi^\varepsilon \in L^p(\Omega)$;

(b)

$$\phi^\varepsilon \rightharpoonup \frac{1}{|Y|}\int_Y \phi(\cdot, y)\,dy \text{ weakly in } L^p(\Omega). \tag{2.1.2}$$

In particular, for $p = 2$, $\|\phi^\varepsilon\|^2_{2,\Omega} \to \frac{1}{|Y|}\|\phi\|^2_{2,\Omega\times Y}$.

Proof. We shall prove the result only for $X_p(\Omega; Y) = L^1(\Omega; C_{\text{per}}(Y))$ because the proof is similar in all other cases. By Theorem 2.1.3, the functions ϕ^ε are Caratheodory (cf. [All92, ET74]) and hence they are measurable. The inequality (2.1.1) is easy to conclude because

$$\|\phi^\varepsilon\|_{1,\Omega} = \int_\Omega \left|\phi\left(x, \frac{x}{\varepsilon}\right)\right| dx \le \int_\Omega \sup_{y\in Y}|\phi(x,y)|\,dx = \|\phi\|_{L^1(\Omega; C_{\text{per}}(Y))}.$$

We shall now prove (2.1.2). Consider the partition $\{Y_i\}_1^k$ of Y and $y_i \in Y_i$ as in Theorem 2.1.3. Let χ_{Y_i} be the characteristic function of Y_i in Y, extended periodically to $\mathbb{R}^n$. Define the step functions

$$\phi_k(x,y) = \sum_{i=1}^{k} \phi(x,y_i)\chi_{Y_i}(y).$$

Note that the map $x \mapsto \phi_k(x,y_i)$ is in $L^1(\Omega)$. Define

$$\phi_k^\varepsilon(x) := \phi_k\left(x, \frac{x}{\varepsilon}\right) = \sum_{i=1}^{k} \phi(x,y_i)\chi_{Y_i}\left(\frac{x}{\varepsilon}\right).$$

But

$$\chi_{Y_i}\left(\frac{x}{\varepsilon}\right) \rightharpoonup \frac{1}{|Y|}\int_Y \chi_{Y_i}(y)\,dy \quad \text{weak-* in } L^\infty(\Omega).$$

Therefore, for each fixed $k \in \mathbb{N}$ and $\psi \in L^\infty(\Omega)$,

$$\begin{aligned}
\lim_{\varepsilon\to 0}\int_\Omega \phi_k^\varepsilon(x)\psi(x)\,dx &= \sum_{i=1}^{k}\lim_{\varepsilon\to 0}\left[\int_\Omega \phi(x,y_i)\chi_{Y_i}\left(\frac{x}{\varepsilon}\right)\psi(x)\,dx\right] \\
&= \frac{1}{|Y|}\sum_{i=1}^{k}\int_\Omega \phi(x,y_i)\psi(x)\int_Y \chi_{Y_i}(y)\,dy\,dx \\
&= \frac{1}{|Y|}\int_\Omega\int_Y \phi_k(x,y)\psi(x)\,dy\,dx.
\end{aligned}$$

Thus, (2.1.2) is true for step functions. But, for each fixed $k \in \mathbb{N}$ and $\psi \in L^\infty(\Omega)$,

$$\begin{aligned}
\left|\int_\Omega \left[\phi^\varepsilon(x) - \frac{1}{|Y|}\int_Y \phi(x,y)\,dy\right]\psi(x)\,dx\right| &\leq \int_\Omega |\phi^\varepsilon(x) - \phi_k^\varepsilon(x)|\,|\psi(x)|\,dx \\
&+ \int_\Omega \left|\phi_k^\varepsilon - \frac{1}{|Y|}\int_Y \phi_k\,dy\right|\,|\psi|\,dx \\
&+ \frac{1}{|Y|}\int_{\Omega\times Y} |\phi_k - \phi|\,|\psi(x)|\,dy\,dx.
\end{aligned}$$

As $\varepsilon \to 0$, the second term, consisting of step functions, converges to zero. In the first and third term are smaller than its supremum w.r.t y-variable.

Thus,

$$\begin{aligned}\lim_{\varepsilon\to 0}\left|\int_\Omega \left[\phi^\varepsilon - \frac{1}{|Y|}\int_Y \phi\, dy\right]\psi\, dx\right| &\le 2\int_\Omega \sup_{y\in Y}|\phi(x,y)-\phi_k(x,y)|\,|\psi(x)|\, dx \\ &= 2\,\|(\phi-\phi_k)\psi\|_{L^1(\Omega;C_{\mathrm{per}}(Y))}.\end{aligned}$$

Define

$$g_k(x) := \sup_{y\in Y}|\phi(x,y)-\phi_k(x,y)|\,|\psi(x)|.$$

By the continuity of ϕ in y-variable, $g_k(x)\to 0$, as $k\to\infty$, pointwise for a.e $x\in\Omega$. Further, $g_k(x)\le 2\sup_{y\in Y}|\phi(x,y)|\,|\psi(x)|$ in $L^1(\Omega)$. Hence, by Lebesgue's dominated convergence result, $\|(\phi-\phi_k)\psi\|_{L^1(\Omega;C_{\mathrm{per}}(Y))}\to 0$. Thus, (2.1.2) is proved. □

Theorem 2.1.5 (cf. [BM]). *Let $1\le p<\infty$. Suppose $\phi(x,y)=\phi_1(x)\phi_2(y)$ such that $\phi_1\in L^s(\Omega)$ and $\phi_2\in L^t_{per}(Y)$ with $1\le s,t<\infty$ and $\frac{1}{s}+\frac{1}{t}=\frac{1}{p}$. Then $\phi^\varepsilon(x)=\phi\left(x,\frac{x}{\varepsilon}\right)\in L^p(\Omega)$ and*

$$\phi^\varepsilon \rightharpoonup \frac{\phi_1(\cdot)}{|Y|}\int_Y \phi_2(y)\, dy \quad \text{weakly in } L^p(\Omega).$$

2.2 Two-scale Convergence

The notion of two-scale convergence was introduced by G. Nguetseng (cf. [Ngu89]) for L^2-spaces and, then, generalized to L^p spaces in [All92, LNW02]. Recall that a sequence $u_\varepsilon(x) := u\left(x,\frac{x}{\varepsilon}\right)$ in $L^2(\Omega)$, with u being Y-periodic in second variable, will weakly converge to $\frac{1}{|Y|}\int_Y u(x,y)\, dy$ (cf. Theorem 1.1.3). Thus, weak limit of an oscillating sequence does not capture the oscillations. Two-scale convergence is a generalization of weak convergence such that the limit captures the oscillations.

Definition 2.2.1. *Let $1<p,q<\infty$ such that $\frac{1}{p}+\frac{1}{q}=1$. A sequence $\{u_\varepsilon\}\subset L^p(\Omega)$ is said to* two-scale converge *to $u\in L^p(\Omega\times Y)$ if*

$$\int_\Omega u_\varepsilon(x)\phi\left(x,\frac{x}{\varepsilon}\right)\, dx \to \frac{1}{|Y|}\int_\Omega\int_Y u(x,y)\phi(x,y)\, dy\, dx, \qquad (2.2.1)$$

for all $\phi\in L^q(\Omega;C_{per}(Y))$. The convergence is denoted as $u_\varepsilon \overset{2s}{\rightharpoonup} u$.

Theorem 2.2.2 (Uniqueness)**.** *The two-scale limit is unique.*

Proof. If $u, v \in L^p(\Omega \times Y)$ are two distinct two-scale limits of a sequence $\{u_\varepsilon\} \subset L^p(\Omega)$, then

$$\int_\Omega \int_Y [u(x,y) - v(x,y)]\phi(x,y)\,dy\,dx = 0$$

for all $\phi \in L^q(\Omega; C_{\text{per}}(Y))$. Thus, $u = v$ a.e. in $\Omega \times Y$. $\square$

Example 2.1. Let $u_\varepsilon(x) := u\left(x, \frac{x}{\varepsilon}\right)$, where $u \in L^p(\Omega \times Y)$ is smooth and Y-periodic in y-variable. For any $\phi \in L^q(\Omega; C_{\text{per}}(Y))$, the product $u\phi \in L^1(\Omega; C_{\text{per}}(Y))$. By the Y-periodicity of $u\phi$ and Theorem 2.1.4, it follows that $u_\varepsilon \overset{2s}{\rightharpoonup} u$.

Example 2.2. Let $u_\varepsilon := u\left(x, \frac{x}{\varepsilon^2}\right)$, where $u \in L^p(\Omega \times Y)$ is a smooth and Y-periodic in the second variable. For any $\phi \in L^q(\Omega; C_{\text{per}}(Y))$, set $\psi(x,y,z) := u(x,z)\phi(x,y)$. Recall that if $\psi : \Omega \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a function which is Y-periodic in both second variable and third variable, then

$$\psi\left(x, \frac{x}{\varepsilon}, \frac{x}{\varepsilon^2}\right) \rightharpoonup \frac{1}{|Y|^2} \int_Y \int_Y \psi(x,y,z)\,dz\,dy \quad \text{weakly in } L^p(\Omega).$$

Therefore,

$$\int_\Omega u\left(x, \frac{x}{\varepsilon^2}\right) \phi\left(x, \frac{x}{\varepsilon}\right)\,dx \to \frac{1}{|Y|^2} \int_\Omega \int_Y \int_Y u(x,z)\phi(x,y)\,dz\,dy\,dx.$$

Equivalently,

$$u\left(x, \frac{x}{\varepsilon^2}\right) \overset{2s}{\rightharpoonup} \frac{1}{|Y|} \int_Y u(x,z)\,dz.$$

Thus, the two-scale limit is same as the L^p weak limit. More generally, if $Y_i = Y$, for all $i = 1, 2, \ldots, k$,

$$u\left(x, \frac{x}{\varepsilon^2}, \ldots, \frac{x}{\varepsilon^k}\right) \overset{2s}{\rightharpoonup} \frac{1}{|Y|^{k-1}} \int_{Y_2} \cdots \int_{Y_k} u(x, y_2, \ldots, y_k)\,dy_2 \ldots dy_k$$

which is same as the L^p weak limit. If $\psi : \Omega \times \mathbb{R}^n \cdots \times \mathbb{R}^n \to \mathbb{R}$ is a $k+1$ variable function which is Y-periodic in each of the $i+1$ variable, for $i = 1, 2, \ldots, k$, then

$$\psi\left(x, \frac{x}{\varepsilon}, \ldots, \frac{x}{\varepsilon^k}\right) \rightharpoonup \int_{Y_1} \cdots \int_{Y_k} \psi(x, y_1, \ldots, y_k)\,dy_1 \ldots dy_k \text{ weakly in } L^p(\Omega).$$

Remark 2.2.3. Recall that $u_\varepsilon \rightharpoonup u$ weakly in $L^p(\Omega)$ if, for all $\phi \in L^q(\Omega)$,

$$\int_\Omega u_\varepsilon(x)\phi(x)\,dx \to \int_\Omega u(x)\phi(x)\,dx.$$

Thus, the usual weak convergence in $L^p(\Omega)$ hides (averages out) the effect of oscillations in u_ε. In order to capture the oscillations of the form $\frac{x}{\varepsilon}$, one has to treat u_ε with test functions of the form $\phi\left(x, \frac{x}{\varepsilon}\right)$. This was the motivation behind the definition of two-scale convergence. Also, note that, as seen in the above example, the test function $\phi\left(x, \frac{x}{\varepsilon}\right)$ is not good enough to capture higher order oscillations of the form $\frac{x}{\varepsilon^k}$ for $k \geq 2$. To capture these oscillations, one may need to use test functions of the form $\phi\left(x, \frac{x}{\varepsilon^k}\right)$. The basic idea is that one has to treat with test functions with same order of oscillations. This is called the *multi-scale* or *reiterated homogenization*.

Remark 2.2.4. Suppose u_ε admits an asymptotic expansion

$$u_\varepsilon(x) = \sum_{i=0}^{\infty} \varepsilon^i u_i\left(x, \frac{x}{\varepsilon}\right)$$

where u_i's are Y-periodic and smooth in the second variable. Then, by Example 2.1, $u_\varepsilon \overset{2s}{\rightharpoonup} u_0$, the first term in the expansion.

Theorem 2.2.5. *If u_ε converges to u strongly in $L^p(\Omega)$ then $u_\varepsilon \overset{2s}{\rightharpoonup} u$. In particular, the two-scale limit is independent of y-variable.*

Proof. Let $u_\varepsilon \to u$ strongly in $L^p(\Omega)$. For any $\phi \in L^q(\Omega; C_{\text{per}}(Y))$, let $\phi^\varepsilon(x) := \phi\left(x, \frac{x}{\varepsilon}\right)$ and $\bar{\phi}(x) := \frac{1}{|Y|}\int_Y \phi(x,y)\,dy$. Then

$$\begin{aligned}\left|\int_\Omega \left[u_\varepsilon(x)\phi^\varepsilon(x) - u(x)\bar{\phi}(x)\right]\,dx\right| &\leq \|u_\varepsilon - u\|_{p,\Omega}\|\phi_\varepsilon\|_{q,\Omega} \\ &\quad + \left|\int_\Omega u(x)\left[\phi^\varepsilon(x) - \bar{\phi}(x)\right]\,dx\right|.\end{aligned}$$

By (2.1.1), $\|\phi_\varepsilon\|_{q,\Omega}$ is uniformly bounded. Moreover, the strong convergence of u_ε implies that the first term goes to zero. By (2.1.2) and $u \in L^p(\Omega)$, the second term goes to 0. □

Example 2.3. The converse of above result is not true, i.e., two-scale convergence need not imply strong convergence. Consider the function $u \in$

$L^2([0,1] \times [0,1])$, defined as $u(x,y) = \sin(2\pi y)$, and define the sequence $u_\varepsilon(x) := \sin\left(\frac{2\pi x}{\varepsilon}\right)$ in $L^2[0,1]$. Because $\{u_\varepsilon\}$ converge weakly to 0 in $L^2[0,1]$ (periodic oscillating function weakly converges to average), if it strong converges then the limit must be 0. But $\|\sin(2\pi x/\varepsilon)\|_{2,[0,1]} = 1/2$ and, hence, do not strongly converge. The two-scale limit of the sequence $u_\varepsilon(x) := \sin\left(\frac{2\pi x}{\varepsilon}\right)$ is $u(x,y) = \sin(2\pi y)$ on $[0,1] \times [0,1]$.

Theorem 2.2.6. *For any sequence $u_\varepsilon \subset L^p(\Omega)$, if $u_\varepsilon \xrightharpoonup{2s} u$ with $u \in L^p(\Omega \times Y)$ then $u_\varepsilon \rightharpoonup \frac{1}{|Y|}\int_Y u(x,y)dy$ weakly in $L^p(\Omega)$. In particular, if the two-scale limit u is independent of y then the two-scale limit and weak limit coincide.*

Proof. Let $u_\varepsilon \xrightharpoonup{2s} u$. Then, in particular, for any $\phi \in L^q(\Omega) \subset L^q(\Omega; C_{\text{per}}(Y))$ (ϕ independent of y),

$$\int_\Omega u_\varepsilon(x)\phi(x)\,dx \to \frac{1}{|Y|}\int_\Omega \int_Y u(x,y)\phi(x)\,dy\,dx.$$

Thus,

$$u_\varepsilon \rightharpoonup \frac{1}{|Y|}\int_Y u(\cdot,y)dy \text{ weakly in } L^p(\Omega).$$

If $u(x,y) = u(x)$ then $\frac{1}{|Y|}\int_Y u(x)\,dy = u(x)$. Thus, weak limit and two-scale limit coincide for y independent functions. □

We have noted that the two-scale convergence is intermediary between strong and weak convergences in $L^p(\Omega)$. If the weak and two-scale limits are different then it means that there is more information in the two-scale limit, than the weak limit, about the oscillations in the sequence.

Example 2.4. The converse of the above result is not true, i.e., weak convergence need not imply two-scale convergence. Consider the sequence $\{u_n\} \subset L^2[0,1]$ defined as

$$u_n(x) = \begin{cases} \sin(2\pi n x) & \text{if } n \text{ is odd} \\ \cos(2\pi n x) & \text{if } n \text{ is even.} \end{cases}$$

This sequence converges weakly to zero in $L^2([0,1]$ but does not two-scale converge.

Corollary 2.2.7. *If a sequence $\{u_\varepsilon\} \subset L^p(\Omega)$ two-scale converges then it is bounded in $L^p(\Omega)$.*

Proof. If u_ε two-scale converges then it also weakly converges in $L^p(\Omega)$. Any weakly convergent sequence in $L^p(\Omega)$ is norm bounded. □

Theorem 2.2.8 (Compactness Theorem)**.** *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$. Then there exists a subsequence of $\{u_\varepsilon\}$ (still denoted by ε) and a $u \in L^p(\Omega \times Y)$ such that $u_\varepsilon \xrightarrow{2s} u$.*

Proof. Step 1 For each u_ε, define $L_\varepsilon : L^q(\Omega; C_{\text{per}}(Y)) \to \mathbb{R}$, as

$$L_\varepsilon(\psi) = \int_\Omega u_\varepsilon(x)\psi\left(x, \frac{x}{\epsilon}\right)\, dx.$$

Note that L_ε is a continuous linear functional on $L^q(\Omega; C_{\text{per}}(Y))$. For any $\psi \in L^q(\Omega; C_{\text{per}}(Y))$, $\psi\left(x, \frac{x}{\varepsilon}\right) \in L^q(\Omega)$. Now, by Hölder's inequality,

$$\begin{aligned} |L_\varepsilon(\psi)| &\leq \left|\int_\Omega u_\varepsilon(x)\psi\left(x, \frac{x}{\epsilon}\right)\, dx\right| \\ &\leq \|u_\varepsilon\|_{p,\Omega}\|\psi(x, \frac{x}{\varepsilon})\|_{q,\Omega} \\ &\leq \|u_\varepsilon\|_{p,\Omega}\|\psi(x,y)\|_{L^q(\Omega;C_{\text{per}}(Y))}. \end{aligned}$$

The last inequality follows from Theorem 2.1.4. Since $\{u_\varepsilon\}$ is bounded in $L^p(\Omega)$, there is a constant $C_0 > 0$ (independent of ε) such that $\|u_\varepsilon\|_{p,\Omega} \leq C_0$. Thus, the sequence $\{L_\varepsilon\}$ is bounded in the dual of $L^q(\Omega; C_{\text{per}}(Y))$.

Step 2 Recall that $L^q(\Omega; C_{\text{per}}(Y))$ is separable. Thus, by Banach-Alaoglu theorem, there is a $L \in [L^q(\Omega; C_{\text{per}}(Y))]^\star$ and a subsequence of $\{L_\varepsilon\}$ such that

$$L_\varepsilon(\psi) \to L(\psi) \quad \forall \psi \in L^q(\Omega; C_{\text{per}}(Y)).$$

Step 3 Passing to the limit, as $\varepsilon \to 0$, in the inequality of Step 1, we obtain

$$|L(\psi)| \leq C_0\|\psi(x,y)||_{L^q(\Omega;C_{\text{per}}(Y))} \quad \forall \psi \in L^q(\Omega; C_{\text{per}}(Y)).$$

Step 4 By the density of $L^q(\Omega; C_{\text{per}}(Y))$ in $L^q(\Omega \times Y)$, we extend L as a bounded linear functional to all of $L^q(\Omega \times Y)$ and denote the extension by $\tilde{L}$. Then

$$|\tilde{L}(\psi)| \leq C_0\|\psi(x,y)\|_{q,\Omega\times Y} \quad \forall \psi \in L^q(\Omega \times Y).$$

By Riesz representation theorem, $\tilde{L} \in [L^q(\Omega \times Y)]^\star$, may be identified with an element $v \in L^p(\Omega \times Y)$. Then, for every $\psi \in L^q(\Omega; C_{\text{per}}(Y))$,

$$\begin{aligned}\lim_{\varepsilon \to 0} \int_\Omega u_\varepsilon(x)\psi\left(x, \frac{x}{\epsilon}\right) dx &= \lim_{\varepsilon \to 0} L_\varepsilon(\psi) = L(\psi) \\ &= \int_{\Omega \times Y} v(x,y)\psi(x,y)\, dy\, dx.\end{aligned}$$

Thus, u_ε two-scale converges to $u(x,y) := |Y| v(x,y)$.

□

Corollary 2.2.9. *Every weakly convergent sequence in $L^p(\Omega)$ has a two-scale converging subsequence.*

Theorem 2.2.10. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$. Then, along a subsequence,*

$$\int_\Omega u_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_Y u(x,y)\phi(x,y)\, dy\, dx$$

for all ϕ such that $\phi(x,y) = \phi_1(x)\phi_2(y)$ where $\phi_1 \in L^s(\Omega)$ and $\phi_2 \in L^t(\Omega)$ with $1 \le s,t < \infty$ and $\frac{1}{s} + \frac{1}{t} = \frac{1}{q}$.

Proof. If ϕ is in the variable separable form the smoothness hypothesis on one of the variable of ϕ may be relaxed and the measurability of $\phi\left(x, \frac{x}{\varepsilon}\right)$ may be derived. The proof of the result is again by approximation (cf. [LNW02]). □

Recall that the L^p-norm is lower-semicontinuous w.r.t the weak topology, i.e., if u_ε converges weakly to u in $L^p(\Omega)$ then

$$\|u\|_{p,\Omega} \le \liminf_{\varepsilon \to 0} \|u_\varepsilon\|_{p,\Omega}.$$

A similar result holds for two-scale convergence.

Theorem 2.2.11. *Let $\{u_\varepsilon\} \subset L^p(\Omega)$ two-scale converge to $u \in L^p(\Omega \times Y)$. Then*

$$\liminf_{\varepsilon \to 0} \|u_\varepsilon\|_{p,\Omega} \ge \frac{1}{|Y|}\|u\|_{p,\Omega \times Y} \ge \|\bar{u}\|_{p,\Omega}, \tag{2.2.2}$$

where $\bar{u} = \frac{1}{|Y|}\int_Y u(x,y)\, dy$.

Proof. If $u \in L^p(\Omega)$ then $|u|^{p-2}u \in L^q(\Omega \times Y)$. Now, choose a sequence $\psi_k \in L^q(\Omega; C_{\text{per}}(Y))$ that converges to $|u|^{p-2}u$ strongly in $L^q(\Omega \times Y)$. By Young's inequality, we obtain

$$p \int_\Omega u_\varepsilon(x)\psi_k\left(x, \frac{x}{\varepsilon}\right)\,dx \leq \int_\Omega |u_\varepsilon(x)|^p\,dx + (p-1)\int_\Omega \left|\psi_k\left(x, \frac{x}{\varepsilon}\right)\right|^q\,dx.$$

Fix k and pass to limit, as $\varepsilon \to 0$, to obtain

$$\begin{aligned}\frac{p}{|Y|}\int_\Omega\int_Y u(x,y)\psi_k(x,y)\,dx\,dy \quad &\leq \quad \liminf_{\varepsilon\to 0}\int_\Omega |u_\varepsilon(x)|^p\,dx \\ &\quad + \frac{p-1}{|Y|}\int_\Omega\int_Y |\psi_k(x,y)|^q\,dy\,dx.\end{aligned}$$

Now, passing to limit, as $k \to \infty$, we obtain

$$\begin{aligned}\frac{p}{|Y|}\int_\Omega\int_Y |u(x,y)|^p\,dy\,dx \quad &\leq \quad \liminf_{\varepsilon\to 0}\int_\Omega |u_\varepsilon(x)|^p\,dx \\ &\quad + \frac{p-1}{|Y|}\int_\Omega\int_Y |u(x,y)|^p\,dy\,dx \\ \frac{1}{|Y|}\int_\Omega\int_Y |u(x,y)|^p\,dy\,dx \quad &\leq \quad \liminf_{\varepsilon\to 0}\int_\Omega |u_\varepsilon(x)|^p\,dx.\end{aligned}$$

This implies the first inequality of (2.2.2). The second inequality in (2.2.2) follows from Jensen's inequality,

$$|Y|^p\|\bar{u}\|^p_{p,\Omega} = \int_\Omega \left|\int_Y u(x,y)\,dy\right|^p\,dx \leq \int_\Omega\int_Y |u(x,y)|^p\,dy\,dx = \|u\|^p_{p,\Omega\times Y}.$$

□

Let $W^{m,p}_{\text{per}}(Y)$ denote the class of functions in $W^{m,p}(\mathbb{R}^n)$ which are Y-periodic.

Theorem 2.2.12 (Compactness in $W^{1,p}$)**.** *For any given $1 \leq p < \infty$, let u_ε be a bounded sequence in $W^{1,p}(\Omega)$. Then there is exists a $u \in W^{1,p}(\Omega)$ and $u_1 \in L^p(\Omega; W^{1,p}_{per}(Y))$ such that, for a subsequence (still denoted by ε),*

$$\begin{aligned} u_\varepsilon \quad &\rightharpoonup \quad u \text{ weakly in } W^{1,p}(\Omega) \\ u_\varepsilon \quad &\overset{2s}{\rightharpoonup} \quad u \text{ in } L^p(\Omega) \\ \nabla u_\varepsilon \quad &\overset{2s}{\rightharpoonup} \quad \nabla u + \nabla_y u_1 \text{ in } [L^p(\Omega)]^n. \end{aligned}$$

Proof. Let $\{u_\varepsilon\}$ be bounded in $W^{1,p}(\Omega)$. By weak compactness, $u_\varepsilon \rightharpoonup u$ weakly in $W^{1,p}(\Omega)$ and, thus, strongly in $L^p(\Omega)$. Hence, u is also the two-scale limit of $\{u_\varepsilon\}$ in $L^p(\Omega)$. Also, $\{\nabla u_\varepsilon\}$ is bounded in $[L^p(\Omega)]^n$. Hence, by two-scale compactness theorem, there exists $v \in [L^p(\Omega \times Y)]^n$ such that

$$\int_\Omega \frac{\partial u_\varepsilon}{\partial x_i}\phi\left(x, \frac{x}{\varepsilon}\right)\,dx \to \frac{1}{|Y|}\int_\Omega \int_Y v_i(x,y)\phi(x,y)\,dy\,dx, \tag{2.2.3}$$

for all $\phi \in L^q(\Omega; C_{\text{per}}(Y))$. In particular, (2.2.3) is valid for all ϕ in the dense subset $\mathcal{D}(\Omega; C^\infty_{\text{per}}(Y))$. Consider $\Phi \in [\mathcal{D}(\Omega; C^\infty_{\text{per}}(Y))]^n$ such that $\text{div}_y \Phi = 0$. Then

$$\begin{aligned}
\int_\Omega \nabla u_\varepsilon \cdot \Phi\left(x, \frac{x}{\varepsilon}\right)\,dx &= -\int_\Omega u_\varepsilon \left[\text{div}_x \Phi\left(x, \frac{x}{\varepsilon}\right) + \varepsilon^{-1}\text{div}_y \Phi\left(x, \frac{x}{\varepsilon}\right)\right]\,dx \\
&= -\int_\Omega u_\varepsilon \text{div}_x \Phi\left(x, \frac{x}{\varepsilon}\right)\,dx \\
&\overset{\text{as } \varepsilon \to 0}{\longrightarrow} \frac{-1}{|Y|}\int_\Omega \int_Y u(x)\text{div}_x \Phi(x,y)\,dy\,dx \\
&= \frac{1}{|Y|}\int_\Omega \int_Y \nabla u(x) \cdot \Phi(x,y)\,dy\,dx.
\end{aligned}$$

Thus, using (2.2.3), we observe that

$$\int_\Omega \int_Y [v(x,y) - \nabla u(x)] \cdot \Phi(x,y)\,dy\,dx = 0$$

for all $\Phi \in [\mathcal{D}(\Omega; C^\infty_{\text{per}}(Y))]^n$ such that $\text{div}_y \Phi = 0$. By a classical result (cf. [Tem79, GR81]), $v(x,y) - \nabla u(x)$ is a gradient with respect to y, i.e., there is a $u_1 \in L^p(\Omega; W^{1,p}_{\text{per}}(Y))$ such that

$$v(x,y) - \nabla u(x) = \nabla_y u_1(x,y).$$

□

Recall that the strong convergence of u_ε to u in $L^p(\Omega)$ is characterized by both the weak convergence, i.e., $u_\varepsilon \rightharpoonup u$ in $L^p(\Omega)$ and norm convergence, i.e., $\|u_\varepsilon\|_{p,\Omega} \to \|u\|_{p,\Omega}$. This motivates the definition of strong two-scale convergence.

Definition 2.2.13. *We say $\{u_\varepsilon\} \subset L^p(\Omega)$* strongly two-scale converges *to $u \in L^p(\Omega \times Y)$, denoted as $u_\varepsilon \xrightarrow{2s} u$ if $u_\varepsilon \overset{2s}{\rightharpoonup} u$ and $\|u_\varepsilon\|_{p,\Omega} \to \frac{1}{|Y|}\|u\|_{p,\Omega\times Y}$.*

The strong two-scale convergence is weaker than strong convergence in $L^p(\Omega)$ and stronger than two-scale convergence.

