Analysis
MTH-753A

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12 Twelfth Week

- Inverse and Implicit Function Theorem

Purpose of the Course

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- Given the different academic backgrounds students may have come from, the purpose of the course is to the ensure that the student's understanding of concept in Analysis are on equal footing.
- However, to avoid boring repetition, an attempt is being made to present the topics in an application oriented perspective, thus compromising on the usual logical order.

Algebraic and Differential Equations

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- The invention of calculus gave rise to differential equations (DEs).
- Modern topics in Analysis grew out of the attempt to understand and analyse the solutions of DEs.

One Variable Polynomials

While defining the n-th root of a real number, one naturally encounters the following algebraic equation: Given any $a \in \mathbb{R}$ and $n \in \mathbb{N}$, find all $x \in \mathbb{R}$ such that $x^n = a$.

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Definition

A polynomial in one variable of degree n is a map $f : \mathbb{R} \to \mathbb{R}$ defined as

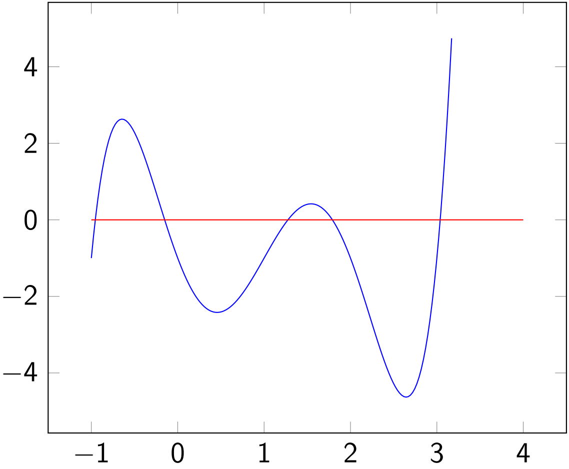
$$f(x) := a_n x^n + a_{n-1} x^{n-1} + \ldots + a_2 x^2 + a_1 x + a_0$$

where $\{a_0, a_1, \ldots, a_{n-1}, a_n\} \subset \mathbb{R}$, the coefficients, and $\mathbb{N} \cup \{0\}$ are given such that $a_n \neq 0$.

A constant function is a polynomial of degree zero.

Zeroes or Roots of Polynomial

One is interested to find all $x \in \mathbb{R}$ where the polynomial attains zero.



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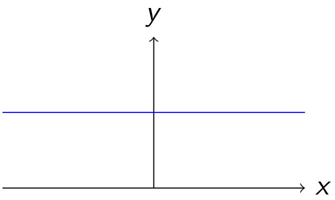
- Every non-zero constant function has no roots!



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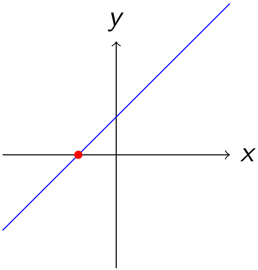
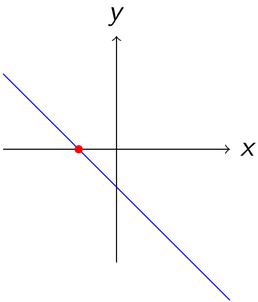
- If f attains zero at some x, then $ax + b = 0$ and hence $x = -b/a$. Thus, there is exactly one zero of f.





Quadratic Equations

- The polynomial in one variable of degree two, called *quadratic function*, is a map $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = ax^2 + bx + c$, for any given $a, b, c \in \mathbb{R}$ with $a \neq 0$.

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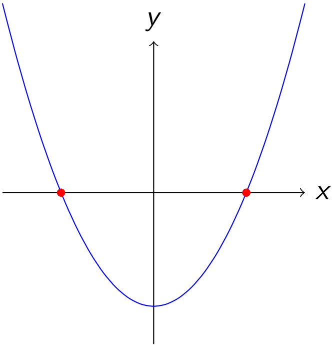
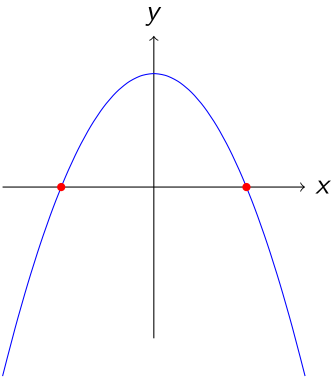
- The case $b^2 - 4ac > 0$ corresponds to two distinct real roots. The graph of the polynomial lies on both the upper and lower plane.





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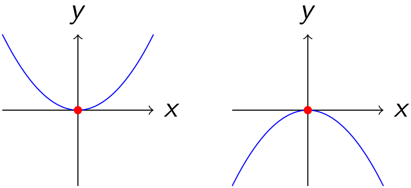

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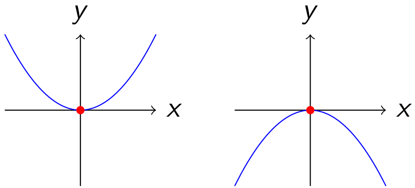
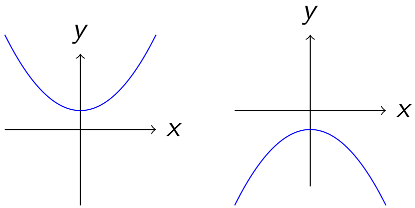


- Observe that in this case the zero is also a zero of the derivative (zero slope tangent). It is a repeated (double) root!

Negative Discriminant

- The case $b^2 - 4ac < 0$ corresponds to no real roots. The graph never intersects/touches the x-axis but lies completely in either the upper or lower plane.



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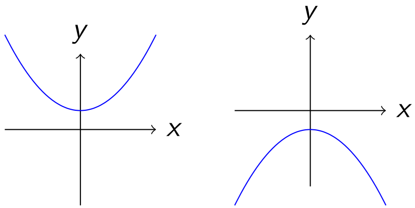

- For example, consider the function $f(x) = x^2 + 1$. Note that for any $x \in \mathbb{R}$, $x^2 + 1 \geq 1 > 0$. Hence the function f never attains zero.

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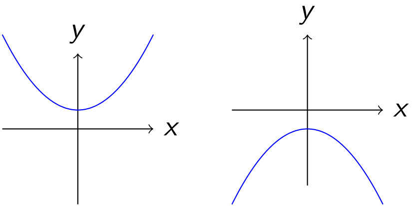

- For example, consider the function $f(x) = x^2 + 1$. Note that for any $x \in \mathbb{R}$, $x^2 + 1 \geq 1 > 0$. Hence the function f never attains zero.
- There is no reason to seek an 'imaginary' solution to $x^2 + 1 = 0$ yet!

Cubic Equations

- The formula for roots of cubic equation were discovered independently by Scipione del Ferro and Nicolo Tartaglia which were orally passed on to Girolamo Cardano who published it in 1545.

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$$q := \left(\frac{b}{3a}\right)^3 (3a - 1) + \frac{3ad - bc}{3a}. \tag{1.2}$$

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- To avoid the confusion that $\sqrt{-1}\sqrt{-1} = -1$ which contradicts the known formula $\sqrt{ab} = \sqrt{a}\sqrt{b}$, we denote $\imath = \sqrt{-1}$ and $\imath^2 = -1$.

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- In contrast to $\mathbb{R}$, $\mathbb{C}$ is algebraically closed, i.e. all complex polynomials admit complex roots? This is the statement of the Fundamental theorem of Algebra.

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where $A = \frac{a^2}{4} - b + y, C = \frac{y^2}{4} - d$ and y is chosen as one of the roots to the cubic equation:

$$y^3 - by^2 + (ac - 4d)y - [d(a^2 - 4b) + c^2] = 0.$$

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- The proof of the Fundamental theorem of Algebra, is a result in Analysis!

Polynomials are Unbounded in $\mathbb{C}$

- Any polynomial $p : \mathbb{C} \to \mathbb{C}$ of degree n has the form $p(z) = \sum_{i=0}^{n} a_i z^i$ where $a_i \in \mathbb{C}$ are given.

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- Above arguments also reveals that $\lim_{|z|\to\infty} \frac{p(z)}{a_n z^n} = 1$.

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- The first correct proof of FTA for real and complex coefficient polynomial was presented by Carl-Friedrich Gauss in 1816 and 1849, respectively.

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If $p : \mathbb{C} \to \mathbb{C}$ *is a non-constant polynomial with constant coefficients then there is a complex number* $z_0 \in \mathbb{C}$ *such that* $p(z_0) = 0$*.*

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Linear Maps

- For any two vector spaces V and W over a field $\mathbb{F}$, the map $T : V \to W$ is said to be *linear* if $T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$ for all $x, y \in V$ and $\alpha, \beta \in \mathbb{F}$.

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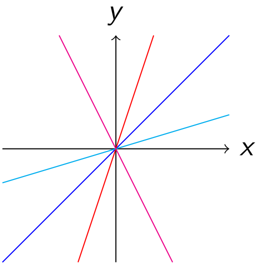
- The dimension of V is the sum of the rank and nullity of T.

- For instance, a map $T : \mathbb{R} \to \mathbb{R}$ is linear iff $Tx = \alpha x$ for some $\alpha \in \mathbb{R}$, i.e. the graphs are straight lines in $\mathbb{R}^2$ passing through origin with slope α and angle of inclination $\tan^{-1}(\alpha)$.



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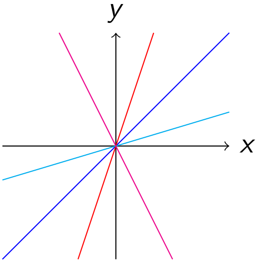

- The real numbers are in on-to-one correspondence with real valued linear maps on $\mathbb{R}$.
- The real linear maps dilates points. i.e. it stretches ($|\alpha| > 1$) or shrinks ($|\alpha| < 1$) points in $\mathbb{R}$.

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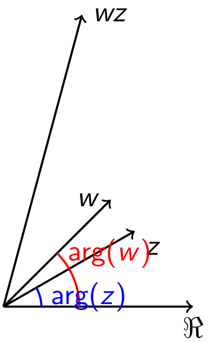

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- For instance, the map $z \mapsto \bar{z}$ is not complex linear while its analogue map in $\mathbb{R}^2$, $(x, y) \mapsto (x, -y)$ is real linear.
- Thus, while the map $(x, y) \mapsto (x, -y)$ is differentiable everywhere and its derivative is itself (being linear) the complex valued function $z \mapsto \bar{z}$ is nowhere complex differentiable.