Recall that the product of two sequences, one converging strongly and the other weakly, is the product of its strong and weak limits. A similar result is valid in the context of two-scale convergence too.

Theorem 2.2.14. *Let u_ε and v_ε be two sequences in $L^p(\Omega)$ and $L^q(\Omega)$, respectively, such that $u_\varepsilon \stackrel{2s}{\to} u$, for $u \in L^p(\Omega \times Y)$, and $v_\varepsilon \stackrel{2s}{\rightharpoonup} v$, for $v \in L^q(\Omega \times Y)$. Then the product $u_\varepsilon v\varepsilon$ converges in the distribution sense, i.e.,*

$$u_\varepsilon v_\varepsilon \rightharpoonup \frac{1}{|Y|}\int_Y u(\cdot,y)v(\cdot,y)\,dy \text{ weak-* in } \mathcal{D}'(\Omega).$$

Further, if $u \in L^p(\Omega; C_{per}(Y))$ then

$$\lim_{\varepsilon\to 0}\left\|u_\varepsilon - u\left(\cdot,\frac{\cdot}{\varepsilon}\right)\right\|_{p,\Omega} = 0.$$

2.3 Classical Definition of Two-Scale Convergence

In the early stages of the development of two-scale convergence, the test function space for ϕ, in the definition of two-scale convergence, was restricted to the sub-class $\mathcal{D}(\Omega; C^\infty_{\text{per}}(Y))$. This was later replaced (cf. [LNW02]) with $L^q(\Omega; C_{\text{per}}(Y))$ to have the implication of weak convergence. Because if $\mathcal{D}(\Omega; C^\infty_{\text{per}}(Y))$ is used as the test function class, then the two-scale convergence of $\{u_\varepsilon\}$ will not imply its weak convergence (and, hence, boundedness in $L^p(\Omega)$).

Example 2.5. Consider $u_n : (0,1) \to \mathbb{R}$ defined as

$$u_n(x) := \begin{cases} n & \text{if } 0 < x < \frac{1}{n} \\ 0 & \text{if } \frac{1}{n} < x < 1. \end{cases}$$

Then, for all $\phi \in \mathcal{D}((0,1); C^\infty_{\text{per}}(0,1))$,

$$\int_0^1 u_n(x)\phi(x,nx)\,dx = n\int_0^{1/n}\phi(x,nx)\,dx = \int_0^1 \phi\left(\frac{z}{n},z\right)\,dz \to 0$$

due to the compact support of ϕ in $(0,1)$. But $\{u_n\}$ is not bounded in $L^2(0,1)$. Also, u_n do not weakly converge to 0 in $L^2(0,1)$ because $\int_0^1 u_n g = 1$ where $g \equiv 1$ is in $L^2(0,1)$.

Example 2.6. The choice of $C(\Omega; C^\infty_{\text{per}}(Y))$ as test function space will, also, not yield any better result. Let $\tilde{u}_\varepsilon$ be the periodic extension of u_ε, defined above, to all of $\mathbb{R}$. Define $v_\varepsilon : (0,1) \to \mathbb{R}$ as

$$v_\varepsilon(x) = \begin{cases} \tilde{u}_\varepsilon(\frac{x}{\epsilon}) & \text{if } \frac{1}{4} < x < \frac{3}{4} \\ 0 & \text{otherwise.} \end{cases}$$

Then, for all $\phi \in C((0,1); C^\infty_{\text{per}}(0,1))$,

$$\int_0^1 v_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) dx \to \int_0^1 \int_0^1 u(x,y)\phi(x,y)\, dy\, dx$$

where

$$u(x,y) = \begin{cases} 1 & \text{if } \frac{1}{4} < x < \frac{3}{4} \\ 0 & \text{otherwise.} \end{cases}$$

Note that $\{v_\varepsilon\}$ is not bounded in $L^2(0,1)$.

Theorem 2.3.1. *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$ and*

$$\int_\Omega u_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) dx \to \frac{1}{|Y|}\int_\Omega \int_Y u(x,y)\phi(x,y)\, dy\, dx \quad \forall \phi \in \mathcal{D}(\Omega; C^\infty_{per}(Y)).$$

Then $u_\varepsilon \stackrel{2s}{\rightharpoonup} u$.

Proof. The idea is to approximate $\phi \in L^q(\Omega; C_{\text{per}}(Y))$ by a sequence $\phi_k \in \mathcal{D}(\Omega; C^\infty_{\text{per}}(Y))$ and use the uniform bound of $\{u_\varepsilon\}$. □

Theorem 2.3.2 (Compactness)**.** *Let $\{u_\varepsilon\}$ be a bounded sequence in $L^p(\Omega)$. Then, along a subsequence,*

$$\int_\Omega u_\varepsilon(x)\phi\left(x, \frac{x}{\varepsilon}\right) dx \to \int_\Omega \int_Y u(x,y)\phi(x,y)\, dy\, dx \quad \forall \phi \in L^q_{per}(Y; C(\bar{\Omega})).$$

Proof. Use the density of $\mathcal{D}(\Omega; C^\infty_{\text{per}}(Y))$ in $L^q_{\text{per}}(Y; C(\bar{\Omega}))$. □

2.4 Homogenization of Second Order Linear Elliptic Problems

Recall from § 1.4 that a Dirichlet problem for a periodic composite material Ω is given as, for a given $f \in H^{-1}(\Omega)$,

$$\left\{ \begin{array}{rll} -\text{div}(A_\varepsilon(x)\nabla u_\varepsilon(x)) & = f(x) & \text{in } \Omega \\ u_\varepsilon & = 0 & \text{on } \partial\Omega \end{array} \right. \tag{2.4.1}$$

where $A_\varepsilon(x) = \left(a_{ij}\left(x, \frac{x}{\varepsilon}\right)\right)$ is in $M(\alpha, \beta, \Omega \times Y)$ and $a_{ij} : \Omega \times Y \to \mathbb{R}$. We know there exists a unique solution $u_\varepsilon \in H_0^1(\Omega)$, by Lax-Milgram result, such that

$$\int_\Omega A_\varepsilon(x)\nabla u_\varepsilon(x) \cdot \nabla v(x)\, dx = \langle f, v\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}, \quad \forall v \in H_0^1(\Omega)$$

and $\|u_\varepsilon\|_{H_0^1(\Omega)} \leq 1/\alpha \|f\|_{H^{-1}(\Omega)}$. Since u_ε is uniformly bounded in $H_0^1(\Omega)$, by Theorem 2.2.12, there exists a $u \in H_0^1(\Omega)$ and $u_1 \in L^2(\Omega; H^1_{\text{per}}(Y))$ such that, for a subsequence

$$\begin{cases} u_\varepsilon \rightharpoonup u & \text{weakly in } H_0^1(\Omega) \text{ and strongly in } L^2(\Omega); \\ u_\varepsilon \stackrel{2s}{\rightharpoonup} u & \text{in } L^2(\Omega); \\ \nabla u_\varepsilon \stackrel{2s}{\rightharpoonup} \nabla u + \nabla_y u_1 & \text{in } [L^2(\Omega)]^n. \end{cases} \tag{2.4.2}$$

Consider the test functions $\phi \in \mathcal{D}(\Omega)$ and $\phi_1 \in \mathcal{D}(\Omega; C^\infty_{\text{per}}(Y))$ and choose $v(x) = \phi(x) + \varepsilon\phi_1\left(x, \frac{x}{\varepsilon}\right)$ in the weak formulation above to obtain

$$\int_\Omega A\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot (\nabla\phi + \varepsilon\nabla\phi_1 + \nabla_y\phi_1)\, dx = \left\langle f, \phi + \varepsilon\phi_1\left(x, \frac{x}{\varepsilon}\right)\right\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Note that $\phi + \varepsilon\phi_1^\varepsilon \rightharpoonup \phi$ weakly in $H_0^1(\Omega)$. Thus, the term in RHS converges, i.e.,

$$\langle f, \phi + \varepsilon\phi_1^\varepsilon\rangle_{H^{-1}(\Omega), H_0^1(\Omega)} \to \langle f, \phi\rangle_{H^{-1}(\Omega), H_0^1(\Omega)}.$$

Note that $\{A_\varepsilon(x)\nabla u_\varepsilon(x)\}$ is bounded in $[L^2(\Omega)]^n$ and, by (2.1.1), $\{\nabla\phi_1^\varepsilon\}$ is bounded in $[L^2(\Omega)]^n$. Using Hölder's inequality, we obtain

$$\left|\int_\Omega A\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot \varepsilon\nabla\phi_1\, dx\right| \leq C\varepsilon.$$

Therefore,

$$\lim_{\varepsilon \to 0} \int_\Omega A\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot \varepsilon\nabla\phi_1\, dx = 0.$$

Let us denote $\psi(x, y) := {}^tA(x, y)(\nabla\phi(x) + \nabla_y\phi_1(x, y))$. Then

$$\int_\Omega A\left(x, \frac{x}{\varepsilon}\right) \nabla u_\varepsilon \cdot \left(\nabla\phi(x) + \nabla_y\phi_1\left(x, \frac{x}{\varepsilon}\right)\right)\, dx = \int_\Omega \nabla u_\varepsilon \cdot \psi\left(x, \frac{x}{\varepsilon}\right)\, dx.$$

Since ψ is a two-scale test function, one may pass to limit in the RHS above. Thus,

$$\int_\Omega A_\varepsilon \nabla u_\varepsilon \cdot (\nabla\phi + \nabla_y\phi_1^\varepsilon)\, dx \stackrel{\varepsilon\to 0}{\to} \frac{1}{|Y|}\int_{\Omega\times Y} [\nabla u + \nabla_y u_1(x, y)] \cdot \psi(x, y)\, dy\, dx$$

and, therefore,

$$\frac{1}{|Y|}\int_{\Omega\times Y} A(x,y)(\nabla u+\nabla_y u_1)\cdot(\nabla\phi+\nabla_y\phi_1(x,y))\,dy\,dx=\langle f,\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} \tag{2.4.3}$$

for all $\phi\in\mathcal{D}(\Omega)$ and $\phi_1\in\mathcal{D}(\Omega;C^\infty_{\text{per}}(Y))$. Thus, by density, for all $\phi\in H_0^1(\Omega)$ and $\phi_1\in L^2(\Omega;H^1_{\text{per}}(Y))$. In particular, by choosing $\phi_1\equiv 0$, we get

$$\begin{cases} -\text{div}_x\left[\frac{1}{|Y|}\int_Y A(x,y)(\nabla u(x)+\nabla_y u_1(x,y))\,dy\right] & =f \quad \text{in } \Omega\\ u & =0 \quad \text{on } \Omega \end{cases} \tag{2.4.4}$$

and by choosing $\phi\equiv 0$ we get

$$\begin{cases} -\text{div}_y\left[A(x,y)(\nabla u(x)+\nabla_y u_1(x,y))\right] & =0 \quad \text{in } \Omega\times Y\\ u_1(x,y) & \text{is} \quad Y\text{-periodic in } y. \end{cases} \tag{2.4.5}$$

Both (2.4.4) and (2.4.5) are called the *coupled two-scale homogenized system* of equations. The system (2.4.4) and (2.4.5) have a unique pair of solution (u,u_1) in $H:=H_0^1(\Omega)\times L^2(\Omega;H^1_{\text{per}}(Y)/\mathbb{R})$, due to Lax-Milgram result. The space H is a Hilbert space with the norm

$$\|(\phi,\phi_1)\|_H^2=\|\nabla\phi\|^2_{L^2(\Omega)}+\|\nabla_y\phi_1\|^2_{L^2(\Omega\times Y)}.$$

Theorem 2.4.1. *Let u_ε be the unique solution of* (2.4.1)*. Then there exists $(u,u_1)\in H$ satisfying* (2.4.2) *and (u,u_1) is the unique solution of the two-scale system* (2.4.4) *and* (2.4.5)*.*

Proof. Observe that (2.4.3) is the weak formulation of (2.4.4) and (2.4.5). We introduce the bilinear form $B:H\times H\to\mathbb{R}$ defined as

$$B[(\phi,\phi_1),(\psi,\psi_1)]=\int_{\Omega\times Y} A(x,y)(\nabla\phi(x)+\nabla_y\phi_1(x,y))\cdot(\nabla\psi(x)+\nabla_y\psi_1(x,y))$$

and the linear form $L:H\to\mathbb{R}$ defined as

$$L(\phi,\phi_1)=\langle f,\phi\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}.$$

Note that B is H-elliptic and continuous. Also, L is continuous on H. Thus, by Lax-Milgram result, (2.4.3) has a unique solution $(u,u_1)\in H$. Further, by uniqueness of solution, the convergence holds for entire sequence. $\square$

The coupled two-scale system (2.4.4) and (2.4.5) can be decoupled. Using (2.4.5), one can represent u_1 in terms of u. This is then used in (2.4.4) to get the homogenized equation of u. Recall that $\{e_j\}_1^n$ denotes the standard basis vectors of $\mathbb{R}^n$. Freezing x as a parameter in (2.4.5) and substituting $\nabla u(x) = \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x) e_j$, we get

$$\begin{cases} -\mathrm{div}_y[A(x,y)\nabla_y u_1(x,y)] & = \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x)\mathrm{div}_y[A(x,y)e_j] \quad \text{in } Y \\ u_1(x,y) & \text{is } Y\text{-periodic in } y. \end{cases}$$

The RHS motivates us to introduce, for each index $j = 1, 2, \ldots, n$, the cell problem. Let $\chi_j(x,y)$ be the solution of

$$\begin{cases} -\mathrm{div}_y[A(x,y)\nabla_y \chi_j(x,y)] & = -\mathrm{div}_y[A(x,y)e_j] \quad \text{in } Y \\ \chi_j(x,y) & \text{is} \qquad Y\text{-periodic in } y. \end{cases} \tag{2.4.6}$$

Observe that above equation is same as (1.5.6) for each fixed $x \in \Omega$. Thus, there is a function $\tilde{u}(x)$ such that

$$u_1(x,y) + \sum_{j=1}^n \frac{\partial u}{\partial x_j}(x)\chi_j(x,y) = \tilde{u}(x).$$

Substituting u_1 in (2.4.4), we get a equation for u

$$\begin{cases} -\mathrm{div}(A_0(x)\nabla u(x)) & = f \quad \text{in } \Omega \\ u & = 0 \quad \text{on } \partial\Omega, \end{cases} \tag{2.4.7}$$

where $A_0(x) = (a_{ij}^0(x))$ is given as

$$a_{ik}^0(x) = \frac{1}{|Y|}\int_Y \left(a_{ik}(x,y) - \sum_{j=1}^n a_{ij}(x,y)\frac{\partial \chi_k(x,y)}{\partial y_j} \right) dy.$$

Note that this is the precise form we obtained in (1.5.7) except that the coefficient depends on x, as well. Further,

$$\begin{aligned} a_{ik}^0(x) &= \frac{1}{|Y|}\int_Y \left(a_{ik}(x,y) - \sum_{j=1}^n a_{ij}(x,y)\frac{\partial \chi_k(x,y)}{\partial y_j} \right) dy \\ &= \frac{1}{|Y|}\int_Y A(x,y)(e_k - \nabla_y \chi_k)\cdot e_i \, dy. \end{aligned}$$

But, by taking χ_i as a test function in (2.4.6), we get

$$\int_Y A(x,y)[e_k - \nabla_y \chi_k] \cdot \nabla_y \chi_i \, dy = 0.$$

Thus, we can write

$$a^0_{ik} = \frac{1}{|Y|} \int_Y A(x,y)[e_k - \nabla_y \chi_k] \cdot [e_i - \nabla_y \chi_i] \, dy.$$

Thus, we have shown the following result.

Theorem 2.4.2. *Let u_ε be the unique solution of (2.4.1). Then u_ε converges to u weakly in $H^1_0(\Omega)$, where u is the unique solution of the homogenized equation (2.4.7).*

Remark 2.4.3. One may replace the smoothness condition on $A = (a_{ij})$ by that of **admissibility (or strong two scale limit)** of A, i.e. the coefficients satisfy

$$\lim_{\varepsilon \to 0} \int_\Omega a_{ij}\left(x, \frac{x}{\varepsilon}\right)^2 \, dx = \frac{1}{|Y|} \int_\Omega \int_Y a_{ij}(x,y)^2 \, dy \, dx$$

for all $1 \le i, j \le n$. In such situations one may appeal to Theorem 2.2.14 for passing to limits. In particular, if $a_{ij}\left(x, \frac{x}{\varepsilon}\right) = a_{ij}\left(\frac{x}{\varepsilon}\right)$, then no assumption is required as the admissibility condition is trivially satisfied.

Remark 2.4.4. The two scale homogenized system (2.4.4) and (2.4.5) is a coupled system of equations with two unknowns, u and u_1, in x and y (macroscopic and microscopic, respectively) variables. Seemingly complicated, it is a well-posed system and has a unique solution. Further, it was possible to decouple the system to recover the homogenized equation. This was due to the simple nature of the equation considered. For other types of equations the decoupling may not be possible or may produce very complicated equations, viz., integro-differential equations. The homogenized equation may pose existential issues while the two-scale form, though with twice the number of unknowns and variables, may give a solution. Thus, the presence of the microscopic variables in the two-scale homogenized problem may double the size of the equation but simplifies the structure. In some cases, decoupling might introduce strange effects, viz., memory or non-local effects.

Theorem 2.4.5 (Corrector)**.** *If $\nabla_y u_1(x, x/\varepsilon) \in [L^2(\Omega)]^n$ then $\|\xi_\varepsilon\|_{2,\Omega} \to 0$, where $\xi_\varepsilon := \nabla u_\varepsilon(x) - \nabla u(x) - \nabla_y u_1(x, x/\varepsilon)$. Thus, if both u_1 and u are in $H^1(\Omega)$ then*

$$\|u_\varepsilon(x) - u(x) - \varepsilon u_1(x, x/\varepsilon)\|_{H^1(\Omega)} \to 0.$$

Proof. If A is smooth, say, $A \in C(\Omega; L^\infty_{\mathrm{per}}(Y))^{n^2}$ then, by the regularity of χ^j, the function $u_1(x, x/\varepsilon) \in L^2(\Omega)$. Thus, u_1 may be used as a test function for the two-scale convergence. Let $\xi_\varepsilon := \nabla u_\varepsilon - \nabla u(x) - \nabla_y u_1\left(x, \frac{x}{\varepsilon}\right)$. Then,

$$\begin{aligned}
\int_\Omega A_\varepsilon(x)\xi_\varepsilon \cdot \xi_\varepsilon \, dx &= \int_\Omega f(x) u_\varepsilon(x)\, dx \\
&+ \int_\Omega A_\varepsilon \left[\nabla u + \nabla_y u_1 \left(x, \frac{x}{\varepsilon}\right)\right] \cdot \left[\nabla u + \nabla_y u_1 \left(x, \frac{x}{\varepsilon}\right)\right] dx \\
&- \int_\Omega (A_\varepsilon + {}^t A_\varepsilon)(x) \nabla u_\varepsilon(x) \cdot \left[\nabla u(x) + \nabla_y u_1 \left(x, \frac{x}{\varepsilon}\right)\right] dx.
\end{aligned}$$

Using the coercivity of A and passing to the two-scale limit, we obtain

$$\begin{aligned}
\alpha \limsup_{\varepsilon \to 0} \|\xi_\varepsilon\|^2_{2,\Omega} &\leq \int_\Omega f(x) u(x)\, dx \\
&- \frac{1}{|Y|} \int_\Omega \int_Y A(x,y)[\nabla u + \nabla_y u_1] \cdot [\nabla u + \nabla_y u_1]\, dy\, dx
\end{aligned}$$

The term on right-hand side is zero (why!), thus completing the proof. □

2.5 Summary

Chapter 3

H-Convergence

In this chapter, we consider the non-periodic situation as opposed to those considered in previous chapters. Let us set the environment for non-periodic case.

3.1 Coercive Operators

Definition 3.1.1. *We say a linear operator $\mathcal{A} : X \to X^\star$ is* bounded *or* continuous*, if there is a constant $0 < \beta < +\infty$ such that*

$$\|\mathcal{A}x\|_{X^\star} \leq \beta \|x\|_X, \quad \forall x \in X. \tag{3.1.1}$$

Let $\mathcal{B}(X, X^\star)$ denote the set of all linear bounded homomorphisms from X to $X^\star$. The norm on $\mathcal{B}(X, X^\star)$ is given as,

$$\|\mathcal{A}\|_{\mathcal{B}(X,X^\star)} = \sup_{x \in X} \frac{\|\mathcal{A}x\|_{X^\star}}{\|x\|_X}.$$

Definition 3.1.2. *An operator $\mathcal{A} : X \to X^\star$ is said to be* coercive *or* X-elliptic*, if there is a constant $0 < \alpha$ such that*

$$\langle \mathcal{A}x, x \rangle_{X^\star, X} \geq \alpha \|x\|_X^2, \quad \forall x \in X. \tag{3.1.2}$$

Theorem 3.1.3. *Let X be a reflexive Banach space. Any coercive operator $\mathcal{A} \in \mathcal{B}(X, X^\star)$ is an isomorphism.*

Proof. It is enough to show $\mathcal{A}$ is bijective. Observe that

$$\alpha\|x\|_X^2 \leq \langle \mathcal{A}x, x\rangle_{X^\star,X} \leq \|\mathcal{A}x\|_{X^\star}\|x\|_X$$

and hence,

$$\alpha\|x\|_X \leq \|\mathcal{A}x\|_{X^\star}. \tag{3.1.3}$$

Step 1 (Claim: $\mathcal{A}$ is injective). Let $\mathcal{A}x_1 = \mathcal{A}x_2$. Then, from (3.1.3), $\alpha\|x_1 - x_2\|_X \leq 0$. Therefore, $x_1 = x_2$.

Step 2 (Claim: $\mathrm{Im}(\mathcal{A})$ is closed). Suppose $\{\mathcal{A}x_n\}$ is a Cauchy sequence in $X^\star$. Then, by (3.1.3), $\{x_n\}$ is a Cauchy sequence in X. Since X is complete, there is a $x_0 \in X$ such that $x_n \to x_0$ (as $n \to \infty$) in X. Thus, from (3.1.1), we have

$$\|\mathcal{A}(x_n - x_0)\|_{X^\star} \to 0.$$

Consequently, $\mathcal{A}x_0$ is the limit of the Cauchy sequence $\{\mathcal{A}x_n\}$. Thus, $\mathrm{Im}(\mathcal{A})$ is closed.

Step 3 (Claim: $\mathcal{A}$ is surjective). Let $\mathrm{Im}(\mathcal{A}) \neq X^\star$. By Hahn-Banach theorem, for the closed subspace $\mathrm{Im}(\mathcal{A})$ of $X^\star$, there is a non-zero functional vanishing on $\mathrm{Im}(\mathcal{A})$. Thus, there is a non-zero $z \in X^{\star\star} \cong X$ such that

$$\langle \mathcal{A}x, z\rangle_{X^\star,X} = 0, \quad \forall x \in X.$$

In particular, $\langle \mathcal{A}z, z\rangle_{X^\star,X} = 0$ and, by (3.1.2), $z = 0$ which is a contradiction. Thus $\mathcal{A}$ is surjective.

□

Corollary 3.1.4. *Let $\mathcal{A} \in \mathcal{B}(X, X^\star)$ be a coercive operator. Then, for any $f \in X^\star$, the equation $\mathcal{A}u = f$ has a unique solution.*

Remark 3.1.5. Note that, by Theorem 3.1.3, $\mathcal{A}^{-1}$ is an isomorphism in $\mathcal{L}(X^\star, X)$ and

$$\|\mathcal{A}^{-1}f\|_X \leq \frac{1}{\alpha}\|f\|_{X^\star}, \quad \forall f \in X^\star.$$

Thus, $\mathcal{A}^{-1} \in \mathcal{B}(X^\star, X)$.

Definition 3.1.6. *Let X be a reflexive Banach space and $X^\star$ be its topological dual. A sequence of coercive operators $\{\mathcal{A}_\varepsilon\}$ in $\mathcal{B}(X, X^\star)$ is said to* G-converge *to $\mathcal{A}_0$ if*

$$\langle g, \mathcal{A}_\varepsilon^{-1} f\rangle \stackrel{\varepsilon\to 0}{\longrightarrow} \langle g, \mathcal{A}_0^{-1} f\rangle \quad \forall f, g \in X^\star.$$

The above definition defines a topology in $\mathcal{B}(X, X^\star)$. Note that G-convergence of a sequence of operators is nothing but the weak operator topology (WOT) convergence of the inverse operators.

Theorem 3.1.7 (G compactness)**.** *Let X be a separable reflexive Banach space and $X^\star$ be its topological dual. Let $\{\mathcal{A}_\varepsilon\}$ be a sequence of equi-coercive, uniformly bounded operators in $\mathcal{B}(X, X^\star)$, then the sequence is G-compact, i.e., there exists a subsequence $\{\mathcal{A}_\delta\}$ of $\{\mathcal{A}_\varepsilon\}$ and $\mathcal{A}_0$ such that G-converges to a coercive $\mathcal{A}_0$, as $\delta \to 0$.*

Proof. By remark 3.1.5, we know that $\{\mathcal{A}_\varepsilon^{-1}\} \subset \mathcal{B}(X^\star, X)$ is uniformly bounded. Let $\{f_k\}_1^\infty \subset X^\star$ be the countable dense subset of $X^\star$.

We shall now construct an operator $L : X^\star \to X$ as follows: Note that, for each fixed k, $\|\mathcal{A}_\varepsilon^{-1} f_k\|_X$ is bounded uniformly w.r.t ε. Thus, by weak-compactness of unit ball (Banach-Alaoglu result), there is a subsequence $\{\mathcal{A}_{\varepsilon_j^1}^{-1} f_1\}$ converging to, say, some u_1. Set $Lf_1 = u_1$. Now, extract a weak convergence subsequence $\{\mathcal{A}_{\varepsilon_j^2}^{-1} f_2\}$ from $\{\mathcal{A}_{\varepsilon_j^1}^{-1} f_2\}$ that weakly converges to u_2. Set $Lf_2 = u_2$. Proceeding this way, by extracting subsequence, at every stage, we define $Lf_k = u_k$. For each k, by choosing the diagonal sequence $\mathcal{A}_{\varepsilon_j^j}^{-1}$, we have

$$\lim_{k\to\infty} \langle g, \mathcal{A}_{\varepsilon_j^j}^{-1} f_k\rangle = \langle g, Lf_k\rangle \quad \forall g \in X^\star.$$

We next show that L is bounded on $\{f_k\}$. Due to the equi-coercivity of $\mathcal{A}_\varepsilon$, we have for each k,

$$\langle f_k, \mathcal{A}_{\varepsilon_j^j}^{-1} f_k\rangle \geq \alpha \|\mathcal{A}_{\varepsilon_j^j}^{-1} f_k\|_X^2.$$

Thus, by weak lower semi-continuity of norm, we get

$$\begin{aligned} \alpha\|Lf_k\|_X^2 &\leq \alpha \liminf_{j\to\infty} \|\mathcal{A}_{\varepsilon_j^j}^{-1} f_k\|_X^2 \\ &\leq \lim_{j\to\infty} \langle f_k, \mathcal{A}_{\varepsilon_j^j}^{-1} f_k\rangle \\ &= \langle f_k, Lf_k\rangle \leq \|f_k\|_{X^\star}\|Lf_k\|_X. \end{aligned}$$

Thus, $\|Lf_k\|_X \leq 1/\alpha\|f_k\|_{X^\star}$ and L is bounded on the dense subset of $X^\star$. Let $f \in X^\star$. Since $\{f_k\}$ is dense there is a sequence $f_m \to f$ in $X^\star$ as $m \to \infty$. Since Lf_m is bounded in X, for a subsequence, it converges weakly in X to, say, $u^\star$. Note that

$$\langle g, \mathcal{A}_{\varepsilon_j^j}^{-1} f\rangle - \langle g, u^\star\rangle = \langle g, \mathcal{A}_{\varepsilon_j^j}^{-1} f - \mathcal{A}_{\varepsilon_j^j}^{-1} f_m + \mathcal{A}_{\varepsilon_j^j}^{-1} f_m - Lf_m + Lf_m - u^\star\rangle.$$

For large m, one can make the RHS as small as possible. Thus,

$$\lim_{j\to\infty} \langle g, \mathcal{A}_{\varepsilon_j^j}^{-1} f\rangle = \langle g, u^\star\rangle.$$

Since the choice of $u^\star$ is independent of the choice of the subsequence $\{f_m\}$, we set $Lf = u^\star$, for all $f \in X^\star$. We leave it as an *exercise* to show that L is linear.

We now claim that L is coercive. We know that L is bounded and $\|L\| \leq 1/\alpha$. By the equi-coercivity of $\mathcal{A}_\varepsilon$, we have

$$\langle f, \mathcal{A}_{\varepsilon_j^j}^{-1} f\rangle \geq \alpha\|\mathcal{A}_{\varepsilon_j^j}^{-1} f\|_X^2 \geq \alpha\frac{1}{\beta^2}\|\mathcal{A}_{\varepsilon_j^j}\mathcal{A}_{\varepsilon_j^j}^{-1} f\|_X^2 \geq \frac{\alpha}{\beta^2}\|f\|_{X^\star}^2.$$

The constant β is the bound of $\mathcal{A}_\varepsilon$. Passing to limit as $j \to \infty$, we get $\langle f, Lf\rangle \geq \frac{\alpha}{\beta^2}\|f\|_{X^\star}^2$. Thus, L is coercive and, by remark 3.1.5, we have $\|L^{-1}\| \leq \frac{\beta^2}{\alpha}$. Moreover, L^{-1} is also coercive since

$$\langle f, \mathcal{A}_{\varepsilon_j^j}^{-1} f\rangle \geq \alpha\|\mathcal{A}_{\varepsilon_j^j}^{-1} f\|_X^2$$

implies that

$$\alpha\|Lf\|_X^2 \leq \langle f, Lf\rangle$$

or equivalently, $\langle L^{-1}u, u\rangle \geq \alpha\|u\|_X^2$. We set $\mathcal{A}_0 = L^{-1}$. $\square$

3.2 H-Convergence

Let Ω be an open bounded subset of $\mathbb{R}^n$. Let $0 < \alpha < \beta$ and $M(\alpha, \beta, \Omega)$ denote the set of all $n \times n$ matrices $A(x) = (a_{ij}(x))$ of functions such that

$$\alpha|\xi|^2 \leq A(x)\xi \cdot \xi \text{ and } |A(x)\xi| \leq \beta|\xi| \text{ for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^n.$$

Observe that the class $M(\alpha, \beta, \Omega)$ is closed under transpose of matrices. Given a sequence of matrices $\{A_\varepsilon\} \subset M(\alpha, \beta, \Omega)$, define the operator $\mathcal{A}_\varepsilon$:

$H^1(\Omega) \to H^{-1}(\Omega)$ as $\mathcal{A}_\varepsilon = -\mathrm{div}(A_\varepsilon \nabla)$. For any $f \in H^{-1}(\Omega)$, the second order elliptic problem

$$\begin{cases} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) & = f \quad \text{in } \Omega \\ u_\varepsilon & = 0 \quad \text{on } \partial\Omega \end{cases} \tag{3.2.1}$$

has a *unique* solution, by Lax-Milgram result, satisfying the estimate

$$\|u_\varepsilon\|_{H^1_0(\Omega)} \leq \frac{1}{\alpha}\|f\|_{H^{-1}(\Omega)}. \tag{3.2.2}$$

Hence there exists a subsequence such that

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly in } H^1_0(\Omega).$$

The uniqueness of u_ε follows from (3.2.2). The bounded elliptic operator $\mathcal{A}_\varepsilon = -\mathrm{div}(A_\varepsilon \nabla)$ from $H^1_0(\Omega)$ into $H^{-1}(\Omega)$ is an isomorphism and the norm of $(\mathcal{A}_\varepsilon)^{-1}$ is not larger than α^{-1} (cf. (3.2.2)). Moreover, we also know that the solution u_ε of (3.2.3) can be characterized as the minimizer of

$$J_\varepsilon(v) = \frac{1}{2}\int_\Omega A_\varepsilon \nabla v.\nabla v \, dx - \langle f, v\rangle_{H^{-1}(\Omega), H^1_0(\Omega)}$$

in $H^1_0(\Omega)$. Since $A_\varepsilon \in M(\alpha, \beta, \Omega)$, $\mathcal{A}_\varepsilon : H^1_0(\Omega) \to H^{-1}(\Omega)$ is an uniformly bounded, equi-coercive sequence of operators with constants β and α, respectively. Thus, by Theorem 3.1.3 and Theorem 3.1.7, $\mathcal{A}_\varepsilon^{-1}$ exists and there is a $\mathcal{A}_0$ to which $\mathcal{A}_\varepsilon$ G-converges. Does there exist a matrix A_0 such that $\mathcal{A}_0 = -\mathrm{div}(A_0 \nabla)$?

Definition 3.2.1. *A sequence $\{A_\varepsilon\} \subset M(\alpha, \beta, \Omega)$ is said to H-converges to a matrix $A_0 : \Omega \to [L^\infty(\Omega)]^{n\times n}$, denoted as $A_\varepsilon \stackrel{H}{\rightharpoonup} A_0$, if for every sequence $f_\varepsilon \to f$ strongly in $H^{-1}(\Omega)$, the solution u_ε of*

$$\begin{cases} -div(A_\varepsilon \nabla u_\varepsilon) & = f_\varepsilon \quad in\ \Omega \\ u_\varepsilon & = 0 \quad on\ \partial\Omega \end{cases} \tag{3.2.3}$$

is such that

$$u_\varepsilon \rightharpoonup u_0 \ weakly\ in\ H^1_0(\Omega) \quad and \tag{3.2.4a}$$

$$A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_0 \nabla u_0 \ weakly\ in\ (L^2(\Omega))^n, \tag{3.2.4b}$$

where u_0 is the unique solution of

$$\left\{ \begin{array}{rll} -\mathrm{div}(A_0 \nabla u_0) & = f & \textit{in } \Omega \\ u_0 & = 0 & \textit{on } \partial\Omega. \end{array} \right. \tag{3.2.5}$$

The matrix A_0 is called the H-limit of the sequence $\{A_\varepsilon\}$.

Before we embark on the problem of finding H-limit of a given sequence, we highlight some useful properties of H-convergence.

Theorem 3.2.2 (Uniqueness)**.** *The H-limit of a sequence $\{A_\varepsilon\}$ is unique.*

Proof. Let A_0 be a H-limit of $\{A_\varepsilon\}$. Let $\omega \subset\subset \omega_0 \subset \Omega$ and $\phi \in \mathcal{D}(\omega_0)$ such that $\phi \equiv 1$ on ω. For each $\lambda \in \mathbb{R}^n$, we define $f_\lambda = -\mathrm{div}[A_0 \nabla\{(\lambda \cdot x)\phi(x)\}]$ and let u_ε^λ be the unique solution

$$\left\{ \begin{array}{rll} -\mathrm{div}(A_\varepsilon \nabla u_\varepsilon^\lambda) & = f_\lambda & \text{in } \omega_0 \\ u_\varepsilon^\lambda & = 0 & \text{on } \partial\omega_0. \end{array} \right.$$

Then, by definition of H-convergence,

$$u_\varepsilon^\lambda \rightharpoonup (\lambda \cdot x)\phi(x) \text{ weakly in } H_0^1(\omega_0)$$

$$A_\varepsilon \nabla u_\varepsilon^\lambda \rightharpoonup A_0 \nabla\{(\lambda \cdot x)\phi(x)\} \text{ weakly in } (L^2(\omega_0))^n.$$

If A_1 is another H-limit of $\{A_\varepsilon\}$ then, corresponding to the same f_λ and by H-convegence

$$u_\varepsilon^\lambda \rightharpoonup v_0^\lambda \text{ weakly in } H_0^1(\omega_0)$$

$$A_\varepsilon \nabla u_\varepsilon^\lambda \rightharpoonup A_1 \nabla v_0^\lambda \text{ weakly in } (L^2(\omega_0))^n.$$

By uniqueness of weak limits, $v_0^\lambda(x) = (\lambda \cdot x)\phi(x)$ and $A_0 \nabla\{(\lambda \cdot x)\phi(x)\} = A_1 \nabla\{(\lambda \cdot x)\phi(x)\}$ in ω_0. Thus, in ω, $\nabla\{(\lambda \cdot x)\phi(x)\} = \lambda$ and, hence, $A_0 = A_1$ in ω. □

Corollary 3.2.3 (Local Property)**.** *If $\{A_\varepsilon\}$ and $\{B_\varepsilon\}$ are two sequences in $M(\alpha, \beta, \Omega)$ which H-converges to A_0 and B_0, respectively, such that $A_\varepsilon = B_\varepsilon$ in $\omega \subset \Omega$, for all ε, then $A_0 = B_0$ in ω.*

Theorem 3.2.4 (Transpose)**.** *If $A_\varepsilon \overset{H}{\rightharpoonup} A_0$ then ${}^tA_\varepsilon \overset{H}{\rightharpoonup} {}^tA_0$.*

Proof. Let $\omega \subset\subset \Omega$ and $g \in H^{-1}(\omega)$. Let v_ε be the solution of

$$\left\{ \begin{array}{rll} -\mathrm{div}({}^tA_\varepsilon \nabla v_\varepsilon) & = g & \text{in } \omega \\ v_\varepsilon & = 0 & \text{on } \partial\omega. \end{array} \right.$$

Then, upto a subsequence, there is a $v \in H^1_0(\omega)$ and $\eta \in [L^2(\omega)]^n$ such that

$$v_\varepsilon \rightharpoonup v \text{ weakly in } H^1_0(\omega) \quad \text{and}$$

$${}^tA_\varepsilon \nabla v_\varepsilon \rightharpoonup \eta \text{ weakly in } (L^2(\omega))^n.$$

Further, $-\mathrm{div}(\eta(x)) = g(x)$ a.e. in ω. For any $f \in H^{-1}(\omega)$, let u_ε be the solution of (3.2.3) and, by H-convergence, there is a unique $u_0 \in H^1_0(\omega)$ satisfying (3.2.5). For any $\phi \in C^\infty_c(\omega)$, using $v_\varepsilon\phi$ as a test function in (3.2.3), one obtains

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_\omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon \phi \, dx &= -\lim_{\varepsilon\to 0} \langle \mathrm{div}(A_\varepsilon \nabla u_\varepsilon), \phi v_\varepsilon \rangle_{H^{-1}(\omega), H^1_0(\omega)} \\ &\quad - \lim_{\varepsilon\to 0} \int_\omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla \phi v_\varepsilon \, dx \\ &= -\langle \mathrm{div}(A_0 \nabla u_0), \phi v \rangle_{H^{-1}(\omega), H^1_0(\omega)} \\ &\quad - \int_\omega A_0 \nabla u_0 \cdot \nabla \phi v \, dx \\ &= \int_\omega A_0 \nabla u_0 \cdot \nabla v \phi \, dx. \end{aligned}$$

Thus, $A_\varepsilon \nabla u_\varepsilon \cdot \nabla v_\varepsilon \rightharpoonup A_0 \nabla u_0 \cdot \nabla v$ weak-* in $\mathcal{D}'(\Omega)$. Using $u_\varepsilon \phi$ as a test function in the equation for v_ε, one obtains

$$\begin{aligned} \lim_{\varepsilon\to 0} \int_\omega {}^tA_\varepsilon \nabla v_\varepsilon \cdot \nabla u_\varepsilon \phi \, dx &= -\lim_{\varepsilon\to 0} \langle \mathrm{div}({}^tA_\varepsilon \nabla v_\varepsilon), \phi u_\varepsilon \rangle_{H^{-1}(\omega), H^1_0(\omega)} \\ &\quad - \lim_{\varepsilon\to 0} \int_\omega {}^tA_\varepsilon \nabla v_\varepsilon \cdot \nabla \phi u_\varepsilon \, dx \\ &= -\langle \mathrm{div}(\eta), \phi u_0 \rangle_{H^{-1}(\omega), H^1_0(\omega)} - \int_\omega \eta \cdot \nabla \phi u_0 \, dx \\ &= \int_\omega \eta \cdot \nabla u_0 \phi \, dx. \end{aligned}$$

Thus, ${}^tA_\varepsilon \nabla v_\varepsilon \cdot \nabla u_\varepsilon \rightharpoonup \eta \cdot \nabla u_0$ weak-* in $\mathcal{D}'(\Omega)$. Since $A_\varepsilon(x) \nabla u_\varepsilon(x) \cdot \nabla v_\varepsilon(x) = {}^tA_\varepsilon(x) \nabla v_\varepsilon(x) \cdot \nabla u_\varepsilon(x)$, we have $A_0(x) \nabla u_0(x) \cdot \nabla v(x) = \eta(x) \cdot \nabla u_0(x)$ a.e. in

ω. Because the elliptic operator is an isomorphism, as f varies in $H^{-1}(\omega)$, u_0 varies in $H_0^1(\omega)$. Therefore, for any $\lambda \in \mathbb{R}^n$, we can choose a $u_0 \in H_0^1(\omega)$ such that $\nabla u_0(x) = \lambda$ on $\omega_1 \subset\subset \omega$. Thus, for any $\lambda \in \mathbb{R}^n$, ${}^tA_0(x)\nabla v(x)\cdot\lambda = \eta(x)\cdot\lambda$ a.e. in ω_1 and ${}^tA_0(x)\nabla v(x) = \eta(x)$ a.e. in ω. Hence, the limit v satisfies the equation $-\text{div}({}^tA_0(x)\nabla v(x)) = g$ and ${}^tA_0$ is the H-limit of ${}^tA_\varepsilon$. Moreover, by the uniqueness of H-limit, ${}^tA_0$ is unique and, hence, v is the unique limit for all subsequences. Thus, the convergences is true for the entire sequence. $\square$

Example 3.1 (One Dimension). Let us find the H-limit for a one dimension problem. Let $\Omega = (a,b)$, $f \in L^2(a,b)$ and let $a_\varepsilon : (a,b) \to \mathbb{R}$ be a function satisfying the ellipticity $0 < \alpha \leq a_\varepsilon(x) \leq \beta$ a.e. and is in $L^\infty(a,b)$. Note that the ellipticity condition implies that $1/a_\varepsilon$ is in $L^\infty(a,b)$. The equation to be homogenized is

$$\begin{cases} -\frac{d}{dx}(a_\varepsilon(x)\frac{du_\varepsilon(x)}{dx}) & = f(x) \quad \text{in } (a,b) \\ \qquad\qquad u_\varepsilon(a) & = u_\varepsilon(b) \quad = 0. \end{cases}$$

There exists a unique solution $u_\varepsilon \in H_0^1(a,b)$, by Lax-Milgram result, such that

$$\int_\Omega a_\varepsilon(x)\frac{du_\varepsilon(x)}{dx}\cdot\frac{dv(x)}{dx}\,dx = \langle f, v\rangle_{H^{-1}(a,b),H_0^1(a,b)}, \quad \forall v \in H_0^1(a,b)$$

and $\|u_\varepsilon\|_{H_0^1(a,b)} \leq (C/\alpha)\|f\|_{L^2(a,b)}$. By Eberlein-Šmuljan theorem, there exists a subsequence of $\{u_\varepsilon\}$, also denoted by u_ε, such that

$$u_\varepsilon \rightharpoonup u \text{ weakly in } H_0^1(a,b),$$

for some $u \in H_0^1(a,b)$. We need to find the homogenized equation which u solves. Set $\xi_\varepsilon := a_\varepsilon u_\varepsilon'$. Note that ξ_ε is bounded in $L^2(a,b)$, because a_ε is bounded in $L^\infty(a,b)$ and u_ε' is bounded in $L^2(a,b)$. From the equation, we have that $-\xi_\varepsilon'(x) = f(x)$ and hence ξ_ε is bounded in $H^1(a,b)$. Therefore, there exists a $\xi \in H^1(a,b)$ such that for a subsequence of ξ_ε (denoted by itself),

$$\xi_\varepsilon \rightharpoonup \xi \text{ weakly in } H^1(a,b).$$

Thus, by compact imbedding of $H^1(a,b)$ in $L^2(a,b)$, ξ_ε converges to ξ strongly in $L^2(a,b)$. Note that if either a_ε or u_ε' converges strongly, then ξ is the product of the limits of a_ε and u_ε'. But, in general, this need not be the case. Recall the u_ε' converges to u' weakly in $L^2(a,b)$, therefore

$$\frac{1}{a_\varepsilon}\xi_\varepsilon \rightharpoonup u' \text{ weakly in } L^2(a,b).$$

Since $1/a_\varepsilon$ is bounded in $L^\infty(a,b)$, for a subsequence,

$$\frac{1}{a_\varepsilon(x)} \rightharpoonup b(x) \text{ weak-* in } L^\infty(a,b).$$

Then, $u' = b(x)\xi$. Also, the constant sequence ξ'_ε converges weakly to ξ' in $L^2(a,b)$ implies that $-\xi'(x) = f(x)$. Hence,

$$-\frac{d}{dx}\left(\frac{1}{b(x)}\frac{du(x)}{dx}\right) = f(x).$$

u is already in $H_0^1(a,b)$ satisfying the boundary condition and the effective coefficient is $a_0 = 1/b$. The effective coefficient is bounded in $L^\infty(a,b)$ and satisfies the ellipticity condition. Note that the choice of b depends on the subsequence chosen and u is not unique. All the above arguments were for a subsequences, extracted sufficiently.

However, if $a_\varepsilon = a(x/\varepsilon)$ such that a is periodic. Then

$$b = \int_0^1 \frac{1}{a(y)}\,dy$$

for every subsequence of a_ε, and $a_0 = b^{-1}$. Thus, u is a unique solution and all the convergences above are true for the entire sequence and not just for subsequences (cf. Eberlein-Šmuljan result). Thus, in the one dimensional case, the H-limit of a sequence $\{a_\varepsilon\} \subset M(\alpha,\beta,\Omega)$ is the inverse of the weak-* limit in L^∞ of the inverses of a_ε.

Example 3.2 (Layering). Let Ω be a bounded open subset of $\mathbb{R}^n$. Let u^ε be the solution of (3.2.3) with $f_\varepsilon = f$ in $L^2(\Omega)$, for all ε. The coefficient matrix is such that $A_\varepsilon(x) = A_\varepsilon(x_1)$ and $A_\varepsilon = (a^\varepsilon_{ij})$. Therefore, upto a subsequence, there is a $u \in H_0^1(\Omega)$ and $\xi \in [L^2(\Omega)]^n$ such that

$$u^\varepsilon \rightharpoonup u \text{ weakly in } H_0^1(\Omega) \quad \text{and}$$

$$A_\varepsilon \nabla u_\varepsilon \rightharpoonup \xi \text{ weakly in } (L^2(\Omega))^n.$$

Also, $-\operatorname{div}(\xi) = f$. Set $\xi_\varepsilon := A_\varepsilon \nabla u_\varepsilon = \left(\sum_{j=1} a^\varepsilon_{ij}(x_1) u^\varepsilon_{x_j}(x)\right)_i$. Since $-\operatorname{div}(\xi_\varepsilon) = f(x)$, it follows that

$$-\frac{\partial \xi^1_\varepsilon}{\partial x_1} = f + \sum_{i=2}^n \frac{\partial \xi^i_\varepsilon}{\partial x_i}.$$

Let Ω_1 be the projection of Ω in $\mathbb{R}$ and Ω_2 be the projection of Ω in $\mathbb{R}^{n-1}$ such that $\Omega = \Omega_1 \times \Omega_2$. Define $g_\varepsilon : \Omega_1 \to L^2(\Omega_2)$ as $g_\varepsilon(x_1)(x_2, \dots, x_n) := \xi^1_\varepsilon(x_1, \dots, x_n)$. Thus, $g_\varepsilon \in L^2(\Omega_1; L^2(\Omega_2))$. Note that

$$\frac{\partial g_\varepsilon}{\partial x_1} \in L^2(\Omega_1; H^{-1}(\Omega_2))$$

because

$$\frac{\partial g_\varepsilon}{\partial x_1} = \frac{\partial \xi^1_\varepsilon}{\partial x_1}.$$

Thus, $\{g_\varepsilon\}$ is bounded in $H^1(\Omega_1; H^{-1}(\Omega_2))$ and, by Aubin-Lions compactness theorem, relatively compact in $L^2(\Omega_1; H^{-1}(\Omega_2))$. Thus, $g_\varepsilon \to g$ strongly in $L^2(\Omega_1; H^{-1}(\Omega_2))$ and $g = \xi^1$. Using the equation for $\xi^1_\varepsilon(x)$, we get

$$u^\varepsilon_{x_1}(x) = \frac{1}{a^\varepsilon_{11}(x_1)} \xi^1_\varepsilon(x) - \sum_{j=2} \frac{\partial}{\partial x_j} \left(\frac{a^\varepsilon_{1j}(x_1)}{a^\varepsilon_{11}(x_1)} u^\varepsilon(x) \right) \tag{3.2.6}$$

and, for $2 \leq i \leq n$,

$$\xi^i_\varepsilon(x) = \frac{a^\varepsilon_{i1}(x_1}{a^\varepsilon_{11}(x_1)} \xi^1_\varepsilon(x) + \sum_{j=2} \frac{\partial}{\partial x_j} \left[\left(a^\varepsilon_{ij} - \frac{a^\varepsilon_{i1}(x_1) a^\varepsilon_{1j}(x_1)}{a^\varepsilon_{11}(x_1)} \right) u^\varepsilon(x) \right]. \tag{3.2.7}$$

Also, by the ellipticity of A_ε, we have the following weak-* convergences in $L^\infty(\Omega)$

$$\frac{1}{a^\varepsilon_{11}} \rightharpoonup b_{11} := \frac{1}{a_{11}},$$

$$\frac{a^\varepsilon_{i1}}{a^\varepsilon_{11}} \rightharpoonup c_i := \frac{a_{i1}}{a_{11}} \quad \text{for } 2 \leq i \leq n,$$

$$\frac{a^\varepsilon_{1j}}{a^\varepsilon_{11}} \rightharpoonup d_j := \frac{a_{1j}}{a_{11}} \quad \text{for } 2 \leq j \leq n,$$

$$a^\varepsilon_{ij} - \frac{a^\varepsilon_{i1} a^\varepsilon_{1j}}{a^\varepsilon_{11}} \rightharpoonup e_{ij} := a_{ij} - \frac{a_{i1} a_{1j}}{a_{11}} \quad \text{for } 2 \leq i, j \leq n.$$

Multiplying $\phi \in C^\infty_c(\Omega)$ on both sides of (3.2.6), passing to limit, we get

$$\int_\Omega u_{x_1}(x) \phi(x)\, dx = \langle \xi^1, \frac{\phi}{a_{11}} \rangle_{L^2(\Omega_1; H^{-1}(\Omega_2)), L^2(\Omega_1; H^1_0(\Omega_2))} - \sum_{j=2}^n \langle u(x), \phi_{x_j} \frac{a_{1j}}{a_{11}} \rangle.$$

This yields $\xi^1(x) = \sum_{j=1}^n a_{1j} u_{x_j}$. Similarly, multiplying $\phi \in C_c^\infty(\Omega)$ on both sides of (3.2.7) and, passing to limit, one gets for $2 \leq i \leq n$

$$\begin{aligned} \xi^i(x) &= \frac{a_{i1}(x_1)}{a_{11}(x_1)}\xi^1(x) + \sum_{j=2}\left(a_{ij} - \frac{a_{i1}(x_1)a_{1j}(x_1)}{a_{11}(x_1)}\right)u_{x_j}(x) \\ &= \sum_{j=1}^n a_{ij}(x_1)u_{x_j}. \end{aligned}$$

Thus, $\xi = A_0 \nabla u$ where $A_0 = (a_{ij})$.

Theorem 3.2.5 (H-compactness)**.** *For any given sequence $A_\varepsilon \subset M(\alpha, \beta, \Omega)$ there is a $A_0 \in M(\alpha, \beta^2/\alpha, \Omega)$ and a subsequence $\{A_{\varepsilon'}\}$ such that $A_{\varepsilon'}$ H-converges to A_0, i.e., $A_{\varepsilon'} \xrightarrow{H} A_0$.*

Proof. Let O be an open bounded subset of $\mathbb{R}^n$ such that $\Omega \subset O$. Let $\{M_\varepsilon\} \subset M(\alpha, \beta, O)$ such that $M_\varepsilon = {}^t A_\varepsilon$ in Ω. For instance, one may choose $M_\varepsilon = \alpha I$ in $O \setminus \Omega$. Define the operator $T_\varepsilon : H^1(O) \to H^{-1}(O)$ as $T_\varepsilon = -\text{div}(M_\varepsilon \nabla)$. Since $M_\varepsilon \in M(\alpha, \beta, O)$, $T_\varepsilon : H_0^1(O) \to H^{-1}(O)$ is a uniformly bounded, equi-coercive operator with constants β and α, respectively. By Theorem 3.1.7, there is a $T_0 : H_0^1(O) \to H^{-1}(O)$, with ellipticity constant α and bound β^2/α, such that, for a subsequence,

$$\langle g, T_\varepsilon^{-1} f \rangle \xrightarrow{\varepsilon \to 0} \langle g, T_0^{-1} f \rangle \quad \forall f, g \in H^{-1}(O).$$

Consider the projection function $P_i : O \to \mathbb{R}$ defined as $P_i(x) := x_i$, for all $i = 1, 2, \dots, n$. Note that $P_i \in H^1(O) \setminus H_0^1(O)$. For any $\phi \in \mathcal{D}(O)$, $\phi P_i \in H_0^1(O)$. Define $F_i := T_0(\phi P_i) \in H^{-1}(O)$ and $w_\varepsilon^i := T_\varepsilon^{-1} F_i \in H_0^1(O)$. This means that, for every $g \in H^{-1}(O)$,

$$\langle g, w_\varepsilon^i \rangle = \langle g, T_\varepsilon^{-1} F_i \rangle \xrightarrow{\varepsilon \to 0} \langle g, T_0^{-1} F_i \rangle = \langle g, \phi P_i \rangle,$$

i.e., $w_\varepsilon^i \rightharpoonup \phi P_i$ weakly in $H_0^1(O)$. Choose $\phi \in \mathcal{D}(O)$ such that $\phi \equiv 1$ in Ω, then

(i) $\phi P_i = P_i$ in Ω;

(ii) $w_\varepsilon^i \rightharpoonup P_i$ weakly in $H^1(\Omega)$;

(iii) $-\text{div}({}^t A_\varepsilon \nabla w_\varepsilon^i) = F_i$ in Ω.

The last equation implies that there is a $\eta_i \in [L^2(\Omega)]^n$ such that, for a subsequence, ${}^tA_\varepsilon \nabla w^i_\varepsilon \rightharpoonup \eta_i$ weakly in $[L^2(\Omega)]^n$. Also, $-\text{div}(\eta_i) = F_i$ in Ω. The fact that ∇w^i_ε weakly converges to e_i motivates to define ${}^tA_0 e_i = \eta_i$. Thus, set $A_0(x) = a_{ij}(x)$ where $a_{ij}(x) = (\eta_i)_j$. Let $\omega \subset\subset \Omega$ be compactly contained in Ω. Define the operator $\mathcal{A}_\varepsilon : H^1_0(\omega) \to H^{-1}(\omega)$ as $\mathcal{A}_\varepsilon = -\text{div}(A_\varepsilon \nabla)$. By Theorem 3.1.7, there is a $\mathcal{A}_0 : H^1_0(\omega) \to H^{-1}(\omega)$ such that, for a subsequence,

$$\langle g, \mathcal{A}_\varepsilon^{-1} f\rangle \xrightarrow{\varepsilon\to 0} \langle g, \mathcal{A}_0^{-1} f\rangle \quad \forall f, g \in H^{-1}(\omega).$$

For a fixed $f \in H^{-1}(\omega)$, set $u_\varepsilon := \mathcal{A}_\varepsilon^{-1} f$ and $u_0 := \mathcal{A}_0^{-1} f$. Then $u_\varepsilon \rightharpoonup u_0$ weakly in $H^1_0(\omega)$. Set $\xi_\varepsilon := A_\varepsilon \nabla(\mathcal{A}_\varepsilon^{-1}) : H^{-1}(\omega) \to [L^2(\omega)]^n$. Note that

$$\|\xi_\varepsilon f\|_{2,\omega} \le \beta \|\mathcal{A}_\varepsilon^{-1} f\|_{1,2,\omega} \le \frac{\beta}{\alpha}\|f\|_{-1,2,\omega}$$

is bounded in $[L^2(\omega)]^n$. Arguing as in the proof of Theorem 3.1.7, one can conclude that there is a $\xi_0 : H^{-1}(\omega) \to [L^2(\omega)]^n$ such that

$$\xi_\varepsilon f \rightharpoonup \xi_0 f \text{ weakly in } [L^2(\omega)]^n.$$

We claim that $\xi_0 = A_0 \nabla(\mathcal{A}_0^{-1})$. Observe that

$$\int_\omega \xi_\varepsilon f \cdot \nabla w^i_\varepsilon \, dx = \int_\omega A_\varepsilon \nabla u_\varepsilon \cdot \nabla w^i_\varepsilon \, dx = \int_\omega \nabla u_\varepsilon \cdot {}^tA_\varepsilon \nabla w^i_\varepsilon \, dx.$$

But both the LHS and RHS converge weak-* in $\mathcal{D}'(\omega)$. For all $\phi \in C^\infty_c(\omega)$, using integration by parts

$$\int_\omega (\xi_\varepsilon f \cdot \nabla w^i_\varepsilon)\phi \, dx \rightharpoonup \int_\omega (\xi_0 f \cdot e_i)\phi \, dx$$

and

$$\int_\omega (\nabla u_\varepsilon \cdot {}^tA_\varepsilon \nabla w^i_\varepsilon)\phi \, dx \rightharpoonup \int_\omega (\nabla u_0 \cdot {}^tA_0 e_i)\phi \, dx.$$

Thus, for a.e. in ω and any $f \in H^{-1}(\omega)$,

$$\xi_0 f = A_0 \nabla u_0 = A_0 \nabla(\mathcal{A}_0^{-1} f).$$

Hence, $\xi_0 = A_0 \nabla(\mathcal{A}_0^{-1})$. □

Corollary 3.2.6. *Let $\{A_\varepsilon\} \subset M(\alpha, \beta, \Omega)$ and $f_\varepsilon \to f$ strongly in $H^{-1}(\Omega)$. If u_ε is a solution of (3.2.3) then both (3.2.4a) and (3.2.4b) are satisfied where u_0 solves (3.2.5) and A_0 is a H-limit of A_ε.*

Proof. We know that $u_\varepsilon \in H_0^1(\Omega)$, by Lax-Milgram result, is such that

$$\int_\Omega A_\varepsilon(x)\nabla u_\varepsilon(x)\cdot\nabla v(x)\,dx = \langle f_\varepsilon, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall v\in H_0^1(\Omega)$$

and $\|u_\varepsilon\|_{H_0^1(\Omega)} \leq (1/\alpha)\|f_\varepsilon\|_{H^{-1}(\Omega)}$. By Eberlein-Šmuljan theorem, there exists a subsequence of $\{u_\varepsilon\}$, also denoted by u_ε, such that (3.2.4a) is satisfied, for some $u_0 \in H_0^1(\Omega)$.