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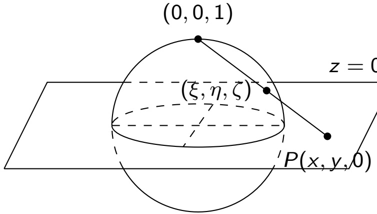

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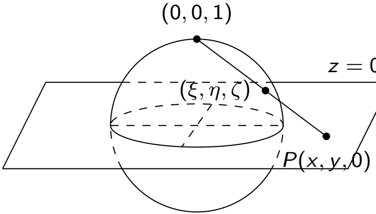
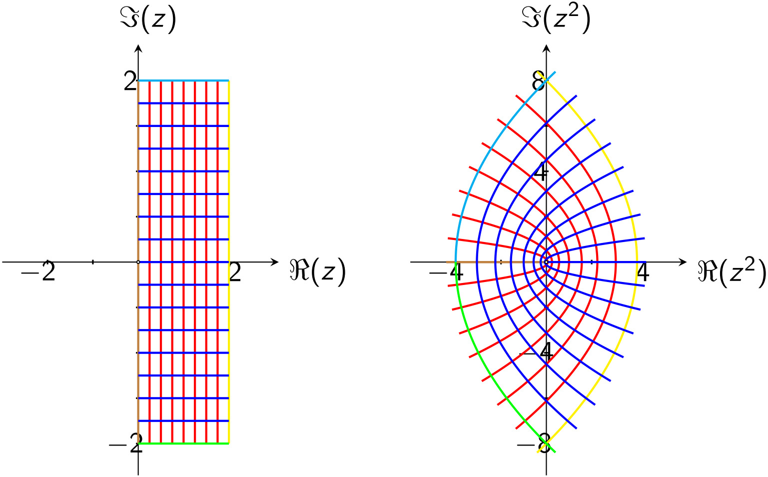


- For functions that are not injective or is multi-valued can be visualised using the concept of Riemann surfaces!

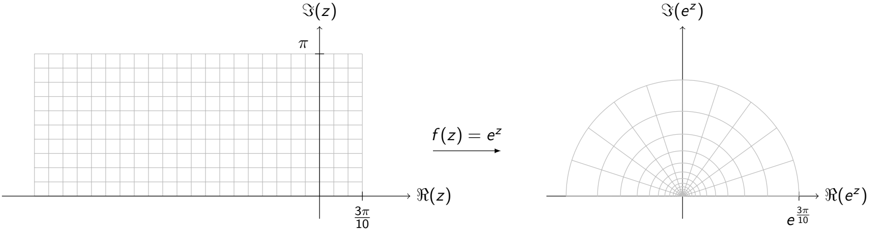
Plot for z^2

$z^2 = (x^2 - y^2) + \imath 2xy$ is not injective.



Plot for e^z

$e^z = e^x e^{\imath y}$ is not injective because $e^{z+\imath 2\pi k} = e^z$ for integral k.



The inversion map $\frac{1}{z}$

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- Thus, the composition of linear and inverse maps also preserve the family of circles and lines.
- More generally, the fractional linear maps given by

$$f(z) = \frac{az + b}{cz + d}$$

  such that $ad - bc \neq 0$ (to exclude constant functions) preserve the family of circles and lines because $f(z) = \frac{a}{c} + \frac{1}{cz+d}\left(b - \frac{ad}{c}\right)$, composition of linear and inverse map.

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- The map $z \mapsto \bar{z}$ is not conformal because it reflects tangent vectors changing its orientation!

Real Differentiation

Definition

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be *differentiable* at a, denoted as $f'(a)$ or $\frac{df}{dx}(a)$, if the limit

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Example

The real valued function $x \mapsto |x|$ is not differentiable at 0.

Definition

Let $\Omega \subset E$ be an open subset of the normed linear space E. We say $f : \Omega \to F$, where F is another normed linear space, is said to be Fréchet differentiable or, simply, differentiable at $a \in \Omega$ if there exists a linear map $Df(a) \in \mathcal{L}(E, F)$ such that

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- The hypothesis that Ω is open ensures that $Df(a)$ is unique.

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Definition

Let V be a vector space. The directional or Gâteau derivative of $f : V \to \mathbb{R}$ at $a \in V$, along the direction $v \in V \setminus \{0\}$, is defined as

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- Also, $D_v f(a) = Df(a) \cdot v$.

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- Let $J_f(a)$ denote the determinant of the Jacobian matrix $Df(a)$.

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- Real derivatives satisfy the intermediate value theorem, a property weaker than continuity!

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where the unknowns $u, v : \mathbb{R}^2 \to \mathbb{R}$.

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- Observe that the $\pi/2$ rotation matrix corresponds to the complex number $\imath$ and square of the matrix is negative of identity matrix.
- In short, the real and imaginary parts of a holomorphic function cannot be chosen independently.

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- Thus $f'(a) = \partial_x f(a) = -\imath \partial_y f(a)$ and $J_f(a) = |\partial_x f(a)|^2 = |\partial_y f(a)|^2$.

Cauchy-Riemann Equations and Ideal Fluid

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- The incompressibility and irrotational condition is precisely the CR equations satisfied by the pair $(u, -v)$.
- A velocity vector field (u, v) induces an ideal planar fluid flow iff $u - \imath v$ is holomorphic.

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- A map $f : \mathbb{C} \to \mathbb{R}$ is either not holomorphic or is a constant.

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- Because, for any $u \in C^2(\Omega)$, $(T_a u)_{x_i}(x) = u_{x_i}(x + a)$ and $(T_a u)_{x_i x_i}(x) = u_{x_i x_i}(x + a)$. Thus, $\Delta(T_a u)(x) = \Delta u(x + a)$.

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- The class of all radial functions is invariant under Laplacian.

Harmonic Functions

Definition

Let Ω be an open subset of $\mathbb{R}^n$. A function $u \in C^2(\Omega)$ is said to be *harmonic* on Ω if $\Delta u(x) := \sum_{i=1}^n \partial_{x_i}^2 u = 0$ in Ω.

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- Later, in 1813, Poisson discovered that on Ω the Newtonian potential satisfies the equation: $-\Delta u = \rho \quad \text{in } \Omega$. Inhomogeneous Laplace equations are called *Poisson* equations.

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- A function u is harmonic iff $\partial_{z\bar{z}} u = 0$ because the Laplacian $\Delta = 4\partial_{z\bar{z}}$, the complex mixed derivative.

2D Laplacian and Complex Wave Operator

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- If we are seeking real solutions u, then

$$u(x, y) = \frac{1}{2}\left(u(x, y) + \overline{u(x, y)}\right) = \Re[F(z) + G(\bar{z})],$$

  real part of a complex function.

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- Properties of harmonic functions can be obtained from properties of holomorphic functions. Compare (Mean value property with Cauchy Integral formula, Maximum Principle with Maximum Modulus and Liouville theorem etc.)

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- For rational $\alpha = p/q$ with $\gcd(p, q) = 1$, z^α is also multivalued and takes exactly q different values corresponding to the q-th roots of unity.

Exponential, Logarithm and Trigonometric

- The complex exponential e^z is defined using the power series $\sum_{k=0}^{\infty} \frac{z^k}{k!}$. It is many-to-one function because $e^{z+\imath 2\pi k} = e^z$. Its real and imaginary parts $e^x \cos y$ and $e^x \sin y$ are harmonic.

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- For instance, real logarithm of 1 is zero but complex $\log(1) = \imath 2k\pi$ for all $k \in \mathbb{Z}^+$.
- Logarithm of negative real numbers is $\log(x) = \log |x| + \imath\pi(1 + 2k)$ for all $k \in \mathbb{Z}^+$.

Dirichlet Problem

- The boundary value problem of seeking a harmonic function with Dirichlet boundary conditions (prescribed value of the harmonic function on the boundary) is:

$$\begin{cases} \Delta u &= 0 \quad \text{in } \Omega \subset \mathbb{R}^n \\ u &= g \quad \text{on } \partial\Omega. \end{cases} \tag{3.1}$$

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- In two dimensions, the solution to above problem can be reduced to the Dirichlet problem on the unit disk $\mathbb{D} = \{|z| < 1\}$ for large class of Ω!

Theorem (Riemann Mapping Theorem)

Every simply connected proper subset Ω of $\mathbb{C}$ is conformally equivalent to $\mathbb{D}$, i.e. there is a biholomorphism (inverse holomorphic too) $f : \Omega \to \mathbb{D}$. For each $z_0 \in \Omega$ there is a unique biholomorphism such that $f(z_0) = 0$ and $f'(z_0) > 0$.

Note that the above result allows Ω to be unbounded!

Multiplicity of Conformality of Unit Disk to Itself

- For any $z_0 \in \mathbb{D}$, the map $T(z) = \frac{z - z_0}{1 - \overline{z_0} z}$ maps $\mathbb{D}$ onto itself with $T(z_0) = 0$ (verify that $|T(z)| < 1$!).

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- However, once z_0 and θ are fixed, there is a unique biholomorphism on $\mathbb{D}$ such that $T(z_0) = 0$ and $T'(z_0) > 0$.

Poisson Kernel for Disk

Theorem (2D Disk)

Let Ω *be* $\mathbb{D}$*, the unit disk in* $\mathbb{R}^2$*. Let* $g : \partial\Omega \to \mathbb{R}$ *be a continuous function. Then there is a unique solution to* (3.1) *on the unit disk with given boundary value* g*.*

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Theorem (2D Disk)

Let Ω be $\mathbb{D}$, the unit disk in $\mathbb{R}^2$. Let $g : \partial\Omega \to \mathbb{R}$ be a continuous function. Then there is a unique solution to (3.1) on the unit disk with given boundary value g.

Proof: Setting $U(r,\theta) = u(re^{\imath\theta})$, (3.1) is

$$\begin{cases} \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial U}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 U}{\partial \theta^2} & = 0 \quad \text{in } \Omega \\ U(r,\theta+2\pi) & = U(r,\theta) \quad \text{in } \Omega \\ U(1,\theta) & = g(e^{\imath\theta}) \quad \text{on } \partial\Omega \end{cases} \tag{3.2}$$

and the *Poisson* formula

$$u(z) = \frac{1-|z|^2}{2\pi}\int_0^{2\pi} \frac{g(e^{\imath\theta})}{|z-e^{\imath\theta}|^2}\,d\theta.$$

Use method of separation of variable, Fourier series and uniqueness of Dirichlet problem for bounded domains. If g is real valued then u is real valued!

Solution on Arbitrary Simple Connected Set

- Thus, to solve the Dirichlet problem on any arbitrary proper simply connected subset of $\mathbb{R}^2$ it is enough to solve it in the unit disk $\mathbb{D}$ as long as the conformal mapping between Ω and $\mathbb{D}$ is known explicitly.

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- If $u : \Omega_1 \to \mathbb{R}$ is harmonic and $T : \Omega_2 \to \Omega_1$ is holomorphic then $u \circ T$ is harmonic in Ω_2 because $u \circ T$ is the real part of the holomorphic function $(u + \imath v) \circ T$ and composition of holomorphic fuctions are holomorphic.