Set $\xi_\varepsilon(x) = A_\varepsilon(x)\nabla u_\varepsilon(x)$. Note that ξ_ε is bounded in $(L^2(\Omega))^n$, because the entries of A_ε are bounded in $L^\infty(\Omega)$ and ∇u_ε is bounded in $(L^2(\Omega))^n$. Therefore, there exists a $\xi_0 \in (L^2(\Omega))^n$ such that for a subsequence of ξ_ε (denoted by itself),

$$\xi_\varepsilon \rightharpoonup \xi \text{ weakly in } (L^2(\Omega))^n.$$

Note that, in contrast, in the one-dimensional case we had the strong convergence in $L^2(\Omega)$, because we had the boundedness of ξ_ε in $H^1(\Omega)$.

Passing to the limit, as $\varepsilon \to 0$, in the weak formulation of (3.2.3)

$$\int_\Omega \xi_\varepsilon \nabla v\,dx = \langle f_\varepsilon, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}, \quad \forall v\in H_0^1(\Omega),$$

we get $\int_\Omega \xi_0\nabla v\,dx = \langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$ for all $v\in H_0^1(\Omega)$. Thus, $-\operatorname{div}(\xi_0) = f$ in Ω. Our proof is done if we show that $\xi_0 = A_0\nabla u_0$.

The compactness of H-convergence implies the existence of a matrix $A_0 \in M(\alpha, \frac{\beta^2}{\alpha}, \Omega)$ such that, for a subsequence (still denoted by ε), $A_\varepsilon \overset{H}{\rightharpoonup} A_0$. Further, ${}^tA_\varepsilon \overset{H}{\rightharpoonup} {}^tA_0$. Note that, as usual, ξ_ε is a product of two weak converging sequences and finding their limit is not trivial. This was cleverly overcome in one dimensional case. For the general case, Tartar came up with idea of using the adjoint of A_ε to define some useful test functions, called the *method of oscillating test function*.

For each $1 \leq i \leq n$, let $w_\varepsilon^i \in H^1(\Omega)$ be a solution of

$$\begin{cases} -\operatorname{div}({}^tA_\varepsilon\nabla w_\varepsilon^i) &= -\operatorname{div}({}^tA_0 e_i) \quad \text{in } \Omega \\ \qquad\qquad w_\varepsilon^i &= x_i \qquad\qquad\qquad \text{on } \partial\Omega. \end{cases} \tag{3.2.8}$$

The function $w_\varepsilon^i \in H^1(\Omega)$, for all $1 \leq i \leq n$ and satisfies the following properties

$$\begin{cases} w_\varepsilon^i \rightharpoonup x_i \text{ weakly in } H^1(\Omega), \\ {}^tA_\varepsilon\nabla w_\varepsilon^i \rightharpoonup {}^tA_0 e_i \text{ weakly in } (L^2(\Omega))^n, \\ \operatorname{div}({}^tA_\varepsilon\nabla w_\varepsilon^i) \text{ converges strongly in } H^{-1}(\Omega). \end{cases}$$

Note that for any $\phi \in \mathcal{D}(\Omega)$, $\phi w_\varepsilon^i \in H_0^1(\Omega)$. Thus, in particular choosing $v = \phi w_\varepsilon^i$ in the weak formulation of (3.2.3), we get

$$\begin{aligned}\left\langle f_\varepsilon, \phi w_\varepsilon^i \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &= \int_\Omega \xi_\varepsilon \cdot \nabla(\phi w_\varepsilon^i)\,dx \\ &= \int_\Omega \xi_\varepsilon \cdot (\nabla\phi) w_\varepsilon^i\,dx + \int_\Omega \xi_\varepsilon \cdot (\nabla w_\varepsilon^i)\phi\,dx.\end{aligned}$$

Note that the last term involves product of two weak converging sequences in $(L^2(\Omega))^n$. To overcome this difficulty, we use ϕu_ε as a test function in (3.2.8) to get

$$\begin{aligned}\int_\Omega {}^tA_0 e_i \nabla(\phi u_\varepsilon)\,dx &= \int_\Omega {}^tA_\varepsilon \nabla w_\varepsilon^i \nabla(\phi u_\varepsilon)\,dx \\ \int_\Omega {}^tA_0 e_i (\nabla\phi) u_\varepsilon\,dx + \int_\Omega {}^tA_0 e_i (\nabla u_\varepsilon)\phi\,dx &= \int_\Omega {}^tA_\varepsilon \nabla w_\varepsilon^i \nabla\phi u_\varepsilon\,dx \\ &\quad + \int_\Omega {}^tA_\varepsilon \nabla w_\varepsilon^i \nabla u_\varepsilon \phi\,dx \\ \int_\Omega {}^tA_0 e_i (\nabla\phi) u_\varepsilon\,dx + \int_\Omega {}^tA_0 e_i (\nabla u_\varepsilon)\phi\,dx &= \int_\Omega {}^tA_\varepsilon \nabla w_\varepsilon^i \nabla\phi u_\varepsilon\,dx \\ &\quad + \int_\Omega \xi_\varepsilon \cdot (\nabla w_\varepsilon^i)\phi\,dx\end{aligned}$$

Therefore, we have

$$\begin{aligned}\left\langle f_\varepsilon, \phi w_\varepsilon^i \right\rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &= \int_\Omega \xi_\varepsilon \cdot (\nabla\phi) w_\varepsilon^i\,dx + \int_\Omega {}^tA_0 e_i (\nabla\phi) u_\varepsilon\,dx \\ &\quad + \int_\Omega {}^tA_0 e_i (\nabla u_\varepsilon)\phi\,dx - \int_\Omega {}^tA_\varepsilon \nabla w_\varepsilon^i \nabla\phi u_\varepsilon\,dx.\end{aligned}$$

Now passing to limit, as $\varepsilon \to 0$, both sides we get

$$\begin{aligned}\langle f, \phi x_i \rangle_{H^{-1}(\Omega),H_0^1(\Omega)} &= \int_\Omega \xi_0 \cdot (\nabla\phi) x_i\,dx + \int_\Omega {}^tA_0 e_i (\nabla\phi) u_0\,dx \\ &\quad + \int_\Omega {}^tA_0 e_i (\nabla u_0)\phi\,dx - \int_\Omega {}^tA_0 e_i (\nabla\phi) u_0\,dx \\ \int_\Omega \xi_0 \cdot \nabla(\phi x_i)\,dx &= \int_\Omega \xi_0 \cdot (\nabla\phi) x_i\,dx + \int_\Omega {}^tA_0 e_i (\nabla u_0)\phi\,dx \\ \int_\Omega \xi_0 \cdot e_i \phi\,dx &= \int_\Omega A_0 \nabla u_0 \cdot e_i \phi\,dx.\end{aligned}$$

Hence $\xi_0 = A_0 \nabla u_0$ and u_0 solves (3.2.5).

Note that the choice of A_0 depends on the subsequence chosen and u_0 is not unique. All the above arguments were for a subsequences, extracted suitably. □

Example 3.3 (Periodic Case). If $A_\varepsilon(x) = (a^\varepsilon_{ij}(x))$ where $a^\varepsilon_{ij}(x) = a_{ij}\left(\frac{x}{\varepsilon}\right)$ a.e. $x \in \mathbb{R}^n$ such that $a_{ij} : Y = (0,1)^n \to \mathbb{R}$, extended Y-periodically to all $\mathbb{R}^n$ and restricted to Ω. In the periodic case, one can explicitly compute the matrix A_0, as seen in the informal asymptotic expansion. Note that, in the proof of the above theorem, the matrix A_0 is obtained as a limit of a converging subsequence of $\{A_\varepsilon\}$, which existed due to compactness of H-convergence. This A_0 was then used in defining the functions w^i_ε using (3.2.8). In the periodic case, we expect the function w^i_ε to be periodic and hence solve (3.2.8) in Y, instead of Ω. For each $i = 1, 2, \ldots$, we begin by solving the cell equation

$$\left\{\begin{array}{rll} -\mathrm{div}_y({}^tA\nabla_y(w^i(y) - y_i)) & = 0 & \text{in } Y \\ w_i(y) & \text{is} & Y\text{-periodic in } y \\ \frac{1}{|Y|}\int_Y w_i(y)\,dy & = 0 & \end{array}\right. \tag{3.2.9}$$

Set $w^i_\varepsilon(x) = \varepsilon w^i\left(\frac{x}{\varepsilon}\right)$. The reason behind having a factor of ε while defining w^i_ε is to avoid the factor of $1/\varepsilon$ while computing its first derivative. Thus,

$$\nabla_x w^i_\varepsilon(x) = \nabla_x\left[\varepsilon w^i\left(\frac{x}{\varepsilon}\right)\right] = \varepsilon \nabla_y w^i\left(\frac{x}{\varepsilon}\right)\frac{1}{\varepsilon} = \nabla_y w^i\left(\frac{x}{\varepsilon}\right)$$

and the vector

$${}^tA_\varepsilon(x)\nabla w^i_\varepsilon(x) = {}^tA\left(\frac{x}{\varepsilon}\right)\nabla_y w^i\left(\frac{x}{\varepsilon}\right)$$

in $(L^2(\Omega))^n$ is also Y-periodic. Therefore, by Theorem 1.1.3, we have that

$${}^tA_\varepsilon(x)\nabla w^i_\varepsilon(x) \rightharpoonup \frac{1}{|Y|}\int_Y {}^tA(y)\nabla_y w^i(y)\,dy \quad \text{weakly in } (L^2(\Omega))^n.$$

Recall that our aim is to identify A_0 such that (3.2.8) is satisfied. Consider the function $\phi \in \mathcal{D}(\Omega)$ and set $\phi_\varepsilon(y) = \phi(\varepsilon y)$ for $y \in (0,1)^n$ and extended to all of $\mathbb{R}^n$. Using $\phi_\varepsilon \in \mathcal{D}(\mathbb{R}^n)$ as a test function in the cell equation of w^i above, we get

$$\int_{\mathbb{R}^n} {}^tA(y)\nabla_y w^i(y)\cdot\nabla_y\phi_\varepsilon(y)\,dy = \int_{\mathbb{R}^n} {}^tA(y)e_i\cdot\nabla_y\phi_\varepsilon(y)\,dy$$

Using the change of variable $x = \varepsilon y$ and $\nabla_y = (1/\varepsilon)\nabla_x$, we get

$$\begin{aligned}\int_\Omega {}^tA\left(\frac{x}{\varepsilon}\right)\nabla_x w^i_\varepsilon(x)\cdot\frac{1}{\varepsilon}\nabla_x\phi(x)\,\frac{dx}{\varepsilon} &= \int_\Omega {}^tA\left(\frac{x}{\varepsilon}\right)e_i\cdot\frac{1}{\varepsilon}\nabla_x\phi(x)\,\frac{dx}{\varepsilon}\\ \int_\Omega {}^tA_\varepsilon(x)\nabla w^i_\varepsilon(x)\cdot\nabla\phi(x)\,dx &= \int_\Omega {}^tA_\varepsilon(x)e_i\cdot\nabla\phi(x)\,dx.\end{aligned}$$

By passing to the limit, as $\varepsilon \to 0$,

$$\int_\Omega\left[\frac{1}{|Y|}\int_Y {}^tA(y)\left(\nabla_y w^i(y) - e_i\right)\,dy\right]\cdot\nabla\phi(x)\,dx = 0.$$

We need to define A_0 such that (3.2.8) is satisfied. Thus, we set

$${}^tA_0e_i = \frac{1}{|Y|}\left[\int_Y {}^tA(y)\nabla_y w^i(y)\,dy - \left(\int_Y {}^tA(y)\,dy\right)e_i\right],$$

and, for all $\phi \in \mathcal{D}(\Omega)$

$$\int_\Omega {}^tA_0e_i\cdot\nabla\phi(x)\,dx = 0.$$

By density, the above equality is true for all $\phi \in H^1_0(\Omega)$. Therefore,

$$A_0e_i = \frac{1}{|Y|}\left[\int_Y A(y)\nabla_y\chi^i(y)\,dy - \left(\int_Y A(y)\,dy\right)e_i\right],$$

where χ^i solves the cell equation (3.2.9) where ${}^tA$ is replaced with A. Note that formula is same as (1.5.7).

Note that the A_0 obtained is independent of the choice of the subsequence of A_ε. Thus, u is a unique solution and all the convergences (3.2.4a) and (3.2.4b) are true for the entire sequence and not just for subsequences (cf. Eberlein-Šmuljan).

Recall that in the one dimensional case, we encountered the problem of product of two weak converging sequences (recall the sequence ξ_ε). In the one dimensional case, it was easy overcome this constraint by other means. However, the same idea would fail in higher dimension. The following theorem, popular as *compensated compactness*, is a fix of the problem in higher dimensions and is due to F. Murat and L. Tartar (cf. [Mur78a, Mur79, Tar79]).

Lemma 3.2.7 (div-curl lemma). *Let u_ε and v_ε be two sequences in $(L^2(\Omega))^n$ such that*

$$u_\varepsilon \rightharpoonup u_0 \text{ weakly in } (L^2(\Omega))^n$$

$$v_\varepsilon \rightharpoonup v_0 \text{ weakly in } (L^2(\Omega))^n.$$

If $\{\operatorname{div} u_\varepsilon\}$ is compact in $H^{-1}(\Omega)$ and $\{\operatorname{curl} v_\varepsilon\}$[1] is bounded in $(L^2(\Omega))^{n\times n}$, then

$$u_\varepsilon v_\varepsilon \to u_0 v_0 \text{ weak* in } \mathcal{D}'(\Omega).$$

□

Theorem 3.2.8 (Energy convergence). *If $A_\varepsilon \stackrel{H}{\rightharpoonup} A_0$ then*

$$\int_\Omega A_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx \to \int_\Omega A_0 \nabla u_0 . \nabla u_0 \, dx \tag{3.2.10}$$

where u_ε and u_0 are, respectively, the unique solution of (3.2.3) and (3.2.5).

The energy convergence also amounts to saying that the quadratic forms associated with the operators converge, i.e., $\langle \mathcal{A}_\varepsilon u_\varepsilon, u_\varepsilon\rangle \to \langle \mathcal{A}_0 u_0, u_0\rangle$. In section §4.3 (cf. Lemma 4.1), we will observe that this is actually subject to a special type of convergence called the Γ-convergence.

The energy functional (cf. (3.2.10)) involves a product of two weakly converging sequences and we have shown that the limit of the product is equal to the product of the limit. This property is not true, in general, and was possible due to the div-curl lemma.

3.3 Correctors

We have from (3.2.4a) that

$$\nabla u_\varepsilon \rightharpoonup \nabla u_0 \text{ weakly in } (L^2(\Omega))^n.$$

In general, the above convergence is not strong. However, by adjusting the term ∇u_0, we get a strong convergence (cf. Theorem 3.3.3). This adjustment is done by introducing the corrector matrix.

[1] for any $v \in (L^2(\Omega))^n$, $(\operatorname{curl} v)_{ij} = \left(\frac{\partial v_i}{\partial x_j} - \frac{\partial v_j}{\partial x_i}\right)$

The corrector matrices are obtained by looking for functions $\chi_\varepsilon^i \in H^1(\Omega)$, for $1 \le i \le n$, with the following properties:

$$\begin{cases} \chi_\varepsilon^i \rightharpoonup x_i \text{ weakly in } H^1(\Omega), \\ A_\varepsilon \nabla \chi_\varepsilon^i \rightharpoonup A_0 e_i \text{ weakly in } (L^2(\Omega))^n, \\ \operatorname{div}(A_\varepsilon \nabla \chi_\varepsilon^i) \text{ converges strongly in } H^{-1}(\Omega). \end{cases} \tag{3.3.1}$$

One procedure to build a function with above properties is by defining $\chi_\varepsilon^i \in H^1(\Omega)$, for $1 \le i \le n$, as a solution of

$$\begin{cases} -\operatorname{div}(A_\varepsilon \nabla \chi_\varepsilon^i) & = -\operatorname{div}(A_0 e_i) & \text{in } \Omega \\ \chi_\varepsilon^i & = x_i & \text{on } \partial\Omega. \end{cases} \tag{3.3.2}$$

Then the corrector matrix $D_\varepsilon \in (L^2(\Omega))^{n\times n}$ is defined as $D_\varepsilon e_i = \nabla \chi_\varepsilon^i$ for $1 \le i \le n$. For other choices of χ_ε^i, we may have $\tilde{D}_\varepsilon$. But they are "unique" in the sense that

$$D_\varepsilon - \tilde{D}_\varepsilon \to 0 \text{ in } [L^2_{\text{loc}}(\Omega)]^{n\times n}.$$

Some interesting properties of the corrector functions are given by the following proposition, the proof of which can be found in [CD99, MT97].

Theorem 3.3.1. *Let $A_\varepsilon \in M(\alpha, \beta, \Omega)$, χ_ε^i be a function with properties (3.3.1) and $D_\varepsilon e_i = \nabla \chi_\varepsilon^i$. Also, let A_ε H-converge to A_0, then the following are true:*

(a) $D_\varepsilon \rightharpoonup I$ weakly in $[L^2(\Omega)]^{n\times n}$, where I is the identity matrix.

(b) $A_\varepsilon D_\varepsilon \rightharpoonup A_0$ weakly in $[L^2(\Omega)]^{n\times n}$.

(c) ${}^t D_\varepsilon A_\varepsilon D_\varepsilon \rightharpoonup A_0$ weak in $[\mathcal{D}'(\Omega)]^{n\times n}$.*

Proof. Note that $\{D_\varepsilon\}$ is bounded in $[L^2(\Omega)]^{n\times n}$. Let $\Phi \in [C_c^\infty(\Omega)]^n$. Then

$$\Phi(x) = \sum_{i=1}^n \Phi_i(x) e_i$$

where $\Phi_i \in C_c^\infty(\Omega)$. Note that

$$\int_\Omega D_\varepsilon \Phi \, dx = \sum_{i=1}^n \int_\Omega D_\varepsilon e_i \Phi_i \, dx = \sum_{i=1}^n \int_\Omega \nabla \chi_\varepsilon^i \Phi_i \, dx.$$

Therefore,

$$\lim_{\varepsilon\to 0}\int_\Omega D_\varepsilon \Phi\,dx = \sum_{i=1}^n \int_\Omega e_i \Phi_i\,dx = \int_\Omega \Phi\,dx.$$

□

Lemma 3.3.2. *Let $A_\varepsilon \rightharpoonup A_0$ and $f_\varepsilon \to f$ in $H^{-1}(\Omega)$. If $u_\varepsilon := \mathcal{A}_\varepsilon^{-1} f_\varepsilon$ and $u_0 := \mathcal{A}^{-1} f$ then, for any $\Phi \in [C_c^\infty(\Omega)]^n$ and $\phi \in C_c^\infty(\Omega)$,*

$$\int_\Omega [A_\varepsilon(\nabla u_\varepsilon - D_\varepsilon \Phi)\cdot(\nabla u_\varepsilon - D_\varepsilon\Phi)]\phi\,dx \to \int_\Omega [A_0(\nabla u_0 - \Phi)\cdot(\nabla u_0 - \Phi)]\phi\,dx.$$

Proof. Let $\Phi \in [C_c^\infty(\Omega)]^n$. Then

$$\Phi(x) = \sum_{i=1}^n \Phi_i(x) e_i$$

where $\Phi_i \in C_c^\infty(\Omega)$. Consider

$$\begin{aligned}\int_\Omega [A_\varepsilon(\nabla u_\varepsilon - D_\varepsilon \Phi)\cdot(\nabla u_\varepsilon - D_\varepsilon\Phi)]\phi\,dx &= \int_\Omega (A_\varepsilon \nabla u_\varepsilon \cdot \nabla u_\varepsilon)\phi\,dx \\ &\quad - \sum_{i=1}^n \int_\Omega (A_\varepsilon \nabla u_\varepsilon \cdot D_\varepsilon e_i)\Phi_i \phi\,dx \\ &\quad - \sum_{i=1}^n \int_\Omega (A_\varepsilon D_\varepsilon e_i \cdot \nabla u_\varepsilon)\Phi_i\phi\,dx \\ &\quad + \int_\Omega (A_\varepsilon D_\varepsilon e_i \cdot D_\varepsilon e_j)\Phi_i\Phi_j\phi\,dx\end{aligned}$$

Passing to the limit, as $\varepsilon \to 0$, we have the result. □

The interest of the corrector matrix D_ε is the following theorem:

Theorem 3.3.3 (cf. [CD99]). *Let $A_\varepsilon \xrightarrow{H} A_0$. Let u_ε solve (3.2.3) and f_ε converge strongly to f. If u_ε weakly converge to u_0 in $H^1(\Omega)$, then*

$$\nabla u_\varepsilon - D_\varepsilon \nabla u_0 \to 0 \text{ strongly in } (L^1_{loc}(\Omega))^n.$$

Moreover, if $D_\varepsilon \in (L^r(\Omega))^{n\times n}, \|D_\varepsilon\|_{(L^r(\Omega))^n} \le C_0$ for $2 \le r \le +\infty$ and $\nabla u_0 \in (L^s(\Omega))^n$, $2 \le s < +\infty$, then

$$\nabla u_\varepsilon - D_\varepsilon \nabla u_0 \to 0 \text{ strongly in } (L^t_{loc}(\Omega))^n,$$

where $t = \min\left\{2, \frac{rs}{r+s}\right\}$. *If* u_0 *is a solution of* (3.2.5) *then*

$$\nabla u_\varepsilon - D_\varepsilon \nabla u_0 \to 0 \text{ strongly in } (L^t(\Omega))^n.$$

Proof. Observe that if $u_0 \in C_c^\infty(\Omega)$ then we choose $\Phi = \nabla u_0$ in the previous lemma and the result is true. For any $\delta > 0$, choose $\Phi \in C_c^\infty(\Omega)$ such that

$$\|\nabla u_0 - \phi\|_{(L^s(\Omega))^n} \le \delta \quad \forall s < \infty.$$

Therefore, for $\frac{1}{q} = \frac{1}{s} + \frac{1}{r}$,

$$\|\nabla u_0 - \phi\|_{[L^q(\Omega)]^n} \le C_0 \delta.$$

For $q \ge 1$, we note that both $r, s \ge 2$. Let $\omega \subset\subset \Omega$ and take $\phi \in C_c^\infty(\Omega)$ such that $\phi = 1$ on ω and $0 \le \phi \le 1$. Consider

$$\begin{aligned}
\limsup_{\varepsilon \to 0} \alpha \|\nabla u_\varepsilon - D_\varepsilon \Phi\|_{2,\omega}^2 &\le \lim_{\varepsilon \to 0} \int_\Omega [A_\varepsilon(\nabla u_\varepsilon - D_\varepsilon \Phi) \cdot (\nabla u_\varepsilon - D_\varepsilon \Phi)] \phi \, dx \\
&= \int_\Omega [A_0(\nabla u_0 - \Phi) \cdot (\nabla u_0 - \Phi)] \phi \, dx \\
&\le \frac{\beta^2}{\alpha} \|\nabla u_0 - \Phi\|_{2,\Omega}^2 \le \frac{C_0^2 \beta^2}{\alpha} \delta^2.
\end{aligned}$$

Because

$$\nabla u_\varepsilon - D_\varepsilon \nabla u_0 = (\nabla u_\varepsilon - D_\varepsilon \Phi) - (D_\varepsilon \nabla u_0 - D_\varepsilon \Phi)$$

and, choosing $t = \min\{2, q\}$,

$$\limsup_{\varepsilon \to 0} \|\nabla u_\varepsilon - D_\varepsilon \nabla u_0\|_{[L^t(\omega)]^n} \le \left(\frac{\beta}{\alpha} + 1\right) C_0 \delta.$$

□

3.4 Generalised Energy Convergence

A question of similar interest is to know the limit of $\|\nabla u_\varepsilon\|_{2,\Omega}^2$. One knows that this quantity is uniformly bounded and hence, at least for a subsequence, converges. We know that the limit is *not* $\|\nabla u_0\|_{2,\Omega}^2$, since we know from the above theorem that u_ε does not converge to u_0 strongly in $H_0^1(\Omega)$. We would like to know the limit and whether it can be expressed in terms of the function u_0. More generally, the problem can be framed as identifying the limit of $\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx$ where B_ε is a family of matrices in $\mathcal{M}(c, d, \Omega)$. More precisely, does there exist a matrix $B' \in \mathcal{M}(c', d', \Omega)$ such that, at least for a subsequence, we have

$$\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx \to \int_\Omega B' \nabla u_0 . \nabla u_0 \, dx?$$

The convergence question posed above is answered when $B_\varepsilon = A_\varepsilon$ (cf. (3.2.10)), in which case, it has been observed that $B^\sharp = A_0$, the H-limit of A_ε. The general problem was studied by Kesavan and Rajesh in [KR02].

Theorem 3.4.1. *Let $A_\varepsilon \in \mathcal{M}(a, b, \Omega)$, $B_\varepsilon \in \mathcal{M}(c, d, \Omega)$ and χ_ε^i be a function with properties (3.3.1) and $D_\varepsilon e_i = \nabla \chi_\varepsilon^i$. Also, let A_ε H-converge to A_0, then the following are true:*

(a) There exists a $B^\sharp$ (depending only on $\{A_\varepsilon\}$ and $\{B_\varepsilon\}$) such that

$$ {}^t D_\varepsilon B_\varepsilon D_\varepsilon \rightharpoonup B^\sharp \text{ weak* in } (\mathcal{D}'(\Omega))^{n \times n}. \tag{3.4.1}$$

(b) If $B_\varepsilon = A_\varepsilon$ for all ε, then $B^\sharp = A_0$.

(c) If B_ε's are symmetric, then $B^\sharp$ is symmetric.

(d) $B^\sharp \in \mathcal{M}\left(c, d(\frac{b}{a})^2, \Omega\right)$. □

The existence of the matix $B^\sharp$, mentioned in the above proposition, was shown by Kesavan and Vanninathan, for the periodic case (cf. [KV77]), and by Kesavan and Saint Jean Paulin in the general case (cf. [KP97]), in the process of homogenizing an optimal control problem.

It was observed that the required B' is actually the $B^\sharp$ obtained in Proposition 3.4.1 and thus

$$\int_\Omega B_\varepsilon \nabla u_\varepsilon . \nabla u_\varepsilon \, dx \to \int_\Omega B^\sharp \nabla u_0 . \nabla u_0 \, dx. \tag{3.4.2}$$

Therefore, if C is the positive square root of the matrix $B^\sharp$ when $B_\varepsilon = I$, for all $\varepsilon > 0$, then

$$\|\nabla u_\varepsilon\|_{2,\Omega}^2 \to \|C \nabla u_0\|_{2,\Omega}^2.$$

3.5 Optimal Bounds

The computation of H-limit A_0 is, in general, not easy to characterise for a given sequence of matrices $\{A_\varepsilon\}$. We ask if it is possible to conclude something in the simple case when $A_\varepsilon = a_\varepsilon(x)I$, i.e., are isotropic and the domain is a (non-periodic) mixture of two materials. Let $\Omega = \Omega_1^\varepsilon \cup \Omega_2^\varepsilon$ such that $\Omega_1^\varepsilon \cap \Omega_2^\varepsilon = \emptyset$. Let

$$a_\varepsilon(x) = \begin{cases} a_1 & \text{if } x \in \Omega_1^\varepsilon \\ a_2 & \text{if } x \in \Omega_2^\varepsilon \end{cases}$$

such that $0 < \alpha \leq a_\varepsilon(x) \leq \beta$ a.e. on Ω. Without loss of generality, assume $a_1 \leq a_2$. If $1_{\Omega_1^\varepsilon}$ denotes the characteristic function of Ω_1^ε then $a_\varepsilon(x) = a_1 1_{\Omega_1^\varepsilon}(x) + a_2(1 - 1_{\Omega_1^\varepsilon}(x))$. Since $1_{\Omega_1^\varepsilon}$ is bounded in $L^\infty(\Omega)$, there is a $\theta \in L^\infty(\Omega)$ such that

$$1_{\Omega_1^\varepsilon} \rightharpoonup \theta \text{ weak-* in } L^\infty(\Omega).$$

Therefore, $a_\varepsilon \rightharpoonup a_1\theta(x) + a_2(1 - \theta(x))$ weak-* in $L^\infty(\Omega)$. Note that $A_\varepsilon(x) = a_\varepsilon(x)I$ is symmetric, therefore, its H-limit A_0 (for a subsequence) is also symmetric and is in $M(\alpha, \beta, \Omega)$. Thus, A_0 admits strictly positive eigenvalues $\lambda_1(x), \dots, \lambda_n(x)$ defined in Ω.

Theorem 3.5.1. *The eigenvalues $\lambda_1(x), \dots, \lambda_n(x)$ of A_0 satisfy the following inequalities, a.e. in Ω:*

$$A^*(x) \leq \lambda_i(x) \leq \overline{A}(x) \quad \forall i = 1, 2, \dots, n;$$

$$\sum_{i=1}^{n} \frac{1}{\lambda_i(x) - a_1} \leq \frac{1}{A^*(x) - a_1} + \frac{n-1}{\overline{A}(x) - a_1};$$

and

$$\sum_{i=1}^{n} \frac{1}{a_2 - \lambda_i(x)} \leq \frac{1}{a_2 - A^*(x)} + \frac{n-1}{a_2 - \overline{A}(x)}$$

where

$$A^*(x) := \left(\frac{\theta(x)}{a_1} + \frac{1 - \theta(x)}{a_2} \right)^{-1}$$

and

$$\overline{A}(x) := a_1\theta(x) + (1 - \theta(x))a_2.$$

In the two dimensional case and when $\theta = \frac{\mu(\Omega_1^\varepsilon)}{\mu(\Omega)}$, the above result has a nice geometric interpretation. For each $\theta \in (0,1)$, consider the set of points

$$(X(\theta), Y(\theta)) := \left(\frac{a_1 a_2}{(a_2 - a_1)\theta + a_1}, a_2 - (a_2 - a_1)\theta \right).$$

Note that $(X(0), Y(0)) = (a_2, a_2)$ and $(X(1), Y(1)) = (a_1, a_1)$. The points subscribe to a concave hyperbola H_1 given by $Y = a_1 + a_2 - \frac{a_1 a_2}{X}$, for $a_1 \leq X, Y \leq a_2$, above the line $Y = X$. The reflection of H_1 along $Y = X$ line gives a convex hyperbola H_2 with equation

$$Y = \frac{a_1 a_2}{a_1 + a_2 - X}.$$

Let $G(\theta) := (A^*, A^*)$ and $M(\theta) := (\overline{A}, \overline{A})$ be the points on $Y = X$ line. Then the points $G(\theta), M(\theta), (A^*, \overline{A}) \in H_1$ and $(\overline{A}, A^*) \in H_2$ form a square. The condition

$$A^*(x) \leq \lambda_i(x) \leq \overline{A}(x) \quad \forall i = 1, 2$$

implies that the eigenvalues of the H-limit A_0 are contained in the square describes above. The remaining two inequalities imply that the eigenvalues are contained between two hyperbolas H_3 and H_4 given by the equation

$$H_3 : \frac{1}{X - a_1} + \frac{1}{Y - a_1} = \frac{1}{A^* - a_1} + \frac{1}{\overline{A} - a_1}$$

and

$$H4 : \frac{1}{a_2 - X} + \frac{1}{a_2 - Y} = \frac{1}{a_2 - A^*} + \frac{1}{a_2 - \overline{A}}.$$

The well-known Hashin-Shtrikman (Clausius-Mossoti or Lorentz-Lorenz or Maxwell-Garnett) inequalities is the two dimensional case with A_0 is isotropic. In this case, the inequalities reduce to

$$\left(\frac{\theta a_1 + (1-\theta) a_2 + a_2}{(1-\theta) a_1 + \theta a_2 + a_1} \right) a_1 \leq \lambda \leq \left(\frac{\theta a_1 + (1-\theta) a_2 + a_1}{(1-\theta) a_1 + \theta a_2 + a_2} \right) a_2.$$

Chapter 4

Γ-Convergence

4.1 Motivation

Recall that the weak solution u of (1.2.2) can also be characterized as the minimizer in $H_0^1(\Omega)$ of the functional

$$J(v) = \frac{1}{2}\int_\Omega A\nabla v.\nabla v\,dx - \langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}\,,$$

i.e.,

$$J(u) = \min_{v\in H_0^1(\Omega)} J(v).$$

Thus, the problem of studying the asymptotic behaviour of the second order elliptic problem

$$\begin{cases} -\mathrm{div}(A_\varepsilon\nabla u_\varepsilon) &= f \quad \text{in } \Omega \\ \qquad\qquad u_\varepsilon &= 0 \quad \text{on } \partial\Omega, \end{cases}$$

with $\{A_\varepsilon\} \subset M(\alpha,\beta,\Omega)$ is equivalent to finding a functional J on $H_0^1(\Omega)$ whose minimum is the solution of the homogenized elliptic equation such that both the minimizers and minima of J_ε converge to the minimizers and minima of J. Thus, we need to study the convergence of functionals such that the minimizers and minima converge.

4.2 Direct Method of Calculus of Variation

To motivate the direct method of calculus of variation, we begin by recalling a basic result due to Karl Weierstrass, known as the *extreme value theorem*.

Our aim, in this section, will be to generalise this basic result to an arbitrary topological space. We recall the proof to motivate some definitions.

Theorem 4.2.1 (Extreme value theorem)**.** *Any real valued continuous function f on a closed bounded interval $[a, b]$ attains its minimum.*

Proof. We show f has a lower bound. Suppose f is not bounded below then there exists a sequence $\{x_n\} \subset [a, b]$ such that $f(x_n) > n$. Since the sequence x_n is bounded, by Bolzano-Weierstrass theorem, there exists a convergent subsequence $\{x_{n_k}\}$ and let x be its limit. Thus, by continuity of f, $f(x_{n_k})$ converges to $f(x)$ which is a contradiction since $f(x_{n_k}) > n_k \geq k$. Thus there exists a infimum (greatest lower bound) C such that $f(x) \geq C$ for all $x \in [a, b]$. It now remains to show that there exists a $x \in [a, b]$ such that $f(x) = C$.

Let $\{y_n\} \subset [a, b]$ be a sequence such that $f(y_n) \leq C + 1/n$. Since $C \leq f(y_n)$ for all n, we have that $f(y_n)$ converges to C (thus the sequence y_n is called minimizing sequence). By applying Bolzano-Weierstrass theorem again, there exists a convergent subsequence $\{y_{n_k}\}$ and let y be its limit. Using continuity of f again, $f(y_{n_k})$ converges to $f(y)$. Thus $f(y) = C$. Moreover, since $[a, b]$ is closed $y \in [a, b]$. □

We shall, henceforth, concentrate on the minimum of the function f, since the corresponding result for maximum can be obtained by applying the results to $-f$.

Definition 4.2.2. *Let X be a topological space. A function $F : X \to \overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty, +\infty\}$ is said to be* lower semicontinuous (lsc) *at a point $x \in X$ if*

$$F(x) = \sup_{U \in N(x)} \inf_{y \in U} F(y).$$

F is lower semicontinuous *on X if F is lower semicontinuous at each point $x \in X$.*

Remark 4.2.3. Let X be a topological space satisfying first axiom of countability. Then a function $F : X \to \overline{\mathbb{R}}$ is lower semicontinuous at $x \in X$ iff

$$F(x) \leq \liminf_{n \to \infty} F(x_n)$$

for every sequence $\{x_n\}$ converging to $x \in X$. This is the sequential characterisation of the lower semi-continuity.

Note that one can, in fact, weaken the hypothesis of the extreme value theorem.

Exercise 4.1. Prove the Extreme value theorem when f is lower semicontiuous. Replace lower semicontinuity hypothesis with upper semicontinuity to obtain the maximizer.

Exercise 4.2. Show that if F is lower semicontinuous then the sublevel set $\{F \leq \alpha\} := \{x \in X : F(x) \leq \alpha\}$ is closed for all $\alpha \in \mathbb{R}$.

We have already seen that the continuity property in Extreme Value theorem can be relaxed to lower semicontinuity. Another crucial element of the proof is the Bolzano-Weierstrass theorem which is about the compactness of the interval $[a, b]$.

Definition 4.2.4. *A function $F : X \to \overline{\mathbb{R}}$ is* coercive *on X if the closure of the sublevel set $\{F \leq \alpha\} := \{x \in X : F(x) \leq \alpha\}$ is compact in X for every $\alpha \in \mathbb{R}$.*

Exercise 4.3. Show that if F is a coercive functional on X and $G \geq F$, then G is coercive.

Exercise 4.4. If F is coercive then there is a non-empty compact set K such that

$$\inf_{x \in X} F(x) = \inf_{x \in K} F(x).$$

Definition 4.2.5. *A* minimizing sequence *for $F : X \to \overline{\mathbb{R}}$ in X is a sequence $\{x_n\}$ in X such that*

$$\inf_{y \in X} F(y) = \lim_{n \to \infty} F(x_n)$$

Theorem 4.2.6. *Let X be a topological space. Assume that the function $F : X \to \overline{\mathbb{R}}$ is coercive and lower semicontinuous. Then F has a minimizer in X.*

Proof. If F is identically $+\infty$ or $-\infty$, then every point of X is a minimum point for F. If F takes the value $-\infty$, then all those points are minimizers of F. Suppose now that F is not identically $+\infty$ and $F > -\infty$. Let $\{x_n\}$ be a sequence in X such that

$$\lim_{n \to \infty} F(x_n) = \inf_{y \in X} F(y) := d.$$

The existence of such a sequence is clear. Without loss of generality, we can assume that $F(x_n) < +\infty$ for all n. Let $\alpha := \sup_n F(x_n) < +\infty$. Moreover, since F is coercive, the sublevel set $\{F \leq \alpha\}$ is compact and hence there is a subsequence $\{x_k\}$ of $\{x_n\}$ which converges to a point $x \in X$. Since F is lsc we obtain

$$d = \inf_{y \in X} F(y) \leq F(x) \leq \liminf_{k \to \infty} F(x_k) = d.$$

Thus, $F(x) = d$ and hence is the minimizer of F in X. which proves our theorem. □

Definition 4.2.7. *A family of functionals $\{F_n\}$ on X is said to be* equi-coercive*, if for every $\alpha \in \mathbb{R}$, there is a compact set K_α of X such that the sublevel sets $\{F_n \leq \alpha\} \subseteq K_\alpha$ for all n.*

Exercise 4.5. If $\{F_n\}$ is a family of equi-coercive, then there is a non-empty compact K (independent of n) such that

$$\inf_{x \in X} F_n(x) = \inf_{x \in K} F_n(x).$$

Proposition 4.2.8. *A family of functions F_n on X is equi-coercive if and only if there exists a lower semicontinuous coercive function $\Psi : X \to \overline{\mathbb{R}}$ such that $F_n \geq \Psi$ on X, for every n.*

Proof. Let $\Psi : X \to \overline{\mathbb{R}}$ be a lower semicontinuous coercive function such that $F_n \geq \Psi$ on X, for every n. Set $K_\alpha := \{\Psi \leq \alpha\}$. K_α is closed and compact because of the lsc and coercivity of Ψ, respectively. Moreover, $\{F_n \leq \alpha\} \subseteq K_\alpha$, for all n. Thus, F_n are equi-coercive.

Conversely, let F_n be equi-coercive. Then, for each $\alpha \in \mathbb{R}$, there is a compact set K_α such that $\{F_n \leq \alpha\} \subseteq K_\alpha$, for all n. We shall now define $\Psi : X \to \overline{\mathbb{R}}$ as

$$\Psi(x) = \begin{cases} +\infty, & \text{if } x \notin K_\alpha, \forall \alpha \in \mathbb{R} \\ \inf\{\alpha \mid x \in K_\beta \text{ for all } \beta > \alpha\}. \end{cases}$$

We now show that $\Psi \leq F_n$ for all n. Let $x \in X$. If $F_n(x) = +\infty$, for all n, then by definition, $\Psi(x) = F_n(x) = +\infty$. Otherwise, let F_k be a subfamily such that $F_k(x) = \beta_k < \infty$. Thus, $x \in K_{\beta_k}$ for all k and hence $\Psi(x) = \inf_k\{\beta_k\} \leq F_n(x)$. Thus, $\Psi(x) \leq F(x)$, for every $x \in X$. It now remains to show that Ψ is lsc and coercive. Note that any $x \in \{\Psi \leq \alpha\}$ implies $x \in K_\beta$ for all $\beta > \alpha$. Therefore, the sublevel

$$\{\Psi \leq \alpha\} = \cap_{\beta>\alpha} K_\beta$$

is an arbitrary intersection compact sets and hence is closed and compact. □

Definition 4.2.9. *Let X be a vector space. We say a function $F : X \to \overline{\mathbb{R}}$ is* convex *if*

$$F(tx + (1-t)y) \leq tF(x) + (1-t)F(y)$$

for every $t \in (0,1)$ and for every $x, y \in X$ such that $F(x) < +\infty$ and $F(y) < +\infty$. We say a function $F : X \to \overline{\mathbb{R}}$ is strictly convex *if F is not identically $+\infty$ and*

$$F(tx + (1-t)y) < tF(x) + (1-t)F(y)$$

for every $t \in (0,1)$ and for every $x, y \in X$ such that $x \neq y$, $F(x) < +\infty$ and $F(y) < +\infty$.

If F is constant, then one can see that X is the set of all minimizers of F. We now show that with strict convexity the minimizer, if exists, is unique.

Proposition 4.2.10. *Let X be a vector space. Let $F : X \to \overline{\mathbb{R}}$ be a strictly convex function. Then F has at most one minimizer in X.* □

Proof. If x and y are two minimizers of F in X, then

$$F(x) = F(y) = d := \min_{z \in X} F(z) < +\infty.$$

If $x \neq y$, by strict convexity we have

$$F(tx + (1-t)y) < tF(x) + (1-t)F(y) = d, \quad \forall t \in (0,1).$$

This contradicts the fact that d is a minimum of F. Therefore $x = y$. □

Thus, combining Theorem 4.2.6 and Proposition 4.2.10, we have the following theorem.

Theorem 4.2.11. *Let X be a topological vector space and let $F : X \to \overline{\mathbb{R}}$ be a lower semicontinuous, coercive and strictly convex functional, then F has a unique minimizer.*

4.3 Sequential Γ-Convergence

The notion of Γ-convergence was introduced by Ennio De Giorgi in a sequence of papers (cf. [GS73, Gio75, GF75]). An excellent account of this concept is the book of Dal Maso [DM93] and A. Braides [Bra02].

Definition 4.3.1. *A function F is said to be the* Γ-limit *of F_n (denoted as $F_n \xrightarrow{\Gamma} F$) w.r.t the topology of X, if $F = F^+ = F^-$, where*

(i)
$$F^-(x) = \sup_{U \in N(x)} \liminf_{n\to\infty} \inf_{y \in U} F_n(y).$$

(ii)
$$F^+(x) = \sup_{U \in N(x)} \limsup_{n\to\infty} \inf_{y \in U} F_n(y).$$

We say F^- is the Γ-lower limit and F^+ is the Γ-upper limit.

Remark 4.3.2. If X is a topological space satisfying first axiom of countability, the Γ-limit can be characterised as satisfying the following two conditions:

(i) For every $x \in X$ and for every sequence $\{x_n\}$ converging to x in X, we have
$$\liminf_{n\to\infty} F_n(x_n) \geq F(x).$$

(ii) For every $x \in X$, there exists a sequence $\{x_n\}$ converging to x in X (called the Γ-realising sequence) such that
$$\lim_{n\to\infty} F_n(x_n) = F(x).$$

□

Exercise 4.6. Show that if $F_n \xrightarrow{\Gamma} F$, $G_n \xrightarrow{\Gamma} G$ and $F_n \leq G_n$, for each n, then $F \leq G$.

Exercise 4.7. Show that if F_n Γ-converges to F, then F is lower semicontinuous.

Exercise 4.8. Let X be a topological vector space. Show that if $F_n : X \to \overline{\mathbb{R}}$ is convex for each n, then Γ-$\limsup_n F_n$ is convex. Also show that the Γ-$\liminf_n F_n$ is, in general, not convex.

Exercise 4.9. Compute the Γ-limit of a constant sequence $F_n = F$ on X.

Theorem 4.3.3. *Let X be a topological space and F_n be a family functions on X.*

1. *If U is an open subset of X, then*

$$\inf_{x\in U} F^+(x) \geq \limsup_n \inf_{x\in U} F_n(x).$$

2. *If K is a compact subset of X, then*

$$\inf_{x\in K} F^-(x) \leq \liminf_n \inf_{x\in K} F_n(x).$$

Proof. 1. Let $x \in U$. Then, from the definition of Γ-upper limit which says $F(x)$ is sup over all neighbourhoods of x, we have

$$F^+(x) \geq \limsup_{n\to\infty} \inf_{y\in U} F_n(y).$$

Therefore,

$$\inf_{x\in U} F^+(x) \geq \limsup_{n\to\infty} \inf_{y\in U} F_n(y).$$

2. Since F^- is lsc and by the compactness of K, F^- attains its minimum on K (cf. Theorem 4.2.6). Set $d := \liminf_n \inf_{x\in K} F_n(x)$ and let x_n be a sequence (extracting subsequence, if necessary) in K such that $\lim_n F_n(x_n) = d$. Thus, there is a subsequence x_k which converges to some $x \in K$. Therefore, for every neighbourhood U of x, $\inf_{y\in U} F_k(y) \leq F_k(x_k)$ for infinitely many k. Now, taking $\liminf$ both sides,

$$\liminf_k \inf_{y\in U} F_k(y) \leq \liminf_k F_k(x_k) = d$$

and taking supremum over all neighbourhoods U of x, we still have

$$F^-(x) = \sup_U \liminf_k \inf_{y\in U} F_k(y) \leq d.$$

Now, since $x \in K$, $\inf_{x\in K} F^-(x) \leq d$.

□

Theorem 4.3.4 (Fundamental Theorem of Γ-convergence)**.** *Let X be a topological space. Let $\{F_n\}$ be a equi-coercive family of functions and let F_n Γ-converges to F in X, then* *(i) F is coercive.*

(ii) $\lim_{n\to\infty} d_n = d$, where $d_n = \inf_{x\in X} F_n(x)$ and $d = \inf_{x\in X} F(x)$. That is, the minima converges.

(iii) The minimizers of F_n converge to a minimizer of F.

Proof. Since $\{F_n\}$ are equi-coercive, by Proposition 4.2.8, there is a lsc, coercive function Ψ on X such that $F_n \geq \Psi$. Now, by Exercise 4.6, $F \geq \Psi$ and by Exercise 4.3 F is coercive.

Now, by putting $U = X$ in Theorem 4.3.3, we get $d \geq \limsup_n d_n$. We now need to show that $d \leq \liminf_n d_n$. If F_n are all not identically $+\infty$, then $\liminf_n d_n < +\infty$. Set $\liminf_n d_n = \alpha$. By the equi-coercivity of F_n, there is a compact set K_α such that $\{F_n \leq \alpha\} \subseteq K_\alpha$, for all n. Consider,

$$\begin{aligned} d \leq \inf_{y\in K_\alpha} F(y) &\leq \liminf_n \inf_{y\in K_\alpha} F_n(y) \\ &= \liminf_n \inf_{y\in X} F_n(y) \\ &= \liminf_n d_n. \end{aligned}$$

Thus, $\limsup_n d_n \leq d \leq \liminf_n d_n$ and hence, $\lim_n d_n = d$.

Since F is coercive and lsc (Γ-limit is always lsc), then by Theorem 4.2.6, F attains its minimum. Let x_n^* be a minimizer of F_n, then since F_n are equi-coercive x_n^* belong to a compact set K of X and hence converges up to a subsequence. Let $x_n^* \to x^*$ in X. We need to show that $F(x^*) = d$. By Γ-lower limit,

$$F(x^*) \leq \liminf_n F_n(x_n^*) = \liminf_n d_n = d.$$

But, $d \leq F(x^*)$. Hence $d = F(x^*)$. □

Theorem 4.3.5 (Compactness)**.** *If X is a topological space satisfying second axiom of countability then any sequence of functionals $F_n : X \to \overline{\mathbb{R}}$ has a Γ-convergent subsequence.*

Proof. Let $\{U_k\}_{k\in\mathbb{N}}$ be a countable base for the topology of X. For each k, let $d_k^n = \inf_{y\in U_k} F_n(y)$. Thus, $\{d_k^n\}_n$ is a sequence in $\overline{\mathbb{R}}$ which is compact, hence has a subsequence $\{d_k^m\}_m$ whose limit as $m \to \infty$ exists in $\overline{\mathbb{R}}$. Thus, for each k, we have subsequence $\{d_k^m\}_m$ whose limit as $m \to \infty$ exists in $\overline{\mathbb{R}}$.

Choose the diagonal sequence d_k^k whose limit exists n $\overline{\mathbb{R}}$ as $k \to \infty$. In other words, we have chosen a subsequence F_k of F_n such that

$$\lim_{k\to\infty} d_k^k = \lim_{k\to\infty} \inf_{y\in U_k} F_k(y).$$

Now, define $F(x) = \sup_{U\in N(x)} \lim_{k\to\infty} \inf_{y\in U_k} F_k(y)$ and we have by definition F_k Γ-converges to F. □

Example 4.1. Let $A_\varepsilon \overset{H}{\rightharpoonup} A_0$ then we wish to show that $J_\varepsilon \overset{\Gamma}{\rightharpoonup} J$ in the weak topology of $H_0^1(\Omega)$ where

$$J_\varepsilon(u) = \int_\Omega A_\varepsilon \nabla u.\nabla u \, dx$$

and

$$J(u) = \int_\Omega A_0 \nabla u.\nabla u \, dx.$$

Let $u \in H_0^1(\Omega)$. We need to find a sequence $\{u_\varepsilon\}$ in $H_0^1(\Omega)$ such that u_ε converges to u weakly in $H_0^1(\Omega)$ and $\lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) = J(u)$. Let $u_\varepsilon \in H_0^1(\Omega)$ be the solution of

$$-\mathrm{div}(A_\varepsilon \nabla u_\varepsilon) = -\mathrm{div}(A_0 \nabla u). \tag{4.3.1}$$

Then, it follows from H-convergence that $u_\varepsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$ and $\int_\Omega A_\varepsilon \nabla u_\varepsilon.\nabla u_\varepsilon \, dx \to \int_\Omega A_0 \nabla u.\nabla u \, dx$. Thus, we have shown the existence of a sequence $\{u_\varepsilon\}$ converging weakly to u in $H_0^1(\Omega)$ such that

$$\lim_{\varepsilon\to 0} J_\varepsilon(u_\varepsilon) = J(u).$$

Now, let $w_\varepsilon \in H_0^1(\Omega)$ be a sequence such that $w_\varepsilon \rightharpoonup u$ weakly in $H_0^1(\Omega)$. Then, the solution u_ε obtained in (4.3.1) minimizes the functional

$$\frac{1}{2} J_\varepsilon(v) - \int_\Omega A_0 \nabla u.\nabla v \, dx.$$

Hence, in particular, we have

$$\begin{aligned} \frac{1}{2}\int_\Omega A_\varepsilon \nabla w_\varepsilon.\nabla w_\varepsilon \, dx - \int_\Omega A_0 \nabla u.\nabla w_\varepsilon \, dx \;\; &\geq \;\; \frac{1}{2}\int_\Omega A_\varepsilon \nabla u_\varepsilon.\nabla u_\varepsilon \, dx \\ &\quad - \int_\Omega A_0 \nabla u.\nabla u_\varepsilon \, dx \end{aligned}$$

and taking liminf on both sides of above inequality we have

$$\liminf_{\varepsilon\to 0} J_\varepsilon(w_\varepsilon) \geq J(u).$$

Hence $J_\varepsilon \xrightharpoonup{\Gamma} J$ in the weak topology of $H_0^1(\Omega)$.

In the above example, we assume the H-convergence of the matrix coefficients to describe the Γ-limit. A general question of interest is the following: If for any sequence of functionals, by compactness, there is a Γ-limit, then under what conditions one can get an integral representation of Γ-limit. In the next section, we describe the situation in one-dimension.

4.4 Integral Representation: One Dimension

For any given $1 < p < \infty$ and $c_1, c_2, c_3 > 0$, let $\mathcal{F} = \mathcal{F}(p, c_1, c_2, c_3)$ be the class of all functionals $F : W^{1,p}(\Omega) \to [0, +\infty)$ such that

$$F(u) = \int_\Omega f(x, \nabla u(x))\, dx$$

where $f : \Omega \times \mathbb{R}^n \to [0, +\infty)$

H 1. *is a Borel function such that $\xi \mapsto f(x, \xi)$ is convex for all $x \in \Omega$,*

H 2. *and satisfies the* growth conditions *of order p*

$$c_1|\xi|^p - c_2 \leq f(x, \xi) \leq c_3(1 + |\xi|^p), \quad \forall x \in \Omega, \xi \in \mathbb{R}^n.$$

Exercise 4.10. If f satisfies **H**1 and **H**2, then f satisfies the local Lipschitz condition

$$|f(x, \xi) - f(x, \zeta)| \leq k(1 + |\xi|^{p-1} + |\zeta|^{p-1})|\xi - \zeta| \quad \forall \xi, \zeta \in \mathbb{R}^n.$$

The constant k depends only on c_3 and p.

We take $n = 1$ in the dimension of Euclidean space and set $\Omega = (a, b)$. Observe that any functional in $\mathcal{F}$ is invariant by addition of a constant c, i.e., $F(u + c) = F(u)$. Thus, it is sufficient to characterize in the space

$$X = \{u \in W^{1,p}(\Omega) \mid u(b) = 0\}$$

equipped with L^p norm instead of $W^{1,p}(\Omega)$. Since X is embedded in $L^\infty(a, b)$, $L^1(a, b) \subset X^\star$.