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- Given a conformal mapping $T : \Omega \to \mathbb{D}$ such that $T(\partial\Omega) = \partial\mathbb{D}$ the solution to Dirichlet problem on Ω is given by $u \circ T : \Omega \to \mathbb{R}$

$$u(Tz) = \frac{1 - |Tz|^2}{2\pi} \int_0^{2\pi} \frac{g \circ T^{-1}(e^{\imath\theta})}{|Tz - e^{\imath\theta}|^2}\, d\theta.$$

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- The conformal map $\frac{e^z-1}{e^z+1}$ maps the horizontal strip $-\pi/2 < \Im(z) < \pi/2$ to $\mathbb{D}$ because $z \mapsto e^z$ maps the strip to right half-plane.

Discontinuous Boundary Data

Exercise

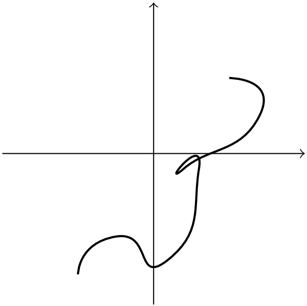
Solve (3.1) in the upper half-plane with discontinuous boundary data

$$g(x,0) = \begin{cases} 0 & x > 0 \\ 1 & x < 0. \end{cases}$$

Verify that $u(x,y) = \frac{\theta}{\pi} = \Re(\frac{1}{\imath\pi}\log(z))$ is a solution, after solving in $\mathbb{D}$ and using the conformal maps.

Curves in Complex Plane

- A *parametrized curve* is a continuous map $\gamma : I \subset \mathbb{R} \to \mathbb{C}$ where I is either an open or closed interval and, possibly, infinite.



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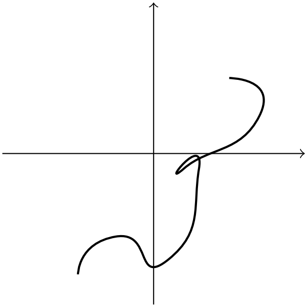

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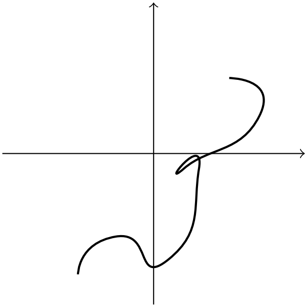

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- A *contour* is a union of finite number of smooth curves.

Simple Loop

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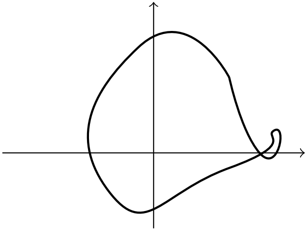

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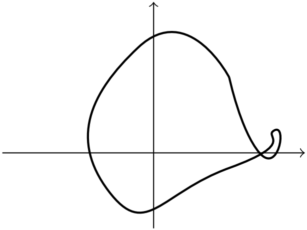
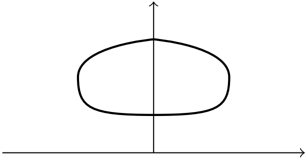


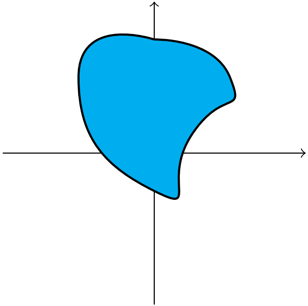
- A loop is *simple* if $\gamma(s) \neq \gamma(t)$ for all $a < s \neq t < b$.



Jordan Curve Theorem

Theorem

The complement of a simple closed curve in $\mathbb{C}$ is a disconnected set and has exactly two connected components, one bounded (interior) component and the other unbounded (exterior).



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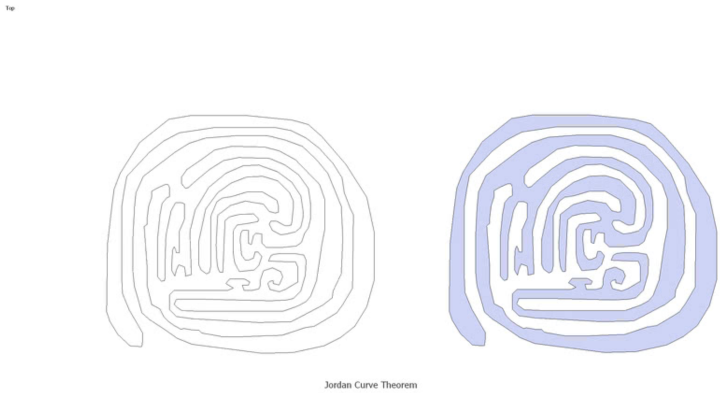

Figure: Image Courtesy: Google Images

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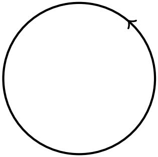
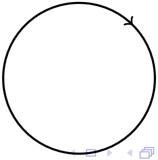
- For instance, for $t \in [0, 1]$, $\gamma(t) = (\cos 2\pi t, \sin 2\pi t)$ is positively oriented while $\gamma(t) = (\cos 2\pi t, -\sin 2\pi t)$ is oriented clockwise (negatively).





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The integral of a function $f : \mathbb{C} \to \mathbb{C}$ *along a path or contour* $\gamma : [a, b] \to \mathbb{C}$ *is defined as*

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- If z is a point on the curve γ then $z = \gamma(t)$ and $dz = \gamma'(t)\, dt$, by usual chain rule.

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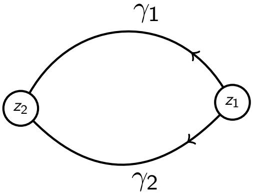
- The parametrisation of $-\gamma$ can be given by the map $\gamma_- : [0,1] \to \mathbb{C}$ defined as $\gamma_-(t) := \gamma[ta + (1-t)b]$.

Path Independence

- Is the contour integral path independent, i.e. for two different paths γ_1 and γ_2 joining z_1 and z_2, is $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$?



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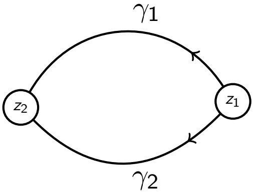

- Set $\gamma := \gamma_1 \cup (-\gamma_2)$ which is a loop at z_1. Then the question on path independence is same as asking: under what conditions on γ and f,

$$\int_\gamma f(z)\,dz = 0.$$

- For a continuous f on a domain Ω, f admits single-valued primitive in Ω iff $\int_\gamma f(z)\,dz = 0$ for every loop in Ω. (Exercise!)

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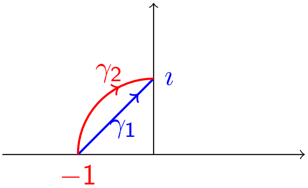
- If γ_1 is the straight line joining -1 and $\imath$, and γ_2 is the arc of unit circle joining -1 and $\imath$ then



  Then

$$\int_{\gamma_1 \cup -\gamma_2} |z|^2 \, dz = \int_{\gamma_1} |z|^2 \, dz - \int_{\gamma_2} |z|^2 \, dz = \frac{2}{3}(1+\imath) - 1 - \imath \neq 0.$$

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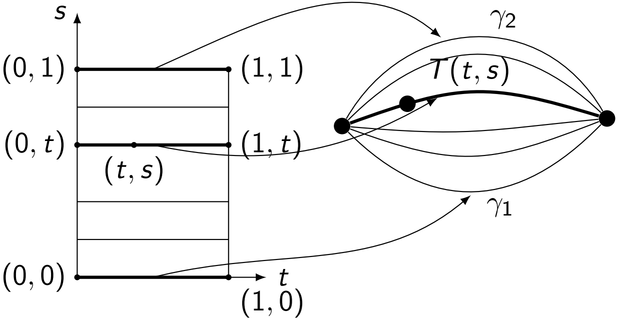

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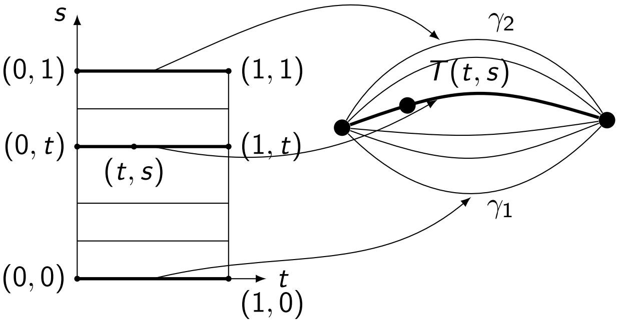

- A topological space X is *simply connected* if every loop or closed path in X is homotopic to a point in X.

Fundamental Theroem of Calculus: Complex Version

- If f admits a primitive F, i.e. $F' = f$ and γ is piecewise differentiable curve then, using the fundamental theorem of calculus, we get

$$\begin{aligned}\int_\gamma f(z)\,dz &= \int_\gamma F'(z)\,dz = \int_a^b F'(\gamma(t))\gamma'(t)\,dt \\ &= \int_a^b \frac{d}{dt}(F\circ\gamma)(t)\,dt = F(\gamma(b)) - F(\gamma(a)).\end{aligned}$$

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- Differentiate F to observe that it is the primitive of f. (For holomorphic functions, this is Morera's Theorem!)

Theorem (Cauchy's Theorem)

Let γ *be a counterclockwise simple loop in a simply connected open set* $\Omega \subset \mathbb{C}$. *If* $f : \Omega \to \mathbb{C}$ *is a holomorphic function then* $\int_\gamma f(z)\, dz = 0$. *Equivalently, every holomorphic function* f *on a simply connected domain has a primitive.*

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Green's Theorem

Theorem

Let γ be a counterclockwise simple loop in $\mathbb{C}$ and U is the bounded region enclosed by γ. If P and Q admit continuous partial derivatives in $U \cup \gamma$ then

$$\int_\gamma (P\,dx + Q\,dy) = \int_U (Q_x - P_y)\,dx\,dy.$$

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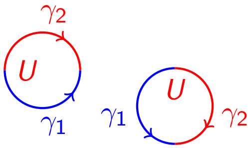

The region U can be interpreted in two ways as above: First one being $U := \cup_{x\in(a,b)} [\{x\} \times (\gamma_1(x), \gamma_2(x))]$.

Proof Continued...

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Generalised Cauchy's Theorem

Theorem (Invariance for Homotopic Curves)

Let γ_1 and γ_2 be two homotopic curves oriented counterclockwise in a domain $\Omega \subset \mathbb{C}$. If $f : \Omega \to \mathbb{C}$ is a holomorphic function then $\int_{\gamma_1} f(z)\,dz = \int_{\gamma_2} f(z)\,dz$.

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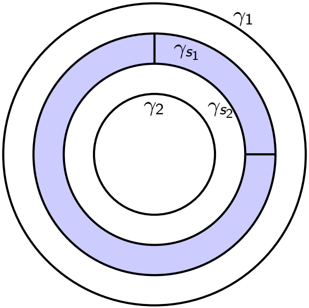
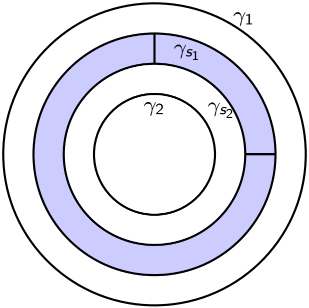

γ_1
γ_{s_1}
γ_2
γ_{s_2}

Proof Continued...