Proposition 4.4.1. *Let* $X = \{u \in W^{1,p}(\Omega) \mid u(b) = 0\}$ *equipped with* L^p *norm. Let* $F \in \mathcal{F}$ *and consider its integrand* f *as a function on* X*, then* $F^\star : X^\star \to \mathbb{R}$ *is given as*

$$F^\star(\phi) = \int_a^b f^\star\left(x, -\int_a^x \phi(t)\,dt\right) dx, \quad \forall \phi \in L^1(a,b).$$

Proof. Let us assume $f(x,\cdot) \in C^1(\mathbb{R})$ for all $x \in (a,b)$. Due to the growth conditions and continuity of f,

$$f^\star(x,\xi^\star) = \sup_{\xi\in\mathbb{R}}\{\xi^\star\cdot\xi - f(x,\xi)\} = \max_{\xi\in\mathbb{R}}\{\xi^\star\cdot\xi - f(x,\xi)\}.$$

Thus, if ζ is the point at which maximum is attained, then

$$f^\star(x,\zeta^\star) = \zeta^\star\cdot\zeta - f(x,\zeta) \quad \text{if and only if } \zeta^\star - \frac{\partial f}{\partial \zeta}(x,\zeta) = 0. \tag{4.4.1}$$

Let $\phi \in L^1(a,b)$, define $\Phi \in W^{1,1}(a,b)$ as,

$$\Phi(x) = -\int_a^x \phi(t)\,dt.$$

Note that $\Phi' = -\phi$ and $\Phi(a) = 0$. Thus, the convex conjugate of F is given as

$$\begin{aligned}
F^\star(\phi) &= \sup_{v\in X}\left\{\int_a^b [\phi(x)v(x) - f(x,v'(x))]\,dx\right\}\\
&= \sup_{v\in X}\left\{\int_a^b [\Phi(x)v'(x) - f(x,v'(x))]\,dx\right\} \quad \text{(integration by parts)}\\
&= \max_{v\in X}\left\{\int_a^b [\Phi(x)v'(x) - f(x,v'(x))]\,dx\right\}\\
&= \int_a^b [\Phi(x)u'(x) - f(x,u'(x))]\,dx.
\end{aligned}$$

By computing Euler equations, we have $\Phi - \frac{\partial f}{\partial u}(x,u') = c$, for some constant c. But $\Phi(a) = 0$ and $\frac{\partial f}{\partial u}(a,u'(a)) = 0$, implies that $c = 0$ and thus, $\Phi = \frac{\partial f}{\partial u}(x,u')$ a.e. on (a,b). By choosing $\zeta^\star = \Phi(x)$ and $\zeta = u'(x)$ in (4.4.1), we have

$$\Phi(x) = \frac{\partial f}{\partial u}(x,u'(x)) \quad \text{if and only if } f^\star(x,\Phi(x)) = \Phi(x)u'(x) - f(x,u'(x)).$$

Hence,

$$\begin{aligned} F^\star(\phi) &= \int_a^b \big(\Phi(x)u'(x) - f(x,u'(x)\big)\,dx \\ &= \int_a^b f^\star(x,\Phi(x)\,dx \\ &= \int_a^b f^\star\left(x, -\int_a^x \phi(t)\,dt\right)\,dx \end{aligned}$$

Now, for any f satisfying **H**1 and **H**2, we define

$$f_\varepsilon(x,\xi) = \int_a^b \rho_\varepsilon(x-y)f(y,\xi)\,dy,$$

where ρ_ε are the sequence of mollifiers. Observe that f_ε are convex in the second variable and, by Jensen's inequality, $f_\varepsilon \geq f$. Also, observe that $\lim_\varepsilon f^\star_\varepsilon(x,\xi^\star) = f^\star(x,\xi^\star)$ for all $x \in (a,b)$ and $\xi^\star \in \mathbb{R}$. We have, for each ε,

$$F^\star_\varepsilon(\phi) = \int_a^b f^\star_\varepsilon\left(x, -\int_a^x \phi(t)\,dt\right)\,dx \quad \forall \phi \in L^1(a,b).$$

Now, by dominated convergence theorem and $F^\star \geq F^\star_\varepsilon$, we get

$$F^\star(\phi) \geq \lim_k F^\star_\varepsilon(\phi) = \int_a^b f^\star\left(x, -\int_a^x \phi(t)\,dt\right)\,dx.$$

Also, by the convex conjugate definition, $f^\star(x,\xi^\star) \geq \xi^\star\xi - f(x,\xi)$ for all $x,\xi,\xi^\star$. Now, choose $\xi^\star = \Phi(x), \xi = v'$, where $v \in X$ and integrate both sides of above inequality,

$$\begin{aligned} \int_a^b f^\star(x,\Phi(x))\,dx &\geq \int_a^b (\Phi(x)v'(x) - f(x,v'(x)))\,dx \\ &= \int_a^b (\phi(x)v(x) - f(x,v'(x)))\,dx. \end{aligned}$$

Taking supremum over $v \in V$, we have $F^\star(\phi) \leq \int_a^b f^\star(x,\Phi(x))\,dx$. $\square$

Proposition 4.4.2. *Let $g_n : \Omega \times \mathbb{R}^n \to [0,+\infty)$ satisfy hypotheses* **H**1 *and* **H**2*, for all n. If $g_n(\cdot,\xi)$ weak* converges to $g(\cdot,\xi)$ for all $\xi \in \mathbb{R}$, then $g_n(\cdot,v(\cdot))$ weak* converges to $g(\cdot,v(\cdot))$, for all $v \in C([a,b])$.*

Proof. Let $v \in C([a,b])$ and $\phi \in L^1(a,b)$. Also, let (x_{i-1}, x_i) be k number of partitions of (a,b) for $i = 1, 2, \ldots, k$ such that $x_0 = a$ and $x_k = b$. Consider,

$$\begin{aligned} \left| \int_a^b (g_n(x,v) - g(x,v))\, \phi \, dx \right| &\leq \sum_{i=1}^{k} \left| \int_{x_{i-1}}^{x_i} [g_n(x,v) - g_n(x, v(x_i))]\, \phi \, dx \right| \\ &+ \sum_{i=1}^{k} \left| \int_{x_{i-1}}^{x_i} [g_n(x,v(x_i)) - g(x, v(x_i))]\, \phi \, dx \right| \\ &+ \sum_{i=1}^{k} \left| \int_{x_{i-1}}^{x_i} [g(x,v(x_i)) - g(x, v(x))]\, \phi \, dx \right| \end{aligned}$$

The second term converges to zero, by hypothesis, and by uniform local Lipschitz continuity (cf. Exercise 4.10 of g_n and g, we have the result. □

Lemma 4.4.3. *Let $g_n : \Omega \times \mathbb{R}^n \to [0, +\infty)$ satisfy hypotheses* **H***1 and* **H***2, for all n. Then, there exists a subsequence of $\{g_n\}$ and a $g : (a,b) \times \mathbb{R} \to [0, +\infty)$ such that $g_n(\cdot, \xi)$ weak* converges to $g(\cdot, \xi)$ for all $\xi \in \mathbb{R}$.*

Theorem 4.4.4. *Let $\{F_n\}$ be a sequence in $\mathcal{F}$ with integrand f_n and $F \in \mathcal{F}$ with integrand f. Then the following statements are equivalent:*

1. *$F_n(\cdot, I)$ Γ-converges to $F(\cdot, I)$ in $W^{1,p}(I)$, for all open intervals I of (a,b).*

2. *$f_n^\star(\cdot, \xi^\star)$ weak* converges to $f^\star(\cdot, \xi^\star)$, for all $\xi^\star \in \mathbb{R}$.*

The proof of above lemma and theorem are being skipped and can be found in [Bra02].

Example 4.2. Let $0 < \alpha \leq a_\varepsilon(x) \leq \beta < +\infty$ and $g \in L^2(a,b)$. Let $F_\varepsilon : H_0^1(a,b) \to \mathbb{R}$ be defined as

$$F_\varepsilon(u) = \int_a^b \left\{ \frac{1}{2} a_\varepsilon(x) |u'|^2 - gu \right\} dx.$$

The Euler-Lagrange equations yields that the minimizers u_ε,

$$\begin{cases} -\frac{d}{dx}\left(a_\varepsilon(x) \frac{du_\varepsilon}{dx} \right) = g \text{ in } (a,b) \\ u_\varepsilon(a) = u_\varepsilon(b) = 0. \end{cases}$$

Now, set $f_\varepsilon(x,\xi) := a_\varepsilon(x)|\xi|^2$. Then, $f_\varepsilon^\star(x,\xi^\star) = \frac{\xi^2}{4a_\varepsilon(x)}$. But, for each $\xi^\star \in \mathbb{R}^n$, $f_\varepsilon^\star(\cdot,\xi^\star)$ converges weak* in $L^\infty(a,b)$ to $f^\star(\cdot,\xi^\star)$, where $f^\star(x,\xi^\star) = \frac{\xi^2}{4b(x)}$ and

$$\frac{1}{a_\varepsilon(x)} \rightharpoonup \frac{1}{b(x)}.$$

□

Chapter 5

Bloch-Floquet Homogenization

5.1 Fourier Transform

In this chapter we assume the functions to be complex valued. Recall that $-\Delta : H^2(\mathbb{R}^n) \subset L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n)$ is an unbounded, self-adjoint operator whose spectral decomposition is well-known. The "generalised" eigenfunctions[1] are the *plane waves* or *Fourier waves* $e^{\imath\xi\cdot x}$, for each $\xi \in \mathbb{R}^n$, and $|\xi|^2$ is an eigenvalue for each $\xi \in \mathbb{R}^n$ giving the spectrum to be $[0, \infty)$. In other words, $-\Delta(e^{\imath x\cdot\xi}) = |\xi|^2 e^{\imath x\cdot\xi}$.

Theorem 5.1.1. *Given any $f \in L^2(\mathbb{R}^n)$ there is a unique $\hat{f} \in L^2(\mathbb{R}^n)$ such that*

$$f(x) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} \hat{f}(\xi) e^{\imath\xi\cdot x}\, d\xi.$$

Also, for any $f, g \in L^2(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx = \int_{\mathbb{R}^n} \hat{f}(\xi)\overline{\hat{g}(\xi)}\, d\xi.$$

In particular, the Fourier transform $f \mapsto \hat{f}$ is an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$.

For any $f \in L^1(\mathbb{R}^n)$, its *Fourier transform* $\hat{f} : \mathbb{R}^n \to \mathbb{C}$ is given as

$$\hat{f}(\xi) = \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-\imath\xi\cdot x}\, dx.$$

[1]For each $\xi \in \mathbb{R}^n$, $e^{\imath\xi\cdot x}$ are not elements of $L^2(\mathbb{R}^n)$ but they span $L^2(\mathbb{R}^n)$

The Fourier transform map $\mathcal{F} : L^1(\mathbb{R}^n) \to L^\infty(\mathbb{R}^n)$ is defined as $\mathcal{F}(f) = \hat{f}$. Note that $\mathcal{F}$ is a bounded linear with $\|\mathcal{F}\| \leq 1$, since $\|\hat{f}\|_\infty \leq \|f\|_1$. The definition is generalised to Schwartz class. In this sense, the Fourier transform can be generalised to $L^2(\mathbb{R}^n)$. The Fourier transform will change a differential equation in to an algebraic equation. For instance, $-\Delta u = f$ will tranform, on applying Fourier transform, to

$$
\begin{aligned}
\hat{f}(\xi) &= \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} f(x) e^{-\imath x\cdot\xi}\, dx = -\frac{1}{(2\pi)^{n/2}} \sum_{j=1}^{n} \int_{\mathbb{R}^n} \frac{\partial^2 u(x)}{\partial x_j^2} e^{-\imath x\cdot\xi}\, dx \\
&= \frac{1}{(2\pi)^{n/2}} \sum_{j=1}^{n} (-\imath\xi_j) \int_{\mathbb{R}^n} \frac{\partial u(x)}{\partial x_j} e^{-\imath x\cdot\xi}\, dx \quad \text{(Integration by parts)} \\
&= -\sum_{j=1}^{n} (-\imath\xi_j)^2 \frac{1}{(2\pi)^{n/2}} \int_{\mathbb{R}^n} u(x) e^{-\imath x\cdot\xi}\, dx \quad \text{(Integration by parts)} \\
&= |\xi|^2 \hat{u}(\xi).
\end{aligned}
$$

More generally, any m-th order linear differential equation with constant coefficients $P(D)u = f$ where $P(D) = \sum_{|\alpha|\leq m} a_\alpha D^\alpha$ will transform in to an algebraic eqaution $P(\imath\xi)\hat{u}(\xi) = \hat{f}(\xi)$. The Laplacian is a particular case of the elliptic operator $-\Delta + c(x)$ with $c \equiv 0$. For $c(x) \neq 0$ (without loss of generality assume $c(x) \geq 0$), the Bloch theorem gives the generalised eigenfunction for $-\Delta + c(x)$ when c is Y-periodic, for any given reference cell $Y \subset \mathbb{R}^n$.

5.2 Schrödinger Operator with Periodic Potential

Definition 5.2.1. *Let $\{e_i\}$ be the canonical basis for $\mathbb{R}^n$. Let $Y = \Pi_{i=1}^n [0, \ell_i)$ be a reference cell (or period) in $\mathbb{R}^n$. A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be* Y-periodic *if $f(x + e_i p_i \ell_i) = f(x)$ for a.e. $x \in \mathbb{R}^n$ and all $p \in \mathbb{Z}^n$, for all $i = 1, 2, \ldots, n$.*

Consider the Schrödinger operator $-\Delta + c(x)$ where c is a periodic function, i.e., for some $\ell = (\ell_i) \in \mathbb{R}^n$ and $p \in \mathbb{Z}^n$, $c(x + e_i \ell_i p_i) = c(x)$. Let $L : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ be the operator $L := -\Delta + c(x)$. The operator L has large symmetry group. Define, for each $p \in \mathbb{Z}^n$,

$$[U(p)v](x) := v(x + \sum_{i=1}^{n} p_i \ell_i).$$

Then $U(p)L = LU(p)$. In fact, $U(p)e^{-\imath Ls} = e^{-\imath Ls}U(p)$. Let us first consider the one dimension situation with $c \in C_c^\infty(\mathbb{R})$ with bounded derivatives and $L : \mathcal{S}(\mathbb{R}) \to \mathcal{S}(\mathbb{R})$ defined as

$$L := -\frac{d^2}{dx^2} + c(x).$$

If c is 2π-periodic and, hence, c admits a uniformly convergent Fourier series

$$c(x) = \sum_{m \in \mathbb{Z}} c_m e^{\imath mx}$$

where

$$c_m = \frac{1}{2\pi} \int_{-\pi}^{\pi} c(x) e^{-\imath mx}\, dx.$$

If $u \in \mathcal{S}(\mathbb{R})$ then

$$\begin{aligned}
\widehat{Lu(x)}(\xi) &= \xi^2 \hat{u}(\xi) + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} c(x)u(x)e^{-\imath \xi x}\, dx \\
&= \xi^2 \hat{u}(\xi) + \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \left(\sum_{m \in \mathbb{Z}} c_m e^{\imath mx} \right) u(x) e^{-\imath \xi x}\, dx \\
&= \xi^2 \hat{u}(\xi) + \sum_{m \in \mathbb{Z}} c_m \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} u(x) e^{-\imath(\xi - m)x}\, dx \\
&= \xi^2 \hat{u}(\xi) + \sum_{m \in \mathbb{Z}} c_m \hat{u}(\xi - m).
\end{aligned}$$

Thus, $\widehat{Lu}(\xi)$ depends only on the values $\hat{u}(\xi - m)$ for all $m \in \mathbb{Z}$. But recall that $\hat{u}(\xi - m) = \widehat{e^{\imath xm} u(x)}(\xi)$. This suggests that the operator L depends on the "modulation" by all $m \in \mathbb{Z}$.

5.2.1 Direct Integral Decomposition

Let H be a separable Hilbert space and (X, μ) be a σ-finite measure space. Let $L^2(X, \mu; H)$ is the Hilbert space of square integrable H-valued functions.

If μ is a sum of point measures at finite set of points $x_1, \dots, x_k$ then, any $f \in L^2(X, \mu; H)$, is determined by the k-tuple $(f(x_1), \dots, f(x_k))$. Thus, $L^2(X, \mu; H)$ is isomorphic to the direct sum $\oplus_{i=1}^{k} H$. For more general μ, one may define a kind of "continuous direct sum" called the *constant fiber direct integral* and write

$$L^2(X, \mu; H) = \int_X^{\oplus} H \, d\mu.$$

Definition 5.2.2. *A function $T(\cdot) : X \to L(H)$ is measurable iff, for each $\phi, \psi \in H$, $\langle \phi, T(\cdot)\psi \rangle$ is measurable. $L^\infty(X, \mu; L(H))$ denotes the equivalence class (with a.e.) of measurable functions from X to $L(H)$ with*

$$\|T\|_\infty = ess \ sup_{x \in X} \|T(x)\|_{L(H)} < \infty.$$

Definition 5.2.3. *A bounded operator T on $\mathcal{H} = \int_X^{\oplus} H \, d\mu$ is said to be decomposed by the direct integral decomposition iff there is $T(\cdot) \in L^\infty(X, \mu; L(H))$ such that, for all $\psi \in \mathcal{H}$,*

$$(T\psi)(x) = T(x)\psi(x).$$

We then say T is decomposable and

$$T = \int_X^{\oplus} T(x) \, d\mu(x).$$

The $T(x)$ are called the fibers *of T.*

Theorem 5.2.4. *Let $H = l_2$ and*

$$\mathcal{H} = \int_{(-\frac{1}{2}, \frac{1}{2}]}^{\oplus} H \, dx.$$

For $\eta \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, let $L_\eta : l_2 \to l_2$ be defined as

$$(L_\eta(z))_k = (\eta + k)^2 z_k + \sum_{m \in \mathbb{Z}} c_m z_{k-m}.$$

Define $T : L^2(\mathbb{R}) \to \mathcal{H}$ by

$$[(Tf)(\eta)]_k = \hat{f}(\eta + k).$$

For $L = -\frac{d^2}{dx^2} + c(x)$ on $L^2(\mathbb{R})$,

$$TLT^{-1} = \int_{(-\frac{1}{2}, \frac{1}{2}]}^{\oplus} L_\eta \, d\eta.$$

When $c \equiv 0$, the eigenvalues and eigenfunctions of L_η are $(\eta+k)^2$ and the Fourier transform of $e^{\imath(\eta+k)x}$, respectively. This suggests that L_η is related to $-\frac{d^2}{dx^2}$ on $[0, 2\pi)$ with the boundary condition $u(2\pi) = e^{\imath 2\pi\eta}u(0)$ and $u'(2\pi) = e^{\imath 2\pi\eta}u'(0)$.

Lemma 5.2.5. *Let $H = L^2[0, 2\pi)$ and*

$$\mathcal{H} = \int^{\oplus}_{(-\frac{1}{2},\frac{1}{2}]} H\, d\eta.$$

Then $T : \mathcal{S}(\mathbb{R}) \to \mathcal{H}$ given by

$$(Tf)_\eta(x) = \sum_{m\in\mathbb{Z}} e^{\imath 2\pi m\eta} f(x + 2\pi m) \quad \eta \in (-\frac{1}{2}, \frac{1}{2}]\, x \in [0, 2\pi)$$

which extends uniquely to an unitary operator on $L^2(\mathbb{R})$. Moreover,

$$T\left(-\frac{d^2}{dx^2}\right)T^{-1} = \int^{\oplus}_{(-\frac{1}{2},\frac{1}{2}]} \left(-\frac{d^2}{dx^2}\right)_\eta d\eta \tag{5.2.1}$$

where $\left(-\frac{d^2}{dx^2}\right)_\eta$ is the operator $-\frac{d^2}{dx^2}$ on $L^2[0, 2\pi)$ with boundary condition

$$u(2\pi) = e^{\imath 2\pi\eta}u(0) \quad u'(2\pi) = e^{\imath 2\pi\eta}u'(0).$$

Proof. Let us note that T is well defined. For any $f \in \mathcal{S}(\mathbb{R})$, the series in RHS is convergent. For any $f \in \mathcal{S}(\mathbb{R})$, $Tf \in \mathcal{S}(\mathbb{R})$ because

$$\begin{aligned}
&\int_{-\frac{1}{2}}^{\frac{1}{2}} \left(\int_0^{2\pi} \left| \sum_{m=-\infty}^{\infty} e^{-\imath 2\pi m\eta} f(x + 2\pi m) \right|^2 dx \right) d\eta \\
= &\int_0^{2\pi} \left[\left(\sum_{m,p\in\mathbb{Z}} \overline{f(x + 2\pi m)} f(x + 2\pi p) \right) \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{-\imath 2\pi(p-m)\eta}\, d\eta \right] dx \\
&\text{(by Fubini's Theorem)} \\
= &\int_0^{2\pi} \left(\sum_{m\in\mathbb{Z}} |f(x + 2\pi m)|^2 \right) dx = \int_{\mathbb{R}} |f(x)|^2\, dx.
\end{aligned}$$

Thus, T is well defined and admits a unique isometry extension. To see that T is onto $\mathcal{H}$, we compute $T^\star$. For any $g \in \mathcal{H}$, $x \in [0, 2\pi]$ and $m \in \mathbb{Z}$

$$(T^\star g)(x + 2\pi m) = \int_{-\frac{1}{2}}^{\frac{1}{2}} e^{\imath 2\pi m\eta} g_\eta(x)\, d\eta.$$

Further,

$$
\begin{aligned}
\|T^\star g\|_2^2 &= \int_{\mathbb{R}} |(T^\star g)(y)|^2\, dy \\
&= \int_0^{2\pi} \left(\sum_{m\in\mathbb{Z}} |(T^\star g)(2\pi m + x)|^2 \right) dx \\
&= \int_0^{2\pi} \left(\sum_{m\in\mathbb{Z}} \left| \int_0^{2\pi} e^{\imath 2\pi m\eta} g_\eta(x)\, d\theta \right|^2 \right) dx \\
&= \int_0^{2\pi} \left(\int_0^{2\pi} |g_\eta(x)|^2\, d\theta \right) dx \quad \text{(Plancherel's identity)} \\
&= \|g\|^2.
\end{aligned}
$$

Finally, to prove (5.2.1), let G be the operator on the right-hand side of (5.2.1). We shall show that if $f \in \mathcal{S}(\mathbb{R})$, then $Tf \in D(G)$ and $T(-f'') = G(Tf)$. Since $-d^2/dx^2$ is essentially self-adjoint on $\mathcal{S}(\mathbb{R})$ and G is self-adjoint, (5.2.1) will follow. So, suppose $f \in \mathcal{S}(\mathbb{R}^n)$, then Tf is given by the convergent sum as in the statement. Thus, $Tf \in C^\infty(0, 2\pi)$ with $(Tf)'_\eta(x) = (Tf'_\eta(x)$ and similarly for higher derivatives. Further, it is clear that

$$
\begin{aligned}
(Tf)_\theta(2\pi) &= \sum_{m\in\mathbb{Z}} e^{-\imath 2\pi m\eta} f(2\pi(m+1)) \\
&= \sum_{m\in\mathbb{Z}} e^{-\imath 2\pi(m-1)\eta} f(2\pi m) = e^{\imath 2\pi\eta}(Tf)_\eta(0).
\end{aligned}
$$

Similarly, $(Tf)'_\eta(2\pi) = e^{\imath 2\pi\eta}(Tf_\eta)'(0)$. Thus, for each η, $(Tf)_\eta \in D((-\frac{d^2}{dx^2})_\eta)$ and

$$\left(-\frac{d^2}{dx^2}\right)_\eta (Tf) = U(-f'')_\eta.$$

We conclude that $Tf \in D(G)$ and $G(Tf) = U(-f'')$. This proves (5.2.1). □

Theorem 5.2.6 (Direct Integral Decomposition of Periodic Schrödinger operator)**.** *Let c be a bounded measurable function on $\mathbb{R}$ with period 2π. For $\eta \in \left(-\frac{1}{2}, \frac{1}{2}\right]$, let*

$$L_\eta = \left(-\frac{d^2}{dx^2}\right)_\eta + c(x)$$

be an operator on $L^2[0, 2\pi]$. *Let* T *be given by*

$$(Tf)_\eta(x) = \sum_{m\in\mathbb{Z}} e^{\imath 2\pi m\eta} f(x + 2\pi m) \quad \eta \in \left(-\frac{1}{2}, \frac{1}{2}\right] \text{ and } x \in [0, 2\pi).$$

Then

$$T\left(-\frac{d^2}{dx^2} + c\right)T^{-1} = \int_{(-\frac{1}{2},\frac{1}{2}]}^{\oplus} L_\eta \, d\eta.$$

Proof. Let c be the η-independent operator acting on the fiber $H = L^2[0, 2\pi)$ by $(c_\eta f)(x) = c(x)f(x)$ for $0 \leq x \leq 2\pi$. It is sufficient to prove that

$$TcT^{-1} = \int_{(-\frac{1}{2},\frac{1}{2}]}^{\oplus} c_\eta \, d\eta.$$

For $f \in \mathcal{S}(\mathbb{R})$,

$$\begin{aligned}
(Tcf)_\eta(x) &= \sum_{m\in\mathbb{Z}} e^{-\imath 2\pi m\eta} c(x + 2\pi m) f(x + 2\pi m) \\
&= c(x) \sum_{m\in\mathbb{Z}} e^{-\imath 2\pi m\eta} f(x + 2\pi m) \\
&= c_\eta (Tf)_\eta(x).
\end{aligned}$$

The second last equality is due to the periodicity of c. □

5.3 Bloch Periodic Functions

The Bloch transform is a generalization of Fourier transform that leaves the periodic functions invariant, in some sense. Let us begin by considering a generalization of periodic functions.

Definition 5.3.1. *Let* $Y = \Pi_{i=1}^n [0, \ell_i)$ *be a reference cell (or period) in* $\mathbb{R}^n$. *For each* $\eta \in \mathbb{R}^n$, *a function* $f : \mathbb{R}^n \to \mathbb{R}$ *is said to be* (η, Y)-Bloch periodic *if* $f(x + \ell \cdot p) = e^{\imath 2\pi p\cdot\eta} f(x)$ *for a.e.* $x \in \mathbb{R}^n$ *and for all* $p \in \mathbb{Z}^n$.

Note that the case $\eta = 0$ corresponds to the usual notion of Y-periodic functions. Note that the boundary condition remains unchanged if η is replaced with $\eta + k$, for any $k \in \mathbb{Z}^n$. Hence, it is sufficient to consider $\eta \in Y^\star$

where $Y^\star = (-\frac{1}{2}, \frac{1}{2}]^n$. The cell $Y^\star$ is called the *reciprocal cell* and, in Physics literature, $Y^\star$ is known as the *first Brillouin zone*.

We shall assume that $Y = [0, 2\pi)^n$ and, for $j, k = 1, 2, \dots, n$, $a_{jk} : Y \to \mathbb{R}$ is such that $a_{jk} \in L^\infty_{\text{per}}(Y)$. Let $A(y) = (a_{jk}(y)) \in M(\alpha, \beta, Y)$ and is a symmetric matrix, i.e., $a_{jk}(y) = a_{kj}(y)$. One can extend a_{jk} to entire $\mathbb{R}^n$ as a Y-periodic function. Also, c is a Y-periodic function such that $c(y) \geq c_3 > 0$. We are interested in the spectral resolution of closure of the operator $\mathcal{A} = -\text{div}(A(y)\nabla) + c(y)$ in $L^2(\mathbb{R}^n)$.

By Bloch Theorem, it is enough to study the (η, Y)-Bloch periodic eigenvalue problem, for each $\eta \in \mathbb{R}^n$, i.e.,

Definition 5.3.2. *For any fixed (momentum) vector $\eta \in Y^\star$, consider the eigenvalue problem: given a symmetric $A \in M(\alpha, \beta, Y)$, find $\lambda(\eta) \in \mathbb{C}$ and non-zero $\psi(\cdot; \eta) : \mathbb{R}^n \to \mathbb{R}$ such that*

$$\begin{cases} \mathcal{A}\psi(y;\eta)) &= \lambda(\eta)\psi(y;\eta) \quad \text{in } \mathbb{R}^n \\ \psi(y + 2\pi\ell) &= e^{2\pi\imath\ell\cdot\eta}\psi(y) \quad \ell \in \mathbb{Z}^n, y \in \mathbb{R}^n. \end{cases} \tag{5.3.1}$$

The eigenvalues ψ are known as Bloch waves *associated with $\mathcal{A}$ and the eigenvalues λ are called* Bloch eigenvalues.

Suppose $\eta \in Y^\star$ have rational components and $\eta = (\eta_1, \dots, \eta_n)$. Recall that there is a homeomorphism from $Y^\star$ to S^1. Thus, $e^{\imath 2\pi\eta_j} \in S^1$. In this sense, the Bloch periodicity condition has the form $e^{2\pi\imath p\cdot\eta} = \omega^p$ where $\omega \in [S^1]^n$ and $\omega^p = \omega_1^{p_1}\omega_2^{p_2}\dots\omega_n^{p_n}$. For any $m \in \mathbb{Z}^n$, let $D_m \subset [S^1]^n$ be the collection of all $\omega \in [S^1]^n$ such that its j-th component is the m_j-th root of unity. Thus, $\omega^m = 1$ for all $\omega \in D_m$. The spectral problem (5.3.1) may be seen as a sequence of spectral problems, i.e., for each $m \in \mathbb{Z}^n$, we define ψ_m as

$$\begin{cases} \mathcal{A}\psi_m(y) &= \lambda_m\psi_m(y) \quad \text{in } \mathbb{R}^n \\ \psi_m(y + 2\pi m) &= \psi(y) \quad y \in \mathbb{R}^n. \end{cases}$$

Note that in the above boundary condition ψ is Y_m-periodic where $Y_m = \prod_{i=1}^n [0, 2\pi m_i)$. The space of spectral decomposition is $L^2_{\text{per}}(Y_m)$ which admits the orthogonal decomposition $L^2_{\text{per}}(Y_m) = \oplus_{\omega\in D_m} L^2_{\text{per}}(\omega, Y)$ where

$$L^2_{\text{per}}(\omega, Y) = \{\psi \in L^2_{\text{loc}}(\mathbb{R}^n) \mid \psi(y + 2\pi\ell) = \omega^\ell\psi(y)\ \forall \ell \in \mathbb{Z}^n, y \in \mathbb{R}^n\}.$$

Thus, we observe that the above space consists of (η, Y)-Bloch Periodic functions. For any irrational η can be approximated by rationals by varying m and noting that the sets of roots of unity is dense in S^1.

5.4 Bloch Transform

Theorem 5.4.1 (Bloch Decomposition)**.** *Let* $Y = [0, 2\pi)^n$ *and* $Y^\star = \left(-\frac{1}{2}, \frac{1}{2}\right]^n$. *Given a* $f \in L^2(\mathbb{R}^n)$ *there is a unique function, called* Bloch Transform, $f_b \in L^2(Y \times Y^\star)$ *such that*

$$f(y) = \int_{Y^\star} f_b(y, \eta) e^{\imath \eta \cdot y}\, d\eta.$$

Also, for any $f, g \in L^2(\mathbb{R}^n)$, *the Parseval formula holds, i.e.,*

$$\int_{\mathbb{R}^n} f(y)\overline{g(y)}\, dy = \int_Y \int_{Y^\star} f_b(y, \eta)\overline{g_b(y, \eta)}\, dy\, d\eta.$$

In particular, the Bloch transform $f \mapsto f_b$ *is an isometry from* $L^2(\mathbb{R}^n)$ *to* $L^2(Y \times Y^\star)$.