Then for each s_1, s_2 such that $|s_1 - s_2| < \delta$,

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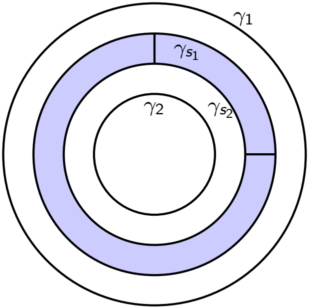

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Extend the argument for $s = 0$ to $s = 1$ in finitely many steps.

Weaker Hypothesis

Theorem

Let γ be a counterclockwise simple loop in a simply connected open set $\Omega \subset \mathbb{C}$. If $f : \Omega \to \mathbb{C}$ is a holomorphic function except at z_0 but continuous everywhere then $\int_\gamma f(z)\,dz = 0$.

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Since ε can be chosen as small as required, we have the result. Recall that $\int_\gamma dz = 0$ and $\int_\gamma |dz| = \text{Length of } \gamma$.

Theorem (Cauchy Integral Formula)

Let $f : \Omega \to \mathbb{C}$ be holomorphic on a simply connected open set $\Omega \subset \mathbb{C}$ and γ be a counter-clockwise simple loop in Ω. Then

$$\frac{1}{2\pi\imath}\int_\gamma \frac{f(w)}{w-z}\,dw = \begin{cases} f(z) & z \in U := Int(\gamma) \\ 0 & z \in \Omega \setminus \bar{U} \\ undefined & z \in \gamma. \end{cases}$$

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Infinite Differentiability

Theorem (Converse to CIF)

Let γ be a counter-clockwise simple loop. If $f : \gamma \to \mathbb{C}$ be any continuous function such that, for all z in the interior of γ,

$$f(z) = \frac{1}{2\pi\imath} \int_\gamma \frac{f(w)}{w - z}\, dw$$

then f is infinitely complex differentiable (and hence holomorphic) and given by the formula

$$f^{(k)}(z) = \frac{k!}{2\pi\imath} \int_\gamma \frac{f(w)}{(w - z)^{k+1}}\, dw.$$

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Proof: Note that

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Theorem

Let $\Omega \subset \mathbb{C}$ is open. A function $f : \Omega \to \mathbb{C}$ is holomorphic at z_0 iff $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ in a neighbourhood of z_0. (The convergence is uniform).

Taylor Series: Holomorphic is Analytic

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Proof: If f admits power series then $f^{(k)}(z_0) = k!a_k$ and, hence holomorphic at z_0. Conversely, if f is holomorphic then choose the neighbourhood $N(z_0)$ centred at z_0 with radius $\text{dist}(z_0, \gamma)$ where γ is any counter clockwise simple loop in Ω enclosing z_0.

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Let $\Omega \subset \mathbb{C}$ is open. A function $f : \Omega \to \mathbb{C}$ is holomorphic at z_0 iff $f(z) = \sum_{k=0}^{\infty} a_k(z - z_0)^k$ in a neighbourhood of z_0. (The convergence is uniform).

Proof: If f admits power series then $f^{(k)}(z_0) = k!a_k$ and, hence holomorphic at z_0. Conversely, if f is holomorphic then choose the neighbourhood $N(z_0)$ centred at z_0 with radius $\text{dist}(z_0, \gamma)$ where γ is any counter clockwise simple loop in Ω enclosing z_0. Then, for all $z \in N(z_0)$ and $w \in \gamma$, we have $|z - z_0| < |w - z_0|$.

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Non-Analytic Infinitely Differentiable Real Function

- Consider the function $f : \mathbb{R} \to \mathbb{R}$ defined as

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- The left side limit of f and its derivative is zero at $x = 0$. Further, the right side limit

$$f^{(k+1)}(0) = \lim_{h \to 0^+} \frac{f^{(k)}(h) - f^{(k)}(0)}{h} = 0.(\text{Exercise!})$$

Therefore, $f \in C^\infty(\mathbb{R})$.

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- Thus, f is not analytic at 0.

Zeroes of Holomorphic Functions

Definition

A $z_0 \in \mathbb{C}$ *is said to be a zero of order* m *if* $f^{(j)}(z_0) = 0$ *for all* $0 \leq j \leq m - 1$*. A zero is simple if* $m = 1$*.*

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- For a non-zero holomorphic function, at least one coefficient of Taylor series is non-zero, say the $f^{(m)}(z_0)$ is first non-zero coefficient, then z_0 is a zero of order m of f.
- If f is holomorphic in Ω with a zero of order m then, from the Taylor series of f in a neighbourhood of z_0, we get $f(z) = (z - z_0)^m g(z)$ where $g(z_0) \neq 0$ and

$$g(z) = \sum_{k=0}^{\infty} \frac{f^{(k+m)}(z_0)}{(k+m)!}(z - z_0)^k$$

where g has the same domain of convergence about z_0 as f.

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- The zeroes of $\sinh z$ are roots of $e^{2z} - 1 = 0$. Thus, the zeroes are $\imath k\pi$ for all $k \in \mathbb{Z}$ (Only imaginary zeroes).

Non-zero Holomorphic has Isolated Zeroes

Theorem

Let f be a non-zero holomorphic function in a domain $\Omega \subset \mathbb{C}$. If z_0 is a zero of f then there is a neighbourhood $N(z_0)$ of z_0 such that $f(z) \neq 0$ for all $z \in N(z_0)$.

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Zero set of Non-zero Holomorphic has no Accumulation

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Let f be holomorphic in a domain $\Omega \subset \mathbb{C}$. If $\{z_n\}$ is a sequence of zeroes of f such that its limit $z_0 \in \Omega$ then $f \equiv 0$ in Ω.

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Theorem

If f is holomorphic in open set $\Omega \subset \mathbb{C}$ except at $z_0 \in \Omega$ then $f(z) = \sum_{k=-\infty}^{\infty} a_k(z - z_0)^k$ in $\Omega \setminus \{|z - z_0| < r\}$ for any $r > 0$ where $a_k = \frac{1}{2\pi\imath} \int_\gamma \frac{f(w)}{(w-z_0)^{k+1}}\, dw$ for any simple loop $\gamma \subset \Omega \setminus \{|z - z_0| < r\}$.

Laurent Series on Annular Domains

Theorem

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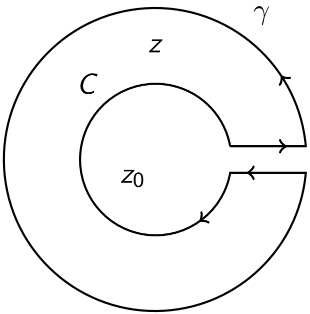

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Definition

let f be holomorphic in Ω except at $z_0 \in \Omega$. The residue of f at z_0 is

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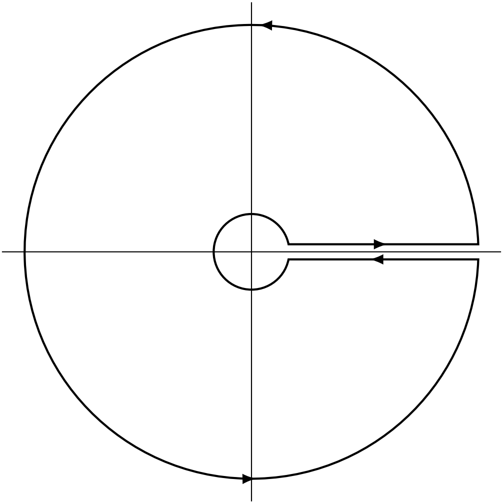
Theorem

Let γ be a simple loop oriented counter-clockwise and f is holomorphic in its interior except at finite number of poles $z_1, \ldots, z_k$. Then

$$\frac{1}{2\pi\imath} \int_\gamma f(z)\, dz = \sum_{j=1}^{k} Res_{z=z_k} f(z).$$

Proof Sketch of Residue Theorem



Simply Periodic Functions

Definition

A holomorphic function $f : \Omega \subset \mathbb{C} \to \mathbb{C}$ *is said to be periodic if there is a non-zero* $\omega \in \mathbb{C}$ *such that* $f(z+\omega) = f(z)$ *for all* $z \in \mathbb{C}$ *and* ω *is called the period of* f*.*

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- More generally, $e^{\imath kz}$, $\sin kz$ and $\cos kz$ are all 2π periodic functions.
- The 2π periodic holomorphic functions f is in one-to-one correspondence with holomorphic functions g on the annulus $\{e^{\pi} < |w| < e^{\pi}\}$. Given f, set $g(w) = f(\log w)$ and given g, set $f(z) = g(e^{\imath z})$.

Theorem

If f is a 2π periodic function in the strip $\{|\Im(z)| < \pi\}$ then f admits the Fourier series representation $f(z) = \sum_{k=-\infty}^{\infty} a_k e^{\imath kz}$ where $a_k = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-\imath k\theta}\,d\theta$.

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$$a_k = \frac{1}{2\pi\imath}\int_0^{2\pi} \frac{g(e^{\imath\theta})}{e^{\imath(k+1)\theta}}\imath e^{\imath\theta}\,d\theta = \frac{1}{2\pi}\int_0^{2\pi} f(\theta)e^{-\imath k\theta}\,d\theta.$$

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- The sinc function $\frac{\sin z}{z}$ has removable singularity at 0 since $\lim_{z\to 0} \frac{\sin z}{z} = 1$.

Removable Singularity

Theorem (Riemann Removable Singularity Theorem)

If f is holomorphic and bounded in $\Omega \setminus \{z_0\}$ then the extension

$$\tilde{f}(z) = \begin{cases} f(z) & z \neq z_0 \\ \lim_{w \to z_0} f(w) & z = z_0. \end{cases}$$

is holomorphic in Ω. Also, f has removable singularity iff $\lim_{z \to z_0} (z - z_0) f(z) = 0$.

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Definition

A pole z_0 is a point at which the function blows-up i.e. it is unbounded in a neighbourhood of z_0. A pole z_0 is of order k if $\lim_{z \to z_0}(z - z_0)^k f(z)$ is finite and non-zero. If no such k exists then z_0 is an essential singularity of f, i.e. pole of infinite order.

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Property of Essential Singularity

Theorem (Casorati-Weierstrass)

If f has an essential singularity at z_0 and is holomorphic in a punctured neighbourhood $U := B_r(z_0) \setminus \{z_0\}$ of z_0 then the image $f(U)$ is dense in $\mathbb{C}$.

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- The radius of convergence of a complex analytic function is the distance from the nearest singularity!

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A topological space is said to be separable if it contains a countable dense subset.

Distance from a Set

Definition

Let (X, d) be a metric space and let E be a subset of X. For any given $x \in X$, we define the distance of E from x, denoted as $d(x, E)$, as:

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Theorem

Let (X, d) be a metric space and $E \subset X$. Then

$$|d(x, E) - d(y, E)| \leq d(x, y) \quad \forall x, y \in X.$$

In particular, the function $x \mapsto d(x, E)$ is uniformly continuous on X.

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$$|f(y) - f(x)| \leq d(x, y) + \varepsilon.$$

Since choice of ε was arbitrary, we get

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Thus, f is Lipschitz and, hence, continuous.

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A subset $E \subset X$ of a topological space is said to be of the first category in X if it is the countable union of no-where dense sets. A subset which is not of the first category is said to be of the second category.

Theorem

Let $\{U_i\}_1^n$ be a finite collection of dense open subsets of a metric space X. Then $U = \cap_{i=1}^n U_i$ is dense in X.

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- By the density of U_1, $B_{\varepsilon_0}(x_0) \cap U_1 \neq \emptyset$ and hence there is a $x_1 \in B_{\varepsilon_0}(x_0) \cap U_1$.

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- By the density of U_1, $B_{\varepsilon_0}(x_0) \cap U_1 \neq \emptyset$ and hence there is a $x_1 \in B_{\varepsilon_0}(x_0) \cap U_1$.
- Further, since $B_{\varepsilon_0}(x_0) \cap U_1$ is open, there is a $\varepsilon_1 > 0$ such that $B_{\varepsilon_1}(x_1) \subset B_{\varepsilon_0}(x_0) \cap U_1$.

Proof Continued...

- Repeat the above argument for x_1, ε_1 and U_2 to obtain a x_2, $\varepsilon_2 > 0$ and $B_{\varepsilon_2}(x_2) \subset B_{\varepsilon_1}(x_1) \cap U_2$.

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- Repeat the above argument for x_1, ε_1 and U_2 to obtain a x_2, $\varepsilon_2 > 0$ and $B_{\varepsilon_2}(x_2) \subset B_{\varepsilon_1}(x_1) \cap U_2$.
- Proceeding this way, we construct $\{x_1, x_2, \ldots, x_n\} \subset X$ and positive numbers $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ such that $B_{\varepsilon_i}(x_i) \subset B_{\varepsilon_{i-1}}(x_{i-1}) \cap U_i$, for all $i = 1, 2, \ldots, n$.

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- Thus, by our construction, $x_n \in B_{\varepsilon_0}(x_0) \cap U$. Since x_0 and ε_0 were arbitrary, we have shown the density of U in X.

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Proof:

- Let $x_0 \in X$ and $\varepsilon > 0$. We have to show that $B_\varepsilon(x_0) \cap U \neq \emptyset$.

Baire Category Theorem

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- Let $x_0 \in X$ and $\varepsilon > 0$. We have to show that $B_\varepsilon(x_0) \cap U \neq \emptyset$.
- Since U_1 is dense, we choose a $x_1 \in X$ and $0 < \varepsilon_1 < 1$ such that $\overline{B}_{\varepsilon_1}(x_1) \subset U_1 \cap B_\varepsilon(x_0)$.

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- For a $n_0 \in \mathbb{N}$ such that $m, n \geq n_0$, we have $0 < \varepsilon_m < 1/m \leq 1/n_0$ and $0 < \varepsilon_n < 1/n \leq 1/n_0$. Therefore,

$$d(x_m, x_n) \leq d(x_m, x_{n_0}) + d(x_{n_0}, x_n) < 2\varepsilon_{n_0} \leq \frac{2}{n_0}.$$

- Hence, $\{x_n\}$ is Cauchy.

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- Observe that, for all $n \geq n_0$, $x_n \in B_{\varepsilon_{n_0}}(x_{n_0})$.

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- But $\overline{B}_{\varepsilon_i}(x_i) \subset U_i \cap B_\varepsilon(x_0)$ for all $i = 1, 2, \ldots$.
- Thus, $x \in U \cap B_\varepsilon(x_0)$.

The Baire category theorem is, in fact, stating that: any complete metric space is second category.

Consequences of Baire's Theorem

Corollary

Let X be a metric space which is countable union of closed sets $\{G_i\}$.

(a) *If $Int(G_i) = \emptyset$, for all n, then X is not complete.*

(b) *If X is complete then, at least, one of the closed sets of $\{G_i\}$ has non-empty interior.*

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Examples

Example

Note that $\mathbb{Q} = \cup_{i \in \mathbb{N}} \{r_i\}$ with usual metric $d(r, s) = |r - s|$. Thus $\mathbb{Q}$ is a countable union of nowhere dense closed subsets. Thus, $\mathbb{Q}$ cannot be complete.

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Example

The plane $\mathbb{R}^2$ cannot be written as countable union of lines. More generally, the space $\mathbb{R}^n$ cannot be written as countable union of hyperplanes.

Consequences of Baire's Theorem

Corollary

In a complete metric space, the intersection of any countable collection of dense G_δ sets is also a dense G_δ set.

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Proof.

The proof is trivial from the fact that G_δ set is a countable intersection of open sets. □

Consequences of Baire's Theorem

Corollary

Let X be a complete metric space with no isolated points. Any countable dense subset of X cannot be a G_δ set.

Proof.

Let $E = \{x_1, x_2, \dots,\}$ be a countable dense subset of X. Suppose E is G_δ set, then $E = \cap_{i=1}^{\infty} U_i$ for a sequence of open sets $\{U_i\}$. Since E is dense in X, U_i is dense in X, for all i. Then the set

$$V_i := U_i \setminus \{x_1, x_2, \dots, x_i\}$$

is also dense (because X has no isolated points) and open in X. But $\cap_i V_i = \emptyset$ is not dense in X which contradicts Baire's theorem. Therefore, E is not a G_δ set. $\square$

Uniform Boundedness Principle

Theorem

Let X be a complete metric space and $\mathcal{F} \subset C(X)$ be a sub-family of the space of continuous functions $f : X \to \mathbb{R}$. Then

1. *either*
$$\sup_{f \in \mathcal{F}} |f(x)| = \infty \tag{5.1}$$
for all x in some dense G_δ subset of X

2. *or there exists a $M > 0$, $r > 0$ and $x_0 \in X$ such that*
$$\sup_{x \in B_r(x_0)} \sup_{f \in \mathcal{F}} |f(x)| \le M. \tag{5.2}$$

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Proof: For each $n \geq 1$, set

$$F_n = \{x \in X \mid \sup_{f \in \mathcal{F}} |f(x)| \leq n\}.$$

Proof Continued

Note that $F_n = \cap_{f \in \mathcal{F}} f^{-1}([-n, n])$ and hence is closed because it is an arbitrary intersection of closed sets (since f is continuous).

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1. F is a first category subset of X. Since X is complete, by Baire category theorem, $F^c := X \setminus F$ is a dense G_δ subset of X. Further, for any $x \in F^c$, (5.1) is satisfied.

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2. F is second category subset of X. Since X is complete, by Baire category theorem, there is a $M > 0$ such that F_M has non-empty interior. Thus, there is a $x_0 \in F_M \subset X$ and $r > 0$ such that $B_r(x_0) \subset F_M$ and (5.2) is satisfied.

Limit

Definition

Let $f : X \to Y$ be any function and X, Y are topological spaces. A $L \in Y$ is called a limit of f at an accumulation point $x_0 \in X$, if for every neighbourhood V of L in Y there exists a neighbourhood U of x_0 in X such that $f(U) \subset V$.

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- In particular, if X and Y are metric spaces with metric d_1 and d_2, respectively, then for any given real number $\varepsilon > 0$ (however small) there exists a $\delta > 0$ such that $d_2(f(x), L) < \varepsilon$, for all x, with $d_1(x, x_0) < \delta$.

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- If Y is Hausdorff then the limit L is unique.

Continuous Functions

Definition

Let X and Y be topological spaces. A function $f : X \to Y$ is continuous at $x_0 \in X$ if for any open set $U \subset Y$ containing $f(x_0)$, its inverse image $f^{-1}(U) \subset X$ containing x_0 is also open.

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- If δ can be chosen independent of x_0 then the function is *uniformly continuous*.

Topology on Space of Continuous Functions

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- For any compact topological space K, the norm of a $f \in C(K)$ is given as $\|f\|_\infty := \sup_{x \in K} |f(x)|$ called the *uniform* or *supremum* norm. Thus, the associated uniform metric is $d(f,g) := \|f - g\|_\infty$ and induces the uniform convergence topology.

Definition

A sequence of functions $\{f_n\} : X \to \mathbb{R}$ is said to *converge pointwise* to a function $f : X \to \mathbb{R}$ if $\lim_{n\to\infty} f_n(x) = f(x)$ for each $x \in X$, i.e. for any given $\varepsilon > 0$ and $x \in X$ there is a positive integer $N \in \mathbb{N}$ (depending on x and ε) such that for all $n \geq N$, $|f_n(x) - f(x)| < \varepsilon$.

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Exercise

Show that for any $\alpha \in [0, 1)$, $\alpha^n \to 0$ as $n \to \infty$.

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Exercise

Show that for any $\alpha \in [0, 1)$, $\alpha^n \to 0$ as $n \to \infty$. Consequently, show that the sequence $\{x^n\}$ indexed by the degree n and defined on $[0, 1]$ pointwise converges to

$$f(x) = \begin{cases} 0 & 0 \leq x < 1 \\ 1 & x = 1. \end{cases}$$

The exercise in the previous slide shows that the pointwise limit of a sequence of continuous functions can be discontinuous.

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$$|f(x) - f(x_0)| \leq |f(x) - f_m(x)| + |f_m(x) - f_m(x_0)| + |f_m(x_0) - f(x_0)|$$

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The choice of $\delta > 0$ comes from the continuity of f_m at x_0. □

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Let $I \subset \mathbb{R}$ be a closed bounded interval of $\mathbb{R}$. If $\{f_n\}$ is a monotone sequence of continuous real valued functions on I which converge point-wise to a continuous function f, then the convergence is uniform on I.

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What is the topology for continuous functions on non-compact Topological Spaces?

Continuous Functions on Open Euclidean Subsets

- For any *open* subset Ω of $\mathbb{R}^n$, there is a sequence K_j of non-empty compact subsets of Ω such that $\Omega = \cup_{j=0}^{\infty} K_j$ and $K_j \subset \mathrm{Int}(K_{j+1})$, for all j. This property is called the *σ-compactness* of Ω.

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- Show that the topology given in $C(\Omega)$ is independent of the choice the exhaustion compact sets $\{K_j\}$ of Ω.

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Lemma

There is a sequence of polynomials $\{p_n\}$ *which converge uniformly to* $|x|$ *on* $[-1, 1]$.

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$$p_{n+1} - |x| = \frac{1}{2}(x^2 - 2|x| + 2p_n - p_n^2) = \frac{1}{2}\left[(1 - |x|)^2 - (1 - p_n)^2\right]$$

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$$|p_n(x) - |x - c|| = n|P_n[(x - c)/n] - |x - c|/n| < 1/n$$

for all $|x - c|/n \leq 1$ or, equivalently, $x \in [c - n, c + n]$. □

Separating Points

Definition

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Lemma

Let $A \subset C(X)$ satisfy the following properties:

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2. *every constant function is in A; and*
3. *A separates points.*

Then, for any $x, y \in X$ with $x \neq y$ and $a, b \in \mathbb{R}$, there exists a $f \in A$ such that $f(x) = a$ and $f(y) = b$.