Proof. For any $f \in \mathcal{D}(\mathbb{R}^n)$ and for each $\eta \in Y^\star$, define

$$f_b(y; \eta) := \sum_{p \in \mathbb{Z}^n} f(y + 2\pi p) e^{-\imath (y + 2\pi p) \cdot \eta}.$$

The sum is well defined because it has finite number of terms because f has compact support. Note that $f_b(y; \eta)$ is Y-periodic in y variable because

$$f_b(y + 2\pi; \eta) := \sum_{p+1 \in \mathbb{Z}^n} f(y + 2\pi p) e^{-\imath (y + 2\pi p) \cdot \eta} = f_b(y; \eta).$$

Similarly, $e^{\imath y \cdot \eta} f_b(y; \eta)$ is $Y^\star$-periodic in η variable because, for $k \in \mathbb{Z}^n$,

$$\begin{aligned}
e^{\imath y \cdot (\eta + k)} f_b(y; \eta + k) &= e^{\imath y \cdot (\eta + k)} \sum_{p \in \mathbb{Z}^n} f(y + 2\pi p) e^{-\imath (y + 2\pi p) \cdot (\eta + k)} \\
&= e^{\imath y \cdot (\eta + k)} \sum_{p \in \mathbb{Z}^n} f(y + 2\pi p) e^{-\imath (y + 2\pi p) \cdot \eta} e^{-\imath (y + 2\pi p) \cdot k} \\
&= e^{\imath y \cdot (\eta + k)} e^{-\imath y \cdot k} \sum_{p \in \mathbb{Z}^n} f(y + 2\pi p) e^{-\imath (y + 2\pi p) \cdot \eta} e^{-\imath 2\pi p \cdot k} \\
&= e^{\imath y \cdot \eta} f_b(y; \eta).
\end{aligned}$$

In the above relation we have used the fact that $e^{\imath 2\pi p \cdot k} = 1$. Observe that

$$e^{\imath y \cdot \eta} f_b(y; \eta) = \sum_{p \in \mathbb{Z}^n} f(y + 2\pi p) e^{-2\imath \pi p \cdot \eta}.$$

Thus,

$$\begin{aligned}
\int_{Y^\star} e^{\imath y\cdot\eta} f_b(y;\eta)\,d\eta &= f(y) + \sum_{\substack{p\in\mathbb{Z}^n\\ p\neq 0}} f(y+2\pi p)\int_{Y^\star} e^{-2\imath\pi p\cdot\eta}\,dy \\
&= f(y) - \sum_{\substack{p\in\mathbb{Z}^n\\ p\neq 0}} f(y+2\pi p)\left[\frac{e^{-\imath\pi p}-e^{\imath\pi p}}{2\imath\pi p_1\dots p_n}\right]\,dy \\
&= f(y).
\end{aligned}$$

Therefore, we have proved the results for all functions in $\mathcal{D}(\mathbb{R}^n)$. Similarly, one can prove the Parseval's formula for functions in $\mathcal{D}(\mathbb{R}^n)$. The Bloch transform is a linear map on $\mathcal{D}(\mathbb{R}^n)$ bounded on $L^2(\mathbb{R}^n)$. Define $\mathcal{B}:\mathcal{D}(\mathbb{R}^n)\to L^2(Y\times Y^\star)$ as $\mathcal{B}f = f_b$. $\mathcal{B}$ is a bounded operator w.r.t L^2-norm. Consider

$$\begin{aligned}
\|\mathcal{B}f\|_2^2 &= \|f_b\|_2^2 \le \int_Y\int_{Y^\star}\sum_{p\in\mathbb{Z}^n}|f(y+2\pi p)e^{-\imath(y+2\pi p)\cdot\eta}|^2\,d\eta\,dy \\
&= \int_Y\sum_{p\in\mathbb{Z}^n}|f(y+2\pi p)|^2\left[\int_{Y^\star}|e^{-\imath(y+2\pi p)\cdot\eta}|^2\,d\eta\right]\,dy \\
&= |Y^\star|\sum_{p\in\mathbb{Z}^n}\int_Y|f(y+2\pi p)|^2\,dy \\
&= \int_{\mathbb{R}^n}|f(y)|^2\,dy = \|f\|_2^2.
\end{aligned}$$

We can unitarily extend $\mathcal{B}$ to all of $L^2(\mathbb{R}^n)$. Thus, by density of $\mathcal{D}(\mathbb{R}^n)$ in $L^2(\mathbb{R}^n)$, the Bloch transform extends to $L^2(\mathbb{R}^n)$ and Parseval's formula holds true. □

Remark 5.4.2. Note that, for each fixed $\eta\in Y^\star$, $y\mapsto f_b(y,\eta)$ is extended Y-periodic to $\mathbb{R}^n$ and, for each fixed $y\in Y$, $\eta\mapsto e^{\imath\eta\cdot y}f_b(y,\eta)$ is extended $Y^\star$-periodic to $\mathbb{R}^n$. Thus, the Bloch transform may be seen as an isometry from $L^2(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n\times\mathbb{R}^n)$.

Remark 5.4.3. The Bloch transform is a "modulation" of Zak transform. The Zak transform for any $f\in\mathcal{D}(\mathbb{R}^n)$ is defined as

$$f_z(y;\eta) := \sum_{p\in\mathbb{Z}^n} f(y+2\pi p)e^{-\imath 2\pi p\cdot\eta}$$

and extended unitarily to to $L^2(\mathbb{R}^n)$. Further, $f_b(y;\eta) = e^{-\imath y\cdot\eta} f_z(y;\eta)$ for all $f \in \mathcal{D}(\mathbb{R}^n)$.

The following theorem explains the sense in which the Bloch transform leaves the periodic functions invariant.

Theorem 5.4.4 (Invariance of Periodic Functions)**.** *Let $Y = [0, 2\pi)^n$ and $c : Y \to \mathbb{C}$ be such that $c \in L^\infty(Y)$ extended Y-periodically to $\mathbb{R}^n$. For any $f \in L^2(\mathbb{R}^n)$, $(cf)_b(y;\eta) = c(y)f_b(y;\eta)$.*

Proof. It is enough to prove the result for $f \in \mathcal{D}(\mathbb{R}^n)$. Consider

$$\begin{aligned}
(cf)_b(y;\eta) &= \sum_{p\in\mathbb{Z}^n} c(y+2\pi p)f(y+2\pi p)e^{-\imath(y+2\pi p)\cdot\eta} \\
&= c(y)\sum_{p\in\mathbb{Z}^n} f(y+2\pi p)e^{-\imath(y+2\pi p)\cdot\eta} \\
&= c(y)f_b(y;\eta).
\end{aligned}$$

By density the result is true for any $f \in L^2(\mathbb{R}^n)$. □

Theorem 5.4.5. *For any $f \in H^1(\mathbb{R}^n)$, $(\nabla_y f)_b(y;\eta) = (\nabla_y + \imath\eta)f_b(y;\eta)$.*

Proof. It is enough to prove the result for $f \in \mathcal{D}(\mathbb{R}^n)$. Consider

$$\begin{aligned}
(\nabla_y f)_b(y;\eta) &= \sum_{p\in\mathbb{Z}^n} \left[\nabla_y f(y+2\pi p)\right] e^{-\imath(y+2\pi p)\cdot\eta} \\
&= \sum_{p\in\mathbb{Z}^n} \nabla_y \left[f(y+2\pi p)e^{-\imath(y+2\pi p)\cdot\eta}\right] \\
&\quad +\imath\eta \sum_{p\in\mathbb{Z}^n} f(y+2\pi p)e^{-\imath(y+2\pi p)\cdot\eta} \\
&= \left[\nabla_y + \imath\eta\right] f_b(y;\eta).
\end{aligned}$$

□

For any $f \in L^2(\mathbb{R}^n)$, consider the equation $\mathcal{A}u = f$ in $\mathbb{R}^n$. Applying Bloch transform to this equation, using Theorems 5.4.4 and 5.4.5, we obtain a family of equations, indexed by $\eta \in Y^\star$, with periodic boundary conditions:

$$\left\{\begin{array}{lll} \mathcal{A}(\eta)u_b(y;\eta) &= f_b(y;\eta) & \text{in } \mathbb{R}^n \\ u_b(y+2\pi\ell;\eta) &= u_b(y;\eta) & \ell \in \mathbb{Z}^n\ y \in \mathbb{R}^n, \end{array}\right. \tag{5.4.1}$$

where $\mathcal{A}(\eta)$ is the *shifted operator*, denoted as

$$\mathcal{A}(\eta) := -\sum_{j,k=1}^{n}\left(\frac{\partial}{\partial y_j}+\imath\eta_j\right)\left[a_{jk}(y)\left(\frac{\partial}{\partial y_k}+\imath\eta_k\right)\right]+c(y).$$

The shifted operator equation admits a solution (being a periodic problem) in $H^1_{\text{per}}(Y)$ and a corresponding Poincaré inequality holds true, i.e., for all $u \in H^1_{\text{per}}(Y)$ and $\eta \in Y^\star$,

$$c\left(\|\nabla u\|_{2,Y}+|\eta|\|u\|_{2,Y}\right) \leq \|\nabla u+\imath u\eta\|_{2,Y} \leq \|\nabla u\|_{2,Y}+|\eta|\|u\|_{2,Y}.$$

5.4.1 Spectrum of Elliptic Operator

The spectral decomposition of $\mathcal{A}$, in one dimension periodic media, was first studied by Floquet (cf. [Flo83]) and much later, in crystal lattice, by Bloch. We shall compute the spectral decomposition of $\mathcal{A}$ in $L^2(\mathbb{R}^n)$ via the spectral decomposition of the shifted operator $\mathcal{A}(\eta)$. Consider the eigenvalue problem

$$\begin{cases} \mathcal{A}(\eta)\phi(y;\eta) &= \lambda(\eta)\phi(y;\eta) \quad \text{in } \mathbb{R}^n \\ \phi(y+2\pi\ell) &= \phi(y) \qquad\qquad \ell\in\mathbb{Z}^n, y\in\mathbb{R}^n \end{cases} \tag{5.4.2}$$

Theorem 5.4.6 (Periodic Eigen Value problem)**.** *There exists a sequence of pairs (λ_m, ϕ_m) satisfying*

$$\begin{cases} \mathcal{A}\phi(y) &= \lambda\phi(y) \quad \textit{in } \mathbb{R}^n \\ \phi(y+2\pi\ell) &= \phi(y) \qquad \ell\in\mathbb{Z}^n, y\in\mathbb{R}^n \end{cases} \tag{5.4.3}$$

where $\{\lambda_m\}$ are positive real eigenvalues and $\{\phi_m(y)\}$ are the corresponding eigenvectors, for each $m \in \mathbb{N}$, such that $\{\phi_m\}$ form an orthonormal basis of $L^2_{per}(Y)$ and $0 \leq \lambda_1 \leq \lambda_2 \leq \dots$ diverges and each eigenvalue has finite multiplicity.

Remark 5.4.7. By Theorem 5.4.6, for each fixed $\eta \in Y^\star$, there exists a sequence of pairs (λ_m, ϕ_m) satisfying (5.4.2) where $\{\lambda_m(\eta)\}$ are positive real eigenvalues and $\{\phi_m(y;\eta)\}$ are the corresponding eigenvectors, for each $m \in \mathbb{N}$, such that $\{\phi_m(\cdot;\eta)\}$ form an orthonormal basis of $L^2_{\text{per}}(Y)$ and $0 \leq \lambda_1(\eta) \leq \lambda_2(\eta) \leq \dots$ diverges and each eigenvalue has finite multiplicity. By varying $\eta \in Y^\star$, we obtain the spectral resolution of $\mathcal{A}$ in $L^2(\mathbb{R}^n)$. The set $\{e^{\imath y\cdot\eta}\phi_m(y,\eta); m\in\mathbb{N}, \eta\in Y^\star\}$forms a 'generalised' basis of $L^2(\mathbb{R}^n)$. As a consequence, $L^2(\mathbb{R}^n)$ can be identified with $L^2(Y^\star; \ell^2(\mathbb{N}))$. $\mathcal{A}$ acts as a multiplication operator: $\mathcal{A}[e^{\imath y\cdot\eta}\phi_m(y,\eta)] = \lambda_m(\eta)e^{\imath y\cdot\eta}\phi_m(y,\eta)$. The spectrum of $\mathcal{A}$, denoted as $\sigma(\mathcal{A})$, coincides with the *Bloch spectrum* and denoted as σ_b. The Bloch spectrum is defined as the union of the images of all the mappings $\lambda_m(\eta)$, i.e.,

$$\sigma_b := \cup_{m=1}^{\infty} \left[\inf_{\eta\in Y^\star} \lambda_m(\eta), \sup_{\eta\in Y^\star} \lambda_m(\eta) \right].$$

The spectrum has a band structure. In contrast to the homogeneous case, $\sigma(\mathcal{A})$ need not fill up the entire $[0,\infty)$ and there may be gaps.

Theorem 5.4.8. *For any $f \in L^2(\mathbb{R}^n)$, its Bloch transform is given as*

$$f_b(y;\eta) = \sum_{m=1}^{\infty} f_b^m(\eta)\phi_m(y;\eta)$$

where, $\{\phi_m\}$ are the eigenfunctions corresponding to the shifted operator $\mathcal{A}(\eta)$ and $f_b^m(\eta)$, for each $\eta \in Y^\star$, is the m-th Bloch coefficient of f defined as

$$f_b^m(\eta) := \int_{\mathbb{R}^n} f(y)e^{-\imath y\cdot\eta}\overline{\phi_m(y;\eta)}\,dy.$$

Proof. It is enough to prove the result for $f \in \mathcal{D}(\mathbb{R}^n)$. Recall that, for each $\eta \in Y^\star$, $f_b(\cdot;\eta) \in L^2_{\text{per}}(Y)$. Hence, by spectral decomposition of $\mathcal{A}(\eta)$,

$$f_b(y;\eta) = \sum_{m=1}^{\infty} f_b^m(\eta)\phi_m(y;\eta),$$

where

$$f_b^m(\eta) = \int_Y f_b(y;\eta)\overline{\phi_m(y;\eta)}\,dy.$$

But,

$$\begin{aligned} f_b^m(\eta) &= \int_Y \sum_{p\in\mathbb{Z}^n} f(y+2\pi p)e^{-\imath(y+2\pi p)\cdot\eta}\overline{\phi_m(y;\eta)}\,dy \\ &= \int_Y \sum_{p\in\mathbb{Z}^n} f(y+2\pi p)e^{-\imath(y+2\pi p)\cdot\eta}\overline{\phi_m(y+2\pi p);\eta)}\,dy \\ &= \int_{\mathbb{R}^n} f(y)e^{-\imath y\cdot\eta}\overline{\phi_m(y;\eta)}\,dy. \end{aligned}$$

$\square$

Remark 5.4.9. The Bloch inversion formula can rewritten as:

$$f(y) = \int_{Y^\star} e^{\imath y\cdot\eta} f_b(y;\eta)\,d\eta = \int_{Y^\star} e^{\imath y\cdot\eta} \sum_{m=1}^{\infty} f_b^m(\eta)\phi_m(y;\eta)\,d\eta.$$

Further, the Plancherel formula holds:

$$\int_{\mathbb{R}^n} |f(y)|^2\,dy = \int_{Y^\star} \sum_{m=1}^{\infty} |f_b^m(\eta)|^2\,d\eta. \tag{5.4.4}$$

Remark 5.4.10 (Algebraic Formula for Solution)**.** For each $m \in \mathbb{N}$ and $\eta \in Y^\star$, multiply $\phi_m(y;\eta)$ on both sides of (5.4.1) to obtain

$$\begin{aligned}
\int_Y \mathcal{A}(\eta)\left[\sum_{k=1}^{\infty} u_b^k(\eta)\phi_k(y;\eta)\right]\phi_m(y;\eta)\,dy &= \int_Y \sum_{k=1}^{\infty} f_b^k(\eta)\phi_k(y;\eta)\phi_m(y;\eta)\,dy \\
\int_Y \sum_{k=1}^{\infty} u_b^k(\eta)\phi_k(y;\eta)\lambda_m(\eta)\phi_m(y;\eta)\,dy &= f_b^m(\eta) \\
u_b^m(\eta)\lambda_m(\eta) &= f_b^m(\eta) \\
u_b^m(\eta) &= \frac{f_b^m(\eta)}{\lambda_m(\eta)}. \quad \square
\end{aligned}$$

Set $\psi_m(y;\eta) := \{e^{\imath y\cdot\eta}\phi_m(y;\eta)\}$. Then, for each $\eta \in Y^\star$, $\psi_m(\cdot;\eta)$ forms a basis of $L^2(\mathbb{R}^n)$. Thus, $L^2(\mathbb{R}^n)$ can be identified with $L^2(Y^\star;\ell^2(\mathbb{N}))$. Let us compute $\psi(y+2\pi\ell)$:

$$\begin{aligned}
\psi_m(y+2\pi\ell) &= e^{\imath y\cdot\eta}e^{2\pi\imath\ell\cdot\eta}\phi_m(y+2\pi\ell) \\
&= e^{\imath y\cdot\eta}e^{2\pi\imath\ell\cdot\eta}\phi_m(y) \\
&= e^{2\pi\imath\ell\cdot\eta}\psi_m(y).
\end{aligned}$$

5.4.2 Regularity of $\lambda_m(\eta)$ and $\phi_1(\cdot,\eta)$

Theorem 5.4.11. *For all $m \geq 1$, $\eta \mapsto \lambda_m(\eta)$ is a Lipschitz function.*

Proof. Consider the quadratic form associated with $\mathcal{A}(\eta)$:

$$a(v,v;\eta) = \int_Y a_{jk}(y)\left(\frac{\partial v}{\partial y_k} + \imath\eta_k v\right)\overline{\left(\frac{\partial v}{\partial y_j} + \imath\eta_j v\right)}\,dy.$$

The quadratic form admits a decomposition as follows:

$$a(v,v;\eta) = a(v,v;\eta^0) + R(v,v;\eta,\eta^0)$$

where

$$\begin{aligned} R = \int_Y a_{jk}(y)\frac{\partial v}{\partial y_k}\overline{(\imath\eta_j - \imath\eta_j^0)}v\,dy \quad &+ \quad \int_Y a_{jk}(y)(\imath\eta_k - \imath\eta_k^0)v\overline{\frac{\partial v}{\partial y_j}}\,dy \\ &+ \quad \int_Y a_{jk}(y)(\eta_k\eta_j - \eta_k^0\eta_j^0)|v|^2\,dy. \end{aligned}$$

By Cauchy-Schwarz's inequality,

$$|R| \leq C_0|\eta - \eta^0| \int_Y (|\nabla v|^2 + |v|^2)\,dy.$$

By min-max principle,

$$\lambda_m(\eta) = \min_{W\subset H^1_{\mathrm{per}}(Y)} \max_{v\in W} \frac{a(v,v;\eta)}{\|v\|^2_{2,Y}}$$

where W is a m-dimensional subspace of $H^1_{\mathrm{per}}(Y)$. Using the estimate on R, we deduce that

$$\lambda_m(\eta) \leq \lambda_m(\eta^0) + C_0|\eta - \eta^0|$$

for a suitable constant C_0. Interchanging η and η^0, we obtain

$$|\lambda_m(\eta) - \lambda_m(\eta^0)| \leq C_0|\eta - \eta^0|.$$

□

Theorem 5.4.12 (Analyticity)**.** *There is a $\delta > 0$ such that $\lambda_1(\eta)$ is analytic in the open ball $B_\delta(0)$ centred at origin and radius δ. Further, one can choose a corresponding unit eigenvector $\phi_1(y;\eta)$ satisfying*

(i) $\eta \mapsto \phi_1(\cdot;\eta)$ *from* $Y^\star$ *to* $H^1_{per}(Y)$ *is analytic on* $B_\delta(0)$.

(ii) $\phi_1(y;0) := |Y|^{-1/2} := (2\pi)^{-n/2}$.

(iii) $\|\phi_1(\cdot;\eta)\|_{2,Y} = 1$ *and* $\int_Y \phi_1(y;\eta)\,dy = 0$ *for each* $\eta \in B_\delta$.

5.4.3 Taylor Expansion of Ground State

Observe that (5.4.1) is a polynomial of degree two w.r.t η variable. Let $T_m(\eta) : L^2(Y) \to L^2(Y)$ be defined as

$$T_m(\eta)(\phi) = \mathcal{A}(\eta)\phi - \lambda_m\phi.$$

For a fixed $m \in \mathbb{N}$, let us compute the j-th first partial derivative of (5.4.2) w.r.t η to get

$$\mathcal{A}(\eta)\frac{\partial \phi_m}{\partial \eta_j} + \frac{\partial \mathcal{A}(\eta)}{\partial \eta_j}\phi_m = \lambda_m\frac{\partial \phi_m}{\partial \eta_j} + \phi_m\frac{\partial \lambda_m}{\partial \eta_j}.$$

Thus,

$$\begin{aligned} T_m(\eta)\frac{\partial \phi_m}{\partial \eta_j} &= -\frac{\partial \mathcal{A}(\eta)}{\partial \eta_j}\phi_m + \phi_m\frac{\partial \lambda_m}{\partial \eta_j} \\ &= \imath e_j A(\nabla_y + \imath\eta)\phi_m + (\nabla_y + \imath\eta)\cdot(\imath A e_j \phi_m) + \phi_m\frac{\partial \lambda_m}{\partial \eta_j}. \end{aligned}$$

There exists a solution to the above equation which is unique upto an additive multiple of ϕ_m. Hence, the RHS satisfies the compatibility condition or Fredhölm alternative. Therefore,

$$\int_Y T_m(\eta)\frac{\partial \phi_m}{\partial \eta_j}\overline{\phi}_m \, dy = 0$$

yields a formula for $\nabla_\eta \lambda_m(\eta^m)$ in terms of ϕ_m. Thus,

$$\frac{\partial \lambda_m}{\partial \eta_j}(\eta) = \left\langle \frac{\partial \mathcal{A}(\eta)}{\partial \eta_j}\phi_m(\cdot;\eta), \phi_m(\cdot;\eta) \right\rangle.$$

Similarly, by computing the j-th second partial derivative of (5.4.2) w.r.t η, we get

$$\begin{aligned} T_m(\eta)\frac{\partial^2 \phi_m}{\partial \eta_j \partial \eta_k} &= \imath e_j A(\nabla_y + \imath\eta)\frac{\partial \phi_m}{\partial \eta_k} + (\nabla_y + \imath\eta)\cdot\left(\imath A e_j \frac{\partial \phi_m}{\partial \eta_k}\right) \\ &\quad + \imath e_k A(\nabla_y + \imath\eta)\frac{\partial \phi_m}{\partial \eta_j} + (\nabla_y + \imath\eta)\cdot\left(\imath A e_k \frac{\partial \phi_m}{\partial \eta_j}\right) \\ &\quad + \frac{\partial \lambda_m}{\partial \eta_j}\frac{\partial \lambda_m}{\partial \eta_k} + \frac{\partial \lambda_m}{\partial \eta_k}\frac{\partial \lambda_m}{\partial \eta_j} - e_j A e_k \phi_m - e_k A e_j \phi_m \\ &\quad + \frac{\partial^2 \lambda_m}{\partial \eta_k \partial \eta_j}\phi_m. \end{aligned}$$

There exists a solution to the above equation which is unique upto an additive multiple of ϕ_m. Hence, the RHS satisfies the compatibility condition or Fredhölm alternative. Therefore,

$$\int_Y T_m(\eta) \frac{\partial^2 \phi_m}{\partial \eta_j \partial \eta_k} \overline{\phi}_m \, dy = 0$$

yields a formula for the Hessian matrix $D^2_\eta \lambda_m(\eta^m)$ in terms of ϕ_m. Thus,

$$\begin{aligned} \frac{1}{2} \frac{\partial^2 \lambda_m}{\partial \eta_j \partial \eta_k}(\eta) &= \langle a_{jk}\phi_m, \phi_m \rangle + \frac{1}{2} \left\langle \left[\frac{\partial \mathcal{A}(\eta)}{\partial \eta_j} - \frac{\partial \lambda_m}{\partial \eta_j} \right] \frac{\partial \phi_m}{\partial \eta_k}, \phi_m \right\rangle \\ &\quad + \frac{1}{2} \left\langle \left[\frac{\partial \mathcal{A}(\eta)}{\partial \eta_k} - \frac{\partial \lambda_m}{\partial \eta_k} \right] \frac{\partial \phi_m}{\partial \eta_j}, \phi_m \right\rangle. \end{aligned}$$

Let us summarise the properties of the eigenvalues $\lambda_m(\eta)$ and eigenvectors $\phi_m(y;\eta)$.

(a) All odd order derivatives of $\lambda_1(\eta)$ at $\eta = 0$ vanish.

(b) All odd order derivatives of $\phi_1(\cdot, \eta)$ at $\eta = 0$ are purely imaginary. For instance, the first order derivatives at $\eta = 0$ are given by

$$\frac{\partial \phi_1}{\partial \eta_j}(y;0) = \imath |Y|^{-1/2} w_j(y),$$

where $w_j \in H^1_{\text{per}}(Y)$ is the unique solution of the cell problem

$$\begin{cases} \mathcal{A} w_j &= \sum_{k=1}^n \frac{\partial a_{jk}}{\partial y_k} \quad \text{in } \mathbb{R}^n, \\ \frac{1}{|Y|} \int_Y w_j(y) \, dy &= 0. \end{cases}$$

(c) All even order derivatives of $\phi_1(\cdot;\eta)$ at $\eta = 0$ are real.

(d) Second order derivatives of $\lambda_1(\eta)$ at $\eta = 0$ are given by

$$\frac{1}{2} \frac{\partial^2 \lambda_1}{\partial \eta_j \partial \eta_k}(0) = a^0_{jk}, \quad \forall j,k = 1,...,n,$$

where a^0_{jk} are the homogenized coefficients defined by

$$\frac{1}{|Y|} \int_Y \left[a_{jk} + \sum_{m=1}^n a_{jm} \frac{\partial w_m}{\partial y_m} \right].$$

Theorem 5.4.13. *The origin is a critical point of the first Bloch eigenvalue, i.e.,* $\frac{\partial \lambda_1}{\partial \eta_j}(0) = 0$ *for all* $j = 1, ..., n$*.Further, the Hessian of* λ_1 *at* $\eta = 0$ *is given by*

$$\frac{1}{2}\frac{\partial^2 \lambda_1}{\partial \eta_j \partial \eta_k}(0) = a^0_{jk} \quad \forall j, k = 1, ..., n.$$

The derivatives of the first Bloch mode can also be calculated and they are as follows:

$$\frac{\partial \phi_1}{\partial \eta_j}(y; 0) = \imath |Y|^{-\frac{1}{2}} w_j(y) \quad \forall j = 1, ..., n.$$

Proof. Use the information $\lambda_1(0) = 0$ and $\phi_1(y; 0) = |Y|^{-\frac{1}{2}}$ in the Taylor expansion with $\eta = 0$. □

5.5 Homogenization of Second order Elliptic Operator

Let $\mathcal{A}_\varepsilon = -\mathrm{div}_x(A(x/\varepsilon)\nabla_x)$ be the elliptic opertor with periodically oscillating coefficients. If ξ corresponds to the Fourier variable corresponding to x then $\varepsilon\xi$ corresponds to the Fourier variable corresponding to x/ε. Recall that, for each $m \in \mathbb{N}$, $\{\lambda_m(\eta)\}$ and $\{e^{\imath y \cdot \eta}\phi_m(y; \eta)\}$ are the eigenvalues and eigenvectors, respectively, of $\mathcal{A} = -\mathrm{div}_y(A(y)\nabla_y)$. We employ the change of variables, $y = x/\varepsilon$ and $\eta = \varepsilon\xi$, in the equation $\mathcal{A}[e^{\imath y\cdot\eta}\phi_m(y; \eta)] = \lambda_m(\eta)e^{\imath y\cdot\eta}\phi_m(y; \eta)$ to obtain

$$\varepsilon^2 \mathcal{A}_\varepsilon \left[e^{\imath x\cdot\xi}\phi_m\left(\frac{x}{\varepsilon}; \varepsilon\xi\right)\right] = \lambda_m(\varepsilon\xi)e^{\imath x\cdot\xi}\phi_m\left(\frac{x}{\varepsilon}; \varepsilon\xi\right).$$

Thus, the eigenvalues and eigenvectors of $\mathcal{A}_\varepsilon$ are $\varepsilon^{-2}\lambda_m(\varepsilon\xi)$ and $e^{\imath x\cdot\xi}\phi_m(x/\varepsilon; \varepsilon\xi)$. Set $\lambda^\varepsilon_m(\xi) := \varepsilon^{-2}\lambda_m(\varepsilon\xi)$ and $\phi^\varepsilon_m(x; \xi) := \phi_m(x/\varepsilon; \varepsilon\xi)$. Hence, the Bloch transform of $f \in L^2(\mathbb{R}^n)$, for each $x \in \mathbb{R}^n$ and $\varepsilon > 0$, is

$$f^\varepsilon_b(x; \xi) = \sum_{m=1}^{\infty} f^{m,\varepsilon}_b(\xi)\phi^\varepsilon_m(x; \xi)$$

where, for each $m \in \mathbb{N}$, $\varepsilon > 0$ and $\xi \in \varepsilon^{-1}Y^\star$, the m-th Bloch coefficient of f is

$$f^{m,\varepsilon}_b(\xi) = \varepsilon^{-n/2}\int_{\mathbb{R}^n} f(x)e^{-\imath x\cdot\xi}\overline{\phi^\varepsilon_m(x; \xi)}\, dx.$$

Thus, the inverse formula is

$$f(x) = \varepsilon^{n/2} \int_{\varepsilon^{-1}Y^\star} \sum_{m=1}^{\infty} f_b^{m,\varepsilon}(\xi) e^{\imath x\cdot\xi} \phi_m^\varepsilon(x;\xi)\, d\xi.$$

The $\varepsilon^{n/2}$ is a normalising factor appearing because the Lebesgue measure of $\varepsilon^{-1}Y^\star$ is ε^{-n}. The Parseval identity holds: for any $f, g \in L^2(\mathbb{R}^n)$

$$\varepsilon^{-n} \int_{\mathbb{R}^n} f(x)\overline{g(x)}\, dx = \int_{\varepsilon^{-1}Y^\star} \sum_{m=1}^{\infty} f_b^{m,\varepsilon}(\xi)\overline{g_b^{m,\varepsilon}(\xi)}\, d\xi.$$

Applying the Bloch transform, the equation $\mathcal{A}_\varepsilon u_\varepsilon = f$ transforms in to a set of algebraic equations, indexed by $m \geq 1$, $\lambda_m^\varepsilon(\xi) u_b^{m,\varepsilon}(\xi) = f_b^{m,\varepsilon}(\xi)$ for all $\xi \in \varepsilon^{-1}Y^\star$ (cf. Remark 5.4.10). Our aim is to pass to the limit in the system of algebraic equations. We first claim that one can neglect all the equations corresponding to $m \geq 2$.