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- Note that $sg + t \in A$ because A is a linear space and the function $1 \in A$.
- Note that if $a = b$ then $s = 0$ and $t = a$, and, hence $f \equiv a$.

$\square$

Dense Subsets of C(X)

Theorem

Let X be a compact topological space and $A \subset C(X)$ satisfies the properties as in Lemma 11 and also is a lattice, i.e., $f \vee g \in A$ and $f \wedge g \in A$ whenever $f, g \in A$. Then A is dense in $C(X)$ under the uniform topology.

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- Fix $x \in X$ and for each $y \in X$ with $y \neq x$, by the continuity of $g_{xy} - f$ at y, for the given $\varepsilon > 0$, there is an open set $U_{xy} \in X$ such that $|g_{xy}(z) - f(z)| < \varepsilon$ for all $z \in U_{xy}$.

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Proof Continued...

- For the fixed $x \in X$, the open sets U_{xy} form an open cover of X and, since X is compact, we have finite collection of $\{y_i\}_{i=1}^n \subset X$ such that $X = \cup_{i=1}^n U_{xy_i}$.

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- Since $g_{xy_i}(z) < f(z) + \varepsilon$, we have $g_x(z) < f(z) + \varepsilon$ for all $z \in X$. Moreover, $g_x(x) = f(x)$.

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$$f \vee g = \frac{f+g}{2} + \frac{|f-g|}{2} \text{ and } f \wedge g = \frac{f+g}{2} - \frac{|f-g|}{2}.$$

□

Theorem (Real Stone-Weierstrass)

Let X be a compact topological space and $A \subset C(X)$ satisfies the properties as in Lemma 11 and, in addition, satisfies the property that $fg \in A$ whenever $f, g \in A$. Then A is dense in $C(X)$ under the uniform topology.

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- Thus, $\overline{A} = C(X)$ and hence A is dense in $C(X)$.

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Complex Stone-Weierstrass

Theorem (Complex Stone-Weierstrass)

Let X be a compact topological space and $A \subset C(X, \mathbb{C})$, all complex valued continuous functions, satisfies the properties as in Theorem 26 and, in addition, satisfies the property that if $f \in A$ then $\bar{f} \in A$, the conjugate of f. Then A is dense in $C(X, \mathbb{C})$ under the uniform topology.

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Thus, $\|f - q\|_\infty \leq \varepsilon$. If the set of all polynomials with rational coefficients is countable then our proof is done. This is left as an exercise! □

Trigonometric Polynomials

- Let $P_\sharp^n([-\pi,\pi])$ denote the space of all 2π periodic trigonometric polynomials on $\mathbb{R}$ of degree n, i.e.,

$$\sum_{k=0}^{n} a_k \cos(k\theta) + \sum_{k=1}^{n} b_k \sin(k\theta) \quad \forall a_k, b_k \in \mathbb{R}, n \in \mathbb{N}.$$

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- Let $P_\sharp([-\pi,\pi])$ denote the space of all 2π periodic trigonometric polynomials on $\mathbb{R}$ of any degree, i.e.,

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The density is not valid for non-periodic $C[-\pi,\pi]$ in uniform norm. For instance, $f(x) = x$ cannot be approximated and $P_\sharp[-\pi,\pi]$ has no function that separates $-\pi$ and π.

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- We use the continuous bijection from $C_\sharp([-\pi,\pi],\mathbb{C})$ to $C(\mathbb{T},\mathbb{C})$ where $\mathbb{T} := \{z \in \mathbb{C} \mid |z|^2 = 1\}$ is a compact subset of $\mathbb{C}$ endowed with the usual Euclidean metric.

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- For each $f \in C_\sharp([-\pi,\pi],\mathbb{C})$, we define $f_\sharp : \mathbb{T} \to \mathbb{C}$ as $f_\sharp(e^{\imath\theta}) := f(\theta)$, for all $-\pi \leq \theta < \pi$.

Proof Continued...

- The continuity of f implies the continuity of $f_\sharp$, composition of continuous functions. (Exercise!)
- Thus, the subspace $P_\sharp(X, \mathbb{C})$ of $C(\mathbb{T}, \mathbb{C})$ satisfies hypotheses of complex Stone-Weierstrass theorem.

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- The continuity of f implies the continuity of $f_\sharp$, composition of continuous functions. (Exercise!)
- Thus, the subspace $P_\sharp(X, \mathbb{C})$ of $C(\mathbb{T}, \mathbb{C})$ satisfies hypotheses of complex Stone-Weierstrass theorem.
- The separation property is satisfied because for any $z, w \in \mathbb{T}$, the image $f_\sharp$ of the $f(\theta) = \exp(\imath\theta)$ satisifes $f_\sharp(z) \neq f_\sharp(w)$.

Fourier Series

Definition

The Fourier Series of a function $f \in L^1(-\pi, \pi)$ *is defined as*

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{int} \tag{7.1}$$

where the Fourier coefficient is given as

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(a) Will the series (7.1) always converge?

(b) If it converges, will it converge to f at some/all points $t \in (-\pi, \pi)$?

We shall show that there is a large class of integrable functions on $[-\pi, \pi]$ which fail to converge on a very large set of points in $[-\pi, \pi]$.

Dirichlet Kernel

To study the convergence of (7.1), we consider the sequence of partial sums

$$S_f^m(t) := \sum_{n=-m}^{m} \hat{f}(n)e^{int}$$

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$$S_f^m(t) := \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\left[\sum_{n=-m}^{m} e^{in(t-x)}\right] dx.$$

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This motivates the definition of *Dirichlet kernel*, $D_m : \mathbb{R} \to \mathbb{R}$, defined as

$$D_m(s) := \sum_{n=-m}^{m} e^{ins}$$

and the partial sum is the convolution $S_f^m(t) = (f * D_m)(t)$.

Proposition

Let $m \in \mathbb{N} \cup \{0\}$. *Then*

$$D_m(s) = \begin{cases} \frac{\sin\left(m+\frac{1}{2}\right)s}{\sin\frac{s}{2}} & \text{if } s \neq 2k\pi \text{ for } k \in \mathbb{N} \cup \{0\} \\ 2m+1 & \text{if } s = 2k\pi \text{ for } k \in \mathbb{N} \cup \{0\}. \end{cases}$$

Further

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_m(s)\, ds = 1.$$

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Further

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} D_m(s)\, ds = 1.$$

Proof: Since $e^{i2k\pi} = 1$ for every $k \in \mathbb{N} \cup \{0\}$, we have $D_m(2k\pi) = 2m+1$. If $s \neq 2k\pi$ for all $k \in \mathbb{N} \cup \{0\}$, then $e^{is} - 1 \neq 0$ and, hence,

$$(e^{is} - 1) D_m(s) = \sum_{n=-m}^{m} \left(e^{i(n+1)s} - e^{ins} \right) = e^{i(m+1)s} - e^{-ims}.$$

Proof Continued...

Multiplying both sides by $e^{-is/2}$, we get

$$(e^{is/2} - e^{-is/2})D_m(s) = e^{i\left(m+\frac{1}{2}\right)s} - e^{-i\left(m+\frac{1}{2}\right)s}.$$

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Thus, we have our desired result.Further,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} D_m(s)\,ds = \sum_{n=-m}^{m}\frac{1}{2\pi}\int_{-\pi}^{\pi} e^{ins}\,ds = 1$$

because for non-zero n,

$$\int_{-\pi}^{\pi} e^{ins}\,ds = \left[\frac{e^{ins}}{in}\right]_{-\pi}^{\pi} = \frac{2\sin(n\pi)}{n} = 0.$$

Exercise

Show that D_m is an even function and is 2π-periodic in $\mathbb{R}$. Also, show that D_m is continuous in $\mathbb{R}$.

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Proposition

$\lim_{m\to\infty} \int_{-\pi}^{\pi} |D_m(s)|\, ds = +\infty.$

Proof: For any $s \in \mathbb{R}$, we have $|\sin s| \leq |s|$.

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$$\int_{-\pi}^{\pi} |D_m(s)|\, ds = 2\int_0^{\pi} |D_m(s)|\, ds = 2\int_0^{\pi} \left| \frac{\sin\left(m + \frac{1}{2}\right) s}{\sin \frac{s}{2}} \right| ds$$

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&= \frac{4}{\pi}\sum_{n=1}^{m}\frac{1}{n}\int_0^{\pi} \sin t\,dt = \frac{8}{\pi}\sum_{n=1}^{m}\frac{1}{n}.
\end{aligned}$$

As $m \to \infty$, the series in RHS diverges, we get our desired result.

Theorem

Let $X = C[-\pi, \pi]$ be the space of continuous functions with the supremum norm and define the linear functionals $\{T_n\} : X \to \mathbb{R}$ as

$$T_n(f) := S_f^n(0),$$

where S_f^n is the n-th partial sum of the Fourier series associated to f. Then T_n continuous (bounded), for each n, and

$$\|T_n\| := \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(s)|\, ds. \tag{7.3}$$

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Therefore,

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To show equality, we shall construct a sequence of continuous functions which converges to the equality case.

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$$f_m(x) := \frac{1 - md(x, E_n)}{1 + md(x, E_n)}.$$

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Note that

$$f_m(x) = \begin{cases} 1 & x \in E_n \\ \frac{1/m - d(x, E_n)}{1/m + d(x, E_n)} & x \in E_n^c \end{cases}$$

Proof Continued ...

and $\{f_m\} \subset C[-\pi, \pi]$ because, for each n, $d(x, E_n)$ is a continuous function on $[-\pi, \pi]$ (cf. Exercise 19).

Proof Continued ...

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$$\lim_{m\to\infty} T_n(f_m) = \lim_{m\to\infty} \frac{1}{2\pi} \int_{-\pi}^{\pi} f_m(x) D_n(x)\, dx$$

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&= \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_n(x)|\, dx.
\end{aligned}$$

Thus, we have proved (7.3).

For the Banach space $X = C[-\pi, \pi]$, the sub-family $\mathcal{F} \subset X^\star$ defined as $T_n(f) = S_f^n(0)$ is such that $\sup_n \|T_n\| = \infty$ using Proposition 2 and Theorem 28.

For the Banach space $X = C[-\pi, \pi]$, the sub-family $\mathcal{F} \subset X^\star$ defined as $T_n(f) = S_f^n(0)$ is such that $\sup_n \|T_n\| = \infty$ using Proposition 2 and Theorem 28. Thus, by Uniform Boundedness Principle (cf. Theorem 22), there is a dense G_δ subset $G_0 \subset C[-\pi, \pi]$ such that $\sup_n \|T_n(f)\| = \infty$ for all $f \in G_0$, For the Banach space $X = C[-\pi, \pi]$, the sub-family $\mathcal{F} \subset X^\star$ defined as $T_n(f) = S_f^n(0)$ is such that $\sup_n \|T_n\| = \infty$ using Proposition 2 and Theorem 28. Thus, by Uniform Boundedness Principle (cf. Theorem 22), there is a dense G_δ subset $G_0 \subset C[-\pi, \pi]$ such that $\sup_n \|T_n(f)\| = \infty$ for all $f \in G_0$, i.e., the Fourier series of all $f \in G_0$ diverges at $x = 0$.