Proposition 5.5.1. *Let*

$$v_\varepsilon(x) = \varepsilon^{n/2} \int_{\varepsilon^{-1}Y^\star} \sum_{m=2}^{\infty} u_b^{m,\varepsilon}(\xi) e^{\imath x\cdot\xi} \phi_m^\varepsilon(x;\xi)\, d\xi.$$

Then $\|v_\varepsilon\|_{2,\mathbb{R}^n} \leq C_0\varepsilon$.

Proof. Since

$$\int_{\mathbb{R}^n} \mathcal{A}_\varepsilon u_\varepsilon \overline{u_\varepsilon}\, dx = \int_{\mathbb{R}^n} f(x)\overline{u_\varepsilon}(x)\, dx.$$

The LHS is bounded and, applying Parseval Identity, we get

$$\begin{aligned}
\beta \int_{\mathbb{R}^n} |\nabla u_\varepsilon|^2\, dx &\geq \varepsilon^n \int_{\varepsilon^{-1}Y^\star} \sum_{m=1}^{\infty} f_b^{m,\varepsilon}(\xi)\overline{u_b^{m,\varepsilon}}(\xi)\, d\xi \\
&= \varepsilon^n \int_{\varepsilon^{-1}Y^\star} \sum_{m=1}^{\infty} \lambda_m^\varepsilon(\xi)|u_b^{m,\varepsilon}(\xi)|^2\, d\xi \\
&= \varepsilon^{n-2} \int_{\varepsilon^{-1}Y^\star} \sum_{m=1}^{\infty} \lambda_m(\eta)|u_b^{m,\varepsilon}(\xi)|^2\, d\xi \\
&\geq \varepsilon^{n-2} \int_{\varepsilon^{-1}Y^\star} \sum_{m=2}^{\infty} \lambda_m(\eta)|u_b^{m,\varepsilon}(\xi)|^2\, d\xi \\
&\geq \varepsilon^{n-2}\lambda_2^{(N)} \int_{\varepsilon^{-1}Y^\star} \sum_{m=2}^{\infty} |u_b^{m,\varepsilon}(\xi)|^2\, d\xi.
\end{aligned}$$

The last inequality is a consequence of the min-max principle yielding, for $m \geq 2$,

$$\lambda_m(\eta) \geq \lambda_2(\eta) \geq \lambda_2^{(N)} > 0 \quad \forall \eta \in Y^\star,$$

where $\lambda_2^{(N)}$ is the second eigenvalue of the eigenvalue problem for $\mathcal{A}$ in the cell Y with Neumann boundary condition on ∂Y. Then

$$\varepsilon^n \int_{\varepsilon^{-1}Y^\star} \sum_{m=2}^{\infty} |u_b^{m,\varepsilon}(\xi)|^2 \, d\xi \leq C_0 \varepsilon^2.$$

By Plancherel's identity, the left side is equal to $\|v_\varepsilon\|_{2,\mathbb{R}^n}$. □

Remark 5.5.2. Consider the algebraic equation corresponding to $m = 1$, i.e.,

$$\lambda_1^\varepsilon(\xi) u_b^{1,\varepsilon}(\xi) = f_b^{1,\varepsilon}(\xi) \quad \forall \xi \in \varepsilon^{-1} Y^\star.$$

Multiplying both sides by $\varepsilon^{n/2}$, we get

$$\varepsilon^{-2} \lambda_1(\varepsilon\xi) \varepsilon^{n/2} u_b^{1,\varepsilon}(\xi) = \varepsilon^{n/2} f_b^{1,\varepsilon}(\xi) \quad \forall \xi \in \varepsilon^{-1} Y^\star.$$

Expanding $\lambda_1(\varepsilon\xi)$ by Taylor's formula around $\xi = 0$, we get

$$\left[\frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \lambda_1}{\partial \eta_j \eta_k}(0) \xi_j \xi_k + O(\varepsilon \xi^3) \right] \varepsilon^{n/2} u_b^{1,\varepsilon}(\xi) = \varepsilon^{n/2} f_b^{1,\varepsilon}(\xi)$$

Passing to the limit as $\varepsilon \to 0$ to get

$$\frac{1}{2} \sum_{j,k=1}^{n} \frac{\partial^2 \lambda_1}{\partial \eta_j \eta_k}(0) \xi_j \xi_k \hat{u}_0(\xi) = \hat{f}(\xi).$$

Setting

$$a_{jk}^0 = \frac{1}{2} \frac{\partial^2 \lambda_1}{\partial \eta_j \eta_k}(0)$$

Then $\sum_{j,k=1}^n a_{jk}^0 \xi_k \xi_j \hat{u_0}(\xi) = \hat{f}(\xi)$ and $\mathcal{A}_0 u_0 := -\sum_{j,k=1}^n a_{jk}^0 \frac{\partial^2 u_0}{\partial x_j \partial x_k} = f(x)$. The only flaw in the above argument is that in passing to limit we have not checked uniform compact support of the sequence. To overcome this difficulty we use cut-off function technique to localize the equation.

Proposition 5.5.3 (First Bloch Transform tends to Fourier Transform)**.** *Let* $\{g_\varepsilon\} \subset L^2(\mathbb{R}^n)$ *be a sequence such that there is a fixed compact set* $K \subset \mathbb{R}^n$ *such that* $supp(g_\varepsilon) \subseteq K$ *for all* ε. *If* $g_\varepsilon \rightharpoonup g$ *weakly in* $L^2(\mathbb{R}^n)$ *then* $\varepsilon^{\frac{n}{2}} g_b^{1,\varepsilon} \rightharpoonup \hat{g}$ *weakly in* $L^2_{loc}(\mathbb{R}^n)$.

Proof. The first Bloch transform $g_b^{1,\varepsilon}(\xi)$, *a priori* defined for

$$\xi \in \varepsilon^{-1} Y^\star = (-\frac{\varepsilon^{-1}}{2}, \frac{\varepsilon^{-1}}{2})^n$$

can be extended by zero outside $\varepsilon^{-1}Y^\star$. We write

$$\begin{aligned} \varepsilon^{\frac{n}{2}} g_b^{1,\varepsilon}(\xi) &= \int_{\mathbb{R}^n} g_\varepsilon(x) e^{-\imath x\cdot\xi} \overline{\phi_1(\frac{x}{\varepsilon};0)}\, dx \\ &\quad + \int_K g_\varepsilon(x) e^{-\imath x\cdot\xi} \left(\overline{\phi_1(\frac{x}{\varepsilon};\varepsilon\xi)} - \overline{\phi_1(\frac{x}{\varepsilon};0)} \right) dx. \end{aligned}$$

Since $\phi_1(y;0) = |Y|^{\frac{-1}{2}} = (2\pi)^{-n/2}$, the first term is nothing but the Fourier transform of g_ε and so it converges weakly to $\hat{g}(\xi)$ in $L^2(\mathbb{R}^n)$. By Cauchy-Schwarz inequality and the regularity of the first Bloch eigenfunction $\eta \mapsto \phi_1(\cdot,\eta) \in L^2_{\text{per}}(Y)$ at $\eta = 0$, the second term is bounded by

$$\|g_\varepsilon\|_{2,\mathbb{R}^n} \left[\int_K |\phi_1(\frac{x}{\varepsilon};\varepsilon\xi) - \phi_1(\frac{x}{\varepsilon};0)|^2\, dx \right]^{\frac{1}{2}} \leq C_0 \|\phi_1(y;\varepsilon\xi) - \phi_1(y;0)\|_{2,Y}.$$

By Lipschitz continuity of $\eta \mapsto \phi_1(\cdot,\eta)$, the second term in the right side is bounded above by $C_0\varepsilon\xi$. Thus, if $|\xi| \leq M$ then it is bounded above by $cM\varepsilon$ and so, in particular, it converges to zero in $L^\infty_{\text{loc}}(\mathbb{R}^n)$. ∎

Theorem 5.5.4. *Let* $\Omega \subset \mathbb{R}^n$ *be an arbitrary, not necessarily bounded, domain. Consider a sequence* $u_\varepsilon \rightharpoonup u_0$ *weakly in* $H^1(\Omega)$ *and* $A_\varepsilon u_0 = f$ *in* Ω *with* $f \in L^2(\Omega)$. *Then* u_0 *satisfies* $A_0 u_0 = f$ *in* Ω. *In fact,* $A_\varepsilon \nabla u_\varepsilon \rightharpoonup A_0 \nabla u_0$ *weakly in* $L^2(\Omega)$.

Proof. Let $\phi \in D(\Omega)$ be arbitrary. If u_ε satisfies $\mathcal{A}_\varepsilon u_\varepsilon = f$ in Ω then consider its localization ϕu_ε satisfies

$$\mathcal{A}_\varepsilon(\phi u_\varepsilon) = \phi f + g_\varepsilon + h_\varepsilon \text{ in } \quad \mathbb{R}^n,$$

where

$$\begin{aligned} g_\varepsilon &= -2\sum_{j=1}^n \sigma_j^\varepsilon \frac{\partial\phi}{\partial x_j} - \sum_{j,k=1}^n a_{jk}^\varepsilon \frac{\partial^2\phi}{\partial x_j \partial x_k} u_\varepsilon, \\ \sigma_j^\varepsilon(x) &= \sum_{k=1}^n a_{jk}^\varepsilon \frac{\partial u_\varepsilon}{\partial x_k}, \\ h_\varepsilon &= -\sum_{j,k=1}^n \frac{\partial a_{jk}^\varepsilon}{\partial x_j}\frac{\partial \phi}{\partial x_k} u_\varepsilon. \end{aligned}$$

Using the arguments given in the remark above, we can pass to the limit above, since ϕu_ε is bounded in $H^1(\mathbb{R}^n)$. Neglecting all the harmonics corresponding to $m \geq 2$ and considering only the $m = 1$ yields at the limit

$$\frac{1}{2}\sum_{j,k=1}^n \frac{\partial^2\lambda_1}{\partial\eta_j\partial\eta_k}(0)\xi_j\xi_k\widehat{(\phi u_0)}(\xi) = \widehat{(\phi f)}(\xi) + \lim_{\varepsilon\to 0}\varepsilon^{\frac{n}{2}} g_b^{1,\varepsilon}(\xi) + \lim_{\varepsilon\to 0}\varepsilon^{\frac{n}{2}}\hat{h}_b^{1,\varepsilon}(\xi). \tag{5.5.1}$$

The sequence σ_j^ε is bounded in $L^2(\Omega)$. Therefore, we can extract a subsequence (still denoted by ε) which is weakly convergent in $L^2(\Omega)$. Let σ_j^0 denote its limit and its extension by zero outside Ω. Using this convergence and the definition of g_ε, we see that

$$g_\varepsilon \rightharpoonup g_0 := -2\sum_{j=1}^n \sigma_j^0 \frac{\partial\phi}{\partial x_j} - \sum_{j,k=1}^n \mathcal{M}(a_{jk})\frac{\partial^2\phi}{\partial x_j\partial x_k} u_0 \text{ weakly in } L^2(\mathbb{R}^n),$$

where $\mathcal{M}(a_{jk})$ is the average of a_{jk} on Y. Therefore,

$$\varepsilon^{\frac{n}{2}} g_b^{1,\varepsilon}(\xi) \rightharpoonup \hat{g}_0(\xi) \text{ weakly in } L^2_{\text{loc}}(\mathbb{R}^n).$$

A similar argument fails for $\{h_b^{1,\varepsilon}\}$ because h_ε is not bounded in $L^2(\mathbb{R}^n)$. We decompose

$$\begin{aligned} \varepsilon^{\frac{n}{2}} h_b^{1,\varepsilon}(\xi) &= \int_{\mathbb{R}^n} h_\varepsilon(x) e^{-\imath x\cdot\xi}\overline{\phi_1\left(\frac{x}{\varepsilon}, 0\right)}\,dx \\ &\quad + \int_{\mathbb{R}^n} h_\varepsilon(x) e^{-\imath x\cdot\xi}\left(\overline{\phi_1\left(\frac{x}{\varepsilon};\varepsilon\xi\right)} - \overline{\phi_1\left(\frac{x}{\varepsilon};0\right)}\right)\,dx. \end{aligned}$$

Using the Taylor expansion of $\phi_1(y;\eta)$ at $\eta = 0$, the second term is equal to

$$-\varepsilon^{-1}\sum_{j,k=1}^{n}\int_{\mathbb{R}^n}\frac{\partial a_{jk}}{\partial y_j}\left(\frac{x}{\varepsilon}\right)\frac{\partial\phi}{\partial x_k}(x)u_\varepsilon(x)e^{-\imath x\cdot\xi}\left[\varepsilon\sum_{\ell=1}^{n}\frac{\partial\overline{\phi_1}}{\partial\eta_\ell}\left(\frac{x}{\varepsilon};0\right)\xi_\ell + O(\varepsilon^2\xi^2)\right]dx,$$

which evidently converges to

$$-\sum_{j,k,\ell=1}^{n}\mathcal{M}\left(\frac{\partial a_{jk}}{\partial y_j}\frac{\partial\overline{\phi_1}}{\partial\eta_\ell}(y;0)\right)\xi_\ell\int_{\mathbb{R}^n}\frac{\partial\phi}{\partial x_k}u_0e^{-\imath x\cdot\xi}\,dx.$$

strongly in $L^\infty_{\text{loc}}(\mathbb{R}^n)$. On the other hand, after integraing by parts, the first term in the RHS of the decomposition of $\varepsilon^{n/2}h_b^{1,\varepsilon}$ becomes

$$\sum_{j,k=1}^{n}\int_{\mathbb{R}^n}a^\varepsilon_{jk}\left[\frac{\partial^2\phi}{\partial x_j\partial x_k}u_\varepsilon + \frac{\partial\phi}{\partial x_k}\frac{\partial u^\varepsilon}{\partial x_j} - \imath\xi_j\frac{\partial\phi}{\partial x_k}u_\varepsilon\right]e^{-\imath x\cdot\xi}\overline{\phi_1}\left(\frac{x}{\varepsilon};0\right)dx.$$

Choosing $\phi_1(y;0) = |Y|^{-\frac{1}{2}}$, it is easily seen that the above integral converges weakly in $L^2(\mathbb{R}^n)$ to

$$\begin{aligned}&|Y|^{-\frac{1}{2}}\sum_{j,k=1}^{n}\int_{\mathbb{R}^n}\left[\mathcal{M}(a_{jk})\frac{\partial^2\phi}{\partial x_j\partial x_k}u_0 - \imath\xi_j\mathcal{M}(a_{jk})\frac{\partial\phi}{\partial x_k}u_0\right]e^{-\imath x\cdot\xi}\,dx\\ +\;&|Y|^{-\frac{1}{2}}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\sigma^0_k\frac{\partial\phi}{\partial x_k}e^{-\imath x\cdot\xi}\,dx.\end{aligned}$$

Using this information in (5.5.1) and using Theorem 5.4.13, we conclude that

$$\begin{aligned}\sum_{j,k=1}^{n}a^0_{jk}\xi_j\xi_k\widehat{(\phi u_0)}(\xi) &= \widehat{(\phi f)}(\xi) - |Y|^{-\frac{1}{2}}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\sigma^0_k\frac{\partial\phi}{\partial x_k}e^{-\imath x\cdot\xi}\,dx\\ &\quad -\imath\sum_{j,k=1}^{n}\xi_j|Y|^{-\frac{1}{2}}a^0_{jk}\int_{\mathbb{R}^n}\frac{\partial\phi}{\partial x_k}u_0e^{-\imath x\cdot\xi}\,dx.\end{aligned}$$

This can be rewritten as

$$\begin{aligned}[\widehat{\mathcal{A}_0(\phi u_0)}](\xi) &= \widehat{(\phi f)}(\xi) - |Y|^{-\frac{1}{2}}\sum_{k=1}^{n}\int_{\mathbb{R}^n}\sigma^0_k\frac{\partial\phi}{\partial x_k}e^{-\imath x\cdot\xi}\,dx\\ &\quad -\imath\sum_{j,k=1}^{n}\xi_j|Y|^{-\frac{1}{2}}a^0_{jk}\int_{\mathbb{R}^n}\frac{\partial\phi}{\partial x_k}u_0e^{-\imath x\cdot\xi}\,dx.\end{aligned}$$

This is the *localized homogenized equation* in the Fourier space. Taking inverse Fourier transform of the above equation, we obtain

$$\mathcal{A}_0(\phi u_0) = \phi f - \sum_{k=1}^{n} \sigma_k^0 \frac{\partial \phi}{\partial x_k} - \sum_{j,k=1}^{n} a_{jk}^0 \frac{\partial}{\partial x_j} \left(\frac{\partial \phi}{\partial x_k} u_0 \right) \text{ in } \mathbb{R}^n.$$

On the other hand, we can calculate $\mathcal{A}_0(\phi u_0)$ directly:

$$\mathcal{A}_0(\phi u_0) = -\sum_{j,k=1}^{n} \left[a_{jk}^0 \frac{\partial^2 \phi}{\partial x_j \partial x_k} u_0 + 2 a_{jk}^0 \frac{\partial \phi}{\partial x_j} \frac{\partial u_0}{\partial x_k} \right] + \phi \mathcal{A}_0 u_0 \text{ in } \mathbb{R}^n.$$

A comparison of the above two equation yields

$$\phi(\mathcal{A}_0 u_0 - f) = \sum_{j=1}^{n} \left(\sum_{k=1}^{n} a_{jk}^0 \frac{\partial u_0}{\partial x_k} - \sigma_j^0 \right) \frac{\partial \phi}{\partial x_j} \text{ in } \mathbb{R}^n.$$

Since the above relation is true for all ϕ in $\mathcal{D}(\Omega)$, the desired conclusions follow. In fact, let us choose $\phi(x) = \phi_0(x) e^{\imath m x \cdot \nu}$, where ν is a unit vector in $\mathbb{R}^n$ and $\phi_0(x) \in \mathcal{D}(\Omega)$ is fixed. Letting $m \to \infty$ in the resuling relation and varying the unit vector ν, we can easily deduce, successively, that $\sigma_j^0 = \sum_{k=1}^{n} a_{jk}^0 \frac{\partial u_0}{\partial x_k}$ in Ω and $\mathcal{A}_0 u_0 = f$ in Ω. □

Appendices

Bibliography

[All92] G. Allaire, *Homogenization and two-scale convergence*, SIAM J. Math. Anal. **23** (1992), no. 6, 1482–1518.

[All94] ______, *Two-scale Convergence: A New Method in Periodic Homogenization*, Nonlinear Partial Differential Equations and Their Applications, Pitman Research Notes Math Series 302, 1994, pp. 1–14.

[Att84] H. Attouch, *Variational convergence for functions and operators*, Pitman, London, 1984.

[Bab76] Ivo Babuška, *Homogenization and its application. mathematical and computational problems*, Numerical solution of Partial Differential Equations-III (Proc. of third SYNSPADE, 1975) (Bert Hubbard, ed.), Academic Press, 1976, pp. 89–116.

[BD98] Andrea Braides and Anneliese Defranceschi, *Homogenization of Multiple Integrals*, Oxford lecture series in mathematics and its applications, no. 12, Oxford University Press, 1998.

[BLP78] A. Bensoussan, J. L. Lions, and G. Papanicolaou, *Asymptotic analysis for periodic structures*, North Holland, Amsterdam, 1978.

[BM] J. Ball and F. Murat, $W^{1,p}$ *quasi convexity and variational problems for multiple integrals*, Journal of Functional Analysis **58**.

[Bra02] Andrea Braides, Γ-*Convergence for Beginners*, Oxford Lecture series in Mathematics and its Applications, vol. 22, Oxford University Press, 2002.

[CD99] Doina Cioranescu and Patrizia Donato, *An Introduction to Homogenization*, Oxford lecture series in mathematics and its applications, no. 17, Oxford University Press, 1999.

[Che75] D. Chenais, *On the existence of a solution in a domain identification problem*, J. Math. Anal. Appl. **52** (1975), 189–219.

[Cla79] R. Clausius, *Die mech. wa̋rme theorie*, Vieweg **Bd 2** (1879), 62.

[CP99] Doina Cioranescu and Jeannine Saint Jean Paulin, *Homogenization of Reticulated Structures*, Applied Mathematical Sciences, no. 136, Springer-Verlag, 1999.

[DM93] Gianni Dal Maso, *An introduction to* Γ*-Convergence*, Progress in Nonlinear Differential Equations and Their Applications, vol. 8, Birkhäuser Boston, 1993.

[ET74] I. Ekeland and R. Temam, *Analyse convexe et problèmes variationnels*, Dunod Gauthier-Villars (English Edition: North-Holland, 1976), 1974.

[Flo83] G. Floquet, *Sur les équations différentielles linéaires à coefficients périodiques*, Ann. École Norm. Sér 2 **12** (1883), 47–89.

[GF75] Ennio De Giorgi and T. Franzoni, *Su un tipo di convergenza variazionale*, Atti Acad. Naz. Lincei Rend. Cl. Sci. Fis. Mat. Natur. **58** (1975), 842–850.

[Gio75] Ennio De Giorgi, *Sulla convergenza di alcune successioni di integrali del tipo dell'area*, Rendi Condi di Mat. **8** (1975), 277–294.

[Gio84] ———, *G-operators and* Γ*-Convergence*, Proceedings of the international congress of mathematicians (Warsazwa, August 1983), PWN Polish Scientific Publishers and North Holland, 1984, pp. 1175–1191.

[GR81] V. Girault and P. A. Raviart, *Finite element methods for navier-stokes equations*, Lecture Notes in Mathematics **749** (1981).

[GS73] Ennio De Giorgi and Sergio Spagnolo, *Sulla convergenza degli integrali dell'energia per operatori ellittici del secondo ordine*, Boll. Un. Mat. It. **8** (1973), 391–411.

[Hor97] Ulrich Hornung (ed.), *Homogenization and Porous Media*, Interdisciplinary applied mathematics, no. 6, Springer-Verlag, 1997.

[JKO94] V. V. Jikov, S. M. Kozlov, and O. A. Oleinik, *Homogenization of Differential Operators and Integral Functionals*, Springer-Verlag, 1994, Translated by G. A. Yosifian.

[KP97] S. Kesavan and Jeannine Saint Jean Paulin, *Homogenization of an optimal control problem*, SIAM J. Control Optim. **35** (1997), no. 5, 1557–1573.

[KR02] S. Kesavan and M. Rajesh, *On the limit matrix obtained in the homogenization of an optimal control problem*, Proc. Indian Acad. Sci. (Math. Sci.) **112** (2002), no. 2, 337–346.

[KS00] David Kinderlehrer and Guido Stampacchia, *An introduction to variational inequalities and their applications*, Classics in Applied Mathematics, vol. 31, SIAM, 2000.

[KV77] S. Kesavan and M. Vanninathan, *L'homogénéisation d'un problème de contrôle optimal*, C. R. Acad. Sci. Paris, Série A **285** (1977), 441–444.

[LNW02] D. Lukkassen, G. Nguetseng, and P. Wall, *Two-scale convergence*, Int. J. of Pure and Appl. Math. **2** (2002), no. 1, 35–86.

[Max73] C. Maxwell, *Treatise on electricity and magnetismus*, Oxford University Press **1** (1873), 365.

[Mos50] O. F. Mosotti, *Discussione analitica sul influenze che l'azione di in mezzo dielettrico ha sulla distribuzione dell' electtricita alla superficie di pin corpi electtici disseminati in esso*, Mem. Di Math et Di Fisica in Modena **24** (1850), no. 2, 49.

[MT97] François Murat and Luc Tartar, *H-convergence*, Topics in the Mathematical modelling of Composite materials (Andrej Cherkaev and Robert Kohn, eds.), Progress in Nonlinear Differential Equations and Their Applications, vol. 31, Birkhäuser, Boston, 1997, English translation of [Tar77, Mur78b], pp. 21–43.

[Mur71] François Murat, *Un contre-exemple pour le problème du contrôle dans les coefficients*, C. R. Acad. Sci. Paris, Série A **273** (1971), 708–711.

[Mur72] ______, *Théorèmes de non-existence pour des problèmes de contrôle dans les coefficients*, C. R. Acad. Sci. Paris, Série A **274** (1972), 395–398.

[Mur78a] ______, *Compacité par compensation*, Ann. Sc. Norm. Sup. Pisa **5** (1978), 489–507.

[Mur78b] ______, *H-convergence*, Séminaire d'Analyse Fonctionnelle et Numérique de l'Université d'Alger, March 1978, mimeographed notes. English translation in [MT97].

[Mur79] ______, *Compacité par compensation ii*, Proceedings of the international meeting on recent methods in non linear analysis (Rome, May 1978) (Ennio De Giorgi, E. Magenes, and U. Mosco, eds.), Pitagora Editrice, Bologna, 1979, pp. 245–256.

[Ngu89] G. Nguetseng, *A general convergence result for a functional related to the theory of homogenization*, SIAM J. Math. Anal. **20** (1989), no. 3, 608–623.

[Poi22] S. D. Poisson, *Second mêm. sur la théorie de magnetisme*, Mem. de L'Acad. de France **5** (1822).

[Ray92] J. W. Rayleigh, *On the influence of obstacles in rectangular order upon the properties of a medium*, Phil. Mag. **34** (1892), no. 5, 481.

[Spa67] Sergio Spagnolo, *Sul limite delle soluzioni di problemi di Cauchy relativi all'equazione del calore*, Ann. Sc. Norm. Sup. Pisa **21** (1967), 657–699.

[Spa68] ______, *Sulla convergenza di soluzioni di equazioni paraboliche ed ellittiche*, Ann. Sc. Norm. Sup. Pisa **22** (1968), 571–597.

[Spa76] ______, *Convergence in energy for elliptic operators*, Numerical solution of Partial Differential Equations-III (Proc. of third SYNSPADE, 1975) (Bert Hubbard, ed.), Academic Press, 1976, pp. 469–498.

[Tar77] Luc Tartar, *Cours Peccot au Collège de France*, partially written in [Mur78b]. English translation in [MT97], March 1977.

[Tar79] ______, *Compensated compactness and applications to partial differential equations*, Non linear analysis and mechanics, Heriot-Watt Symposium, Volume IV (R. J. Knops, ed.), Research Notes in Mathematics, vol. 39, Pitman, Boston, 1979, pp. 136–212.

[Tem79] R. Temam, *Navier-stokes equations*, North-Holland, 1979.

[Yos95] K. Yosida, *Functional analysis*, Springer-Verlag, 1995.

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