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Divergence of Fourier Series

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Convolution

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Definition

Let $f, g \in L^1(\mathbb{R}^n)$. The convolution $f * g$ is defined as,

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The integral on RHS is well-defined, since by Fubini's Theorem and the translation invariance of the Lebesgue measure, we have

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} |f(x-y)g(y)|\,dx\,dy = \int_{\mathbb{R}^n} |g(y)|\,dy \int_{\mathbb{R}^n} |f(x-y)|\,dx = \|g\|_1 \|f\|_1.$$

Thus, for a fixed x, $f(x-y)g(y) \in L^1(\mathbb{R}^n)$.

Properties of Convolution

Exercise

The convolution operation on $L^1(\mathbb{R}^n)$ is both commutative and associative.

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Exercise (Young's inequality)

Let $1 \leq p, q, r < \infty$ such that $(1/p) + (1/q) = 1 + (1/r)$. If $f \in L^p(\mathbb{R}^n)$ and $g \in L^q(\mathbb{R}^n)$, then the convolution $f * g \in L^r(\mathbb{R}^n)$ and

$$\|f * g\|_r \leq \|f\|_p \|g\|_q.$$

In particular, for $1 \leq p < \infty$, if $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, then the convolution $f * g \in L^p(\mathbb{R}^n)$ and

$$\|f * g\|_p \leq \|f\|_1 \|g\|_p.$$

Properties of Convolution

Exercise

Let $f \in L^1(\mathbb{R}^n)$ and $g \in L^p(\mathbb{R}^n)$, for $1 \leq p \leq \infty$. Then

$$\operatorname{supp}(f * g) \subset \overline{\operatorname{supp}(f) + \operatorname{supp}(g)}$$

If both f and g have compact support, then support of $f * g$ is also compact.

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If both f and g have compact support, then support of $f * g$ is also compact.

The convolution operation preserves smoothness.

Exercise

Let $f \in C_c^k(\mathbb{R}^n)$ $(k \geq 1)$ and let $g \in L^1_{\text{loc}}(\mathbb{R}^n)$. Then $f * g \in C^k(\mathbb{R}^n)$ and for all $|\alpha| \leq k$

$$D^\alpha(f * g) = D^\alpha f * g = f * D^\alpha g.$$

Mollifiers

For $\varepsilon > 0$,

$$\rho_\varepsilon(x) = \begin{cases} c\varepsilon^{-n}\exp\left(\frac{-\varepsilon^2}{\varepsilon^2-|x|^2}\right) & \text{if } |x| < \varepsilon \\ 0 & \text{if } |x| \geq \varepsilon \end{cases} \tag{7.4}$$

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Note that $\rho_\varepsilon \geq 0$ and is in $C_c^\infty(\mathbb{R}^n)$ with support in $B(0;\varepsilon)$. The sequence $\{\rho_\varepsilon\}$ is an example of mollifiers, a particular case of the *Dirac sequence*. The notion of mollifiers is also an example for the *approximation of identity* concept in functional analysis and ring theory.

Dirac Sequence and Approximate Identity

Definition

A sequence of functions $\{\rho_k\}$, *say on* $\mathbb{R}^n$, *is said to be a Dirac Sequence if*

1. $\rho_k \geq 0$ *for all* k.
2. $\int_{\mathbb{R}^n} \rho_k(x)\,dx = 1$ *for all* k.
3. *For every given* $r > 0$ *and* $\varepsilon > 0$, *there exists a* $N_0 \in \mathbb{N}$ *such that*

$$\int_{\mathbb{R}^n \setminus B(0;r)} \rho_k(x)\,dx < \varepsilon, \quad \forall k > N_0.$$

Definition

An approximate identity is a sequence (or net) $\{\rho_k$ *in a Banach algebra or ring (possible with no identity),* $(X, \star)$ *such that for any element* a *in the algebra or ring, the limit of* $a \star \rho_k$ *(or* $\rho_k \star a$*) is* a.

Regularization

Theorem

Let $\Omega \subset \mathbb{R}^n$ be an open subset of $\mathbb{R}^n$ and let

$$\Omega_\varepsilon := \{x \in \Omega \mid dist(x, \partial\Omega) > \varepsilon\}.$$

If $f \in L^1_{loc}(\Omega)$ then $f_\varepsilon := \rho_\varepsilon * f$ is in $C^\infty(\Omega_\varepsilon)$.

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Similarly, one can show that, for any tuple α, $D^\alpha f_\varepsilon(x) = (D^\alpha \rho_\varepsilon * f)(x)$. Thus, $u_\varepsilon \in C^\infty(\Omega_\varepsilon)$.

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Proof Continued...

Now, for all $x \in \mathbb{R}^n$,

$$|g_m(x) - g(x)| = \left| \int_{|y| \leq 1/m} g(x-y)\rho_m(y)\, dy - g(x) \int_{|y| \leq 1/m} \rho_m(y)\, dy \right|$$

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Hence, for all $x \in K$ and $m > 1/\delta$, we have

$$\begin{aligned} |g_m(x) - g(x)| &\leq \int_{|y| < \delta} |g(x-y) - g(x)| \rho_m(y)\,dy \\ &\leq \eta \int_{|y| < \delta} \rho_m(y)\,dy = \eta \end{aligned}$$

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Since the δ is independent of $x \in K$, we have $\|g_m - g\|_\infty < \eta$ for all $m > 1/\delta$. Hence, $g_m \to g$ uniformly on K.

Density of Smooth Bump Functions

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Corollary

For any $\Omega \subseteq \mathbb{R}^n$, $C_c^\infty(\Omega)$ is dense in $C(\Omega)$ under the uniform convergence on compact sets topology.

Density of Simple Functions

A simple function ϕ is a non-zero function on $\mathbb{R}^n$ having the (canonical) form

$$\phi(x) = \sum_{i=1}^{k} a_i 1_{E_i}$$

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Proof Continued...

Thus,

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$$\|\chi_F - g\|_p^p = \int_\Omega |\chi_F - g|^p = \int_{\Omega \setminus K} |\chi_F - g|^p \leq \mu(\Omega \setminus K) = \varepsilon.$$

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Therefore, $\|g - f\|_p < \varepsilon$. Thus, $C_c(\Omega)$ is dense in $L^p(\Omega)$.

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The first term has been handled using Young's inequality.

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Theorem (Cut-Off Technique)

For $1 \leq p < \infty$ and $\Omega \subseteq \mathbb{R}^n$, $C_c^\infty(\Omega)$ is dense in $L^p(\Omega)$.

Proof: Any $f \in L^p(\Omega)$ can be viewed as an element in $L^p(\mathbb{R}^n)$ under the extension

$$\tilde{f}(x) = \begin{cases} f(x) & x \in \Omega \\ 0 & x \in \Omega^c. \end{cases}$$

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Proof Continued...

Define $F_m := \phi_m f_m$ and, hence, $F_m \in C_c^\infty(\Omega)$.

Proof Continued...

Define $F_m := \phi_m f_m$ and, hence, $F_m \in C_c^\infty(\Omega)$. Also, $F_m = f_m$ on K_m and $|F_m| \leq |f_m|$ in $\mathbb{R}^n$.
Thus,

$$\|F_m - f\|_{p,\Omega} = \|F_m - \tilde{f}\|_{p,\mathbb{R}^n} \leq \|\phi_m f_m - \phi_m \tilde{f}\|_{p,\mathbb{R}^n} + \|\phi_m \tilde{f} - \tilde{f}\|_{p,\mathbb{R}^n}$$

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Remark

The case $p = \infty$ is ignored in the above results, because the L^∞-limit of $\rho_m * f$ is continuous and we do have discontinuous functions in $L^\infty(\Omega)$.

Total Boundedness

Definition

Let (X, d) be a metric space. A set $E \subset X$ is said to be totally bounded if, for every given $\varepsilon > 0$, there exists a finite collection of points $\{x_1, x_2, \cdots, x_n\} \subset X$ such that $E \subset \cup_{i=1}^{n} B_\varepsilon(x_i)$.

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Exercise

If $E \subset X$ is totally bounded then $E^n \subset X^n$ is also totally bounded.

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Definition

A subset $A \subset C(X)$ is said to be bounded if there exists a $M \in \mathbb{N}$ such that $\|f\|_\infty \leq M$ for all $f \in A$.

Definition

A subset $A \subset C(X)$ is said to be equicontinuous at $x_0 \in X$ if, for every given $\varepsilon > 0$, there is an open set U of x_0 such that

$$|f(x) - f(x_0)| < \varepsilon \quad \forall x \in U; f \in A.$$

A is said to be equicontinuous if it is equicontinuous at every point of X.

Total Boundedness implies Equicontinuity

Theorem

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Proof: Let A be totally bounded. Then, for given $\varepsilon > 0$, there is a collection of $\{f_1, f_2, \cdots, f_m\} \subset C(X)$ such that $A \subset \cup_{j=1}^{m} B_{\varepsilon/3}(f_j)$.

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$$|f(x) - f(y)| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)| < \varepsilon.$$

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Corollary (one implication of Ascoli-Arzela Theorem)

Let X be a compact topological space. If a subset $A \subset C(X)$ is compact then A is closed and equicontinuous.

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Proof.

Since $C(X)$ is a metric space and A is compact we have that A is closed and totally bounded. By above theorem, A is equicontinuous. □

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The converse of the Theorem proved above is true with some restriction on the range.

Equicontinuity implies Total Boundedness

Theorem

Let X be a compact topological space and (Y, d) be a totally bounded metric space. If a subset $A \subset C(X, Y)$ is equicontinuous then A is totally bounded.

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$$|f(y) - f(x)| < \frac{\varepsilon}{3} \quad \forall y \in U_x; f \in A.$$

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Since X is compact, there is a finite set of points $\{x_i\}_1^n \subset X$ such that $X = \cup_{i=1}^n U_{x_i}$. Define the subset E_A of Y^n as,

$$E_A := \{(f(x_1), f(x_2), \cdots, f(x_n)) \mid f \in A\}$$

which is endowed with the product metric, i.e.,

$$d(y, z) = \max_{1 \leq i \leq n} \{|y_i - z_i|\}$$

where $y, z \in Y^n$ are n-tuples.

Proof Continued...

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The first and third term is smaller that $\varepsilon/3$ by the continuity of f and f_j, respectively, and the second term is smaller than $\varepsilon/3$ by choice of f_j. Hence A is totally bounded, i.e., $A \subset \cup_{j=1}^m B_\varepsilon(f_j)$, equivalently, for any $f \in A$ there is a j such that $\|f - f_j\|_\infty < \varepsilon$.

Necessary Conditions for Bounded Subsets of $C(X)$

Lemma

Let X be compact topological space. If $A \subset C(X)$ is bounded then there is a compact subset $K \subset \mathbb{R}$ such that $f(x) \in K$ for all $f \in A$ and $x \in X$.

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Let X be compact topological space. If $A \subset C(X)$ is bounded then there is a compact subset $K \subset \mathbb{R}$ such that $f(x) \in K$ for all $f \in A$ and $x \in X$.

Proof.

Choose an element $g \in A$. Since A is bounded in the uniform topology, there is a M such that $\|f - g\|_\infty < M$ for all $f \in A$.

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Ascoli-Arzela Theorem

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Kolmogorov Compactness Criteria

Theorem (Kolmogorov Compactness Criteria)

Let $p \in [1,\infty)$ and let A be a subset of $L^p(\mathbb{R}^n)$. Then A is relatively compact in $L^p(\mathbb{R}^n)$ iff the following conditions are satisfied:

1. *A is bounded in $L^p(\mathbb{R}^n)$;*
2. $\lim_{r\to+\infty} \int_{\{|x|>r\}} |f(x)|^p \, dx = 0$ *uniformly with respect to $f \in A$;*
3. $\lim_{h\to 0} \|\tau_h f - f\|_p = 0$ *uniformly with respect to $f \in A$, where $\tau_h f$ is the translated function $(\tau_h f)(x) := f(x-h)$.*

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Proof: We shall prove the sufficiency part, i.e, (i), (ii), (iii) implies that A is relatively compact in $L^p(\mathbb{R}^n)$. Equivalently, we have to prove that A is precompact, which means that for any $\varepsilon > 0$, there exists a finite number of balls $B_\varepsilon(f_1), \dots, B_\varepsilon(f_k)$ which cover A.

Proof Continued...

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Let $(\rho_n)_{n\in\mathbb{N}}$ be a mollifier. It follows from Theorem 34 that, for all $n \geq 1$ and $f \in L^p(\mathbb{R}^n)$

$$\|f - f * \rho_n\|_p^p \leq \int_{\mathbb{R}^n} \rho_n(y) \|f - \tau_y f\|_p^p \, dy.$$

Hence

$$\|f - f * \rho_n\|_p \leq \sup_{|y|\leq \frac{1}{n}} \|f - \tau_y f\|_p.$$

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By (iii), there exists an integer $N(\varepsilon) \in \mathbb{N}$ such that, for all $f \in A$,

$$\|f - f * \rho_{N(\varepsilon)}\|_p < \varepsilon.$$

Proof Continued...

On the other hand, for any $x, z \in \mathbb{R}^n$, $f \in L^p(\mathbb{R}^n)$ and $n \in \mathbb{N}$,

$$\begin{aligned}
|(f * \rho_n)(x) - (f * \rho_n)(z)| &\leq \int_{\mathbb{R}^n} |f(x-y) - f(z-y)| \rho_n(y)\, dy \\
&\leq \|\tau_x \check{f} - \tau_z \check{f}\|_p \|\rho_n\|_q \\
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The last inequality follows from the invariance property of the Lebesgue measure.

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Let us consider the family $\mathcal{A} = \{f * \rho_{N(\varepsilon)} : B_r(0) \to \mathbb{R} \mid f \in A\}$.

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Let us consider the family $\mathcal{A} = \{f * \rho_{N(\varepsilon)} : B_r(0) \to \mathbb{R} \mid f \in A\}$. By using (i), (iii) and Ascoli-Arzela result, we observe that $\mathcal{A}$ is relatively compact w.r.t the uniform topology on $C(B_r(0))$. Hence, there exists a finite set $\{f_1, \dots, f_k\} \subset A$ such that

$$\mathcal{A} \subset \cup_{i=1}^{k} B_{\varepsilon r^{-n/p}}(f_i * \rho_{N(\varepsilon)}).$$

Thus, for all $f \in A$, there exists some $j \in \{1, 2, \ldots, k\}$ such that, for all $x \in B_r(0)$

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Thus, for all $f \in A$, there exists some $j \in \{1, 2, \dots, k\}$ such that, for all $x \in B_r(0)$

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Hence,

$$\begin{aligned} \|f - f_j\|_p &\leq \left(\int_{|x|>r} |f|^p \, dx \right)^{1/p} + \left(\int_{|x|>r} |f_j|^p \, dx \right)^{1/p} \\ &+ \|f - f * \rho_{N(\varepsilon)}\|_p + \|f_j - f_j * \rho_{N(\varepsilon)}\|_p \\ &+ \|f * \rho_{N(\varepsilon)} - f_j * \rho_{N(\varepsilon)}\|_{p, B_r(0)}. \end{aligned}$$

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The last term may be treated as follows:

$$\begin{aligned} \|f * \rho_{N(\varepsilon)} - f_j * \rho_{N(\varepsilon)}\|_{p,B_r(0)} &= \left(\int_{B_r(0)} |f * \rho_{N(\varepsilon)}(x) - f_j * \rho_{N(\varepsilon)}(x)|^p \, dx \right) \\ &\leq \varepsilon |B_r(0)|^{-1/p} |B_r(0)|^{1/p} = \varepsilon. \end{aligned}$$

Proof Continued...

Finally,

$$\|f - f_j\|_p \leq 5\varepsilon$$

and, hence, A is precompact in $L^p(\mathbb{R}^n)$.

Continuous Bijection on Intervals

- The function $f : [0,1] \to (0,1)$, defined as

$$f(x) = \begin{cases} \frac{1}{2} & \text{for } x = 0 \\ \frac{1}{n+2} & \text{for } x = \frac{1}{n} \\ x & \text{otherwise} \end{cases}$$

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because its image, by definition, is $(0.134567890123\ldots, 0.2000\ldots)$ which is an image of the element

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With this modification, the function f is a bijection.

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Lemma

Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ be a continuous map.

1. *If $K \subset X$ is a compact subset then $f(K)$ is a compact subset of Y.*
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Theorem

Let (X, d_X) and (Y, d_Y) be metric spaces and $f : X \to Y$ be an injective map. If X is compact and f is continuous, then $f^{-1} : f(X) \subseteq Y \to X$ is continuous.

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Example

Consider $f : [0, 1) \to \mathbb{C}$ defined as $f(x) = e^{i2\pi x}$ which is bijective on to the unit circle $|z| = 1$ of $\mathbb{C}$.

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Consider $f : [0, 1) \to \mathbb{C}$ defined as $f(x) = e^{i2\pi x}$ which is bijective on to the unit circle $|z| = 1$ of $\mathbb{C}$.However, f^{-1} is not continuous at the point $f(0) = 1 \in \mathbb{C}$ because the sequence $f\left(1 - \frac{1}{n}\right)$ converges to $f(0)$ while $1 - \frac{1}{n}$ do not converge in $[0, 1)$.

No Continuous Bijection onto Square

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Let $\{f_n\}$ be a sequence of functions and, for all n, there exists a $M_n \in \mathbb{R}$ such that $|f_n(x)| \leq M_n$ for all x. If $\sum_n M_n$ converges then $\sum_n f_n(x)$ converges uniformly on the domain of consideration.

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Theorem

Let $f(x) := \sum_n f_n(x)$, a uniform limit of the series in its domain. If f_n is continuous at x_0, for all n, then f is also continuous at x_0.

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- Define the function $f : [0,2] \to [0,1]$ as

$$f(t) := \begin{cases} 0 & \text{if } 0 \leq t \leq \frac{1}{3} \text{ and } \frac{5}{3} \leq t \leq 2 \\ 3t - 1 & \text{if } \frac{1}{3} \leq t \leq \frac{2}{3} \\ 1 & \text{if } \frac{2}{3} \leq t \leq \frac{4}{3} \\ -3t + 5 & \text{if } \frac{4}{3} \leq t \leq \frac{5}{3} \end{cases}$$

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- By Weierstrass M-test (cf. Theorem 40), and choosing $M_n = 2^n$, we see that both the series converge uniformly (also absolutely) for all $t \in \mathbb{R}$.

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- Moreover, $0 \leq c \leq 1$ since $2\sum_n 3^{-n} = 1$.

- Consider, for each fixed $k \in \mathbb{N} \cup \{0\}$,

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- We shall now analyse v_k based on c_{k+1}. Recall that c_{k+1} is either 0 or 1.

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- If $c_{k+1} = 0$ then

$$0 = 2\sum_{m=2}^{\infty} \frac{0}{3^m} \leq v_k \leq 2\sum_{m=2}^{\infty} 3^{-m} = \frac{1}{3}$$

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- The lack of differentiability signifies a sharp corner at the point.
- Is there a function which is continuous everywhere but nowhere differentiable, i.e. sharp corners everywhere?

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- A nice application of Baire's category theorem gives a non-constructive existential proof for nowhere differentiable continuous functions.

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and set $Y := \cup_{n=1}^{\infty} F_n$. It is understood that we consider all those non-zero h such that $x + h \in [0,1]$, the domain of f.

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$$\left| \frac{f(x+h) - f(x)}{h} - f'(x) \right| \leq 1.$$

Proof Continued...

Therefore, for all $|h| \leq \delta$,

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- Since $\{x_k\} \subset [0,1]$, by Bolzano-Weierstrass result, there is a subsequence $\{x_j\}$ which converges to, say, x_0.

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$$\sup_{h \neq 0} \left| \frac{f_k(x_k + h) - f_k(x_k)}{h} \right| \leq n.$$

- Since $\{x_k\} \subset [0,1]$, by Bolzano-Weierstrass result, there is a subsequence $\{x_j\}$ which converges to, say, x_0.
- Thus, for any $h \neq 0$, there exists a $n_0 \in \mathbb{N}$ (depending on h) such that $x_0 - |h| < x_j < x_0 + |h|$, for all $j \geq n_0$.

Proof Continued...

- We shall now show that each F_n is closed in $C[0,1]$.
- Consider a sequence $\{f_k\} \subset F_n$ that converges to $f \in C[0,1]$ under supremum metric.
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- Let h_j be such that $x_j + h_j = x_0 + h$. Hence h_j is non-zero for all $j \geq n_0$. Note that, by definition, $h_j \to h$.

Proof Continued...

Consider

$$|f(x_0+h)-f(x_0)| \leq |f(x_0+h)-f_j(x_j+h_j)|+|f_j(x_j)-f(x_0)|+|f_j(x_j+h_j)-f_j(x_j)|.$$

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The last inequality is due to the fact that $f_j \in F_n$ for all $j \geq n_0$. Hence, $f \in F_n$ and F_n is closed.

Proof Continued...

- We now show that each F_n has an empty interior, i.e, given any $f \in F_n$ and $\varepsilon > 0$ there exists a function $g \in C[0,1] \setminus F_n$ such that $\|g - f\|_\infty \leq \varepsilon$.

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- We construct a piecewise affine function g, starting from $(0, p(0))$, such that $\|g - p\|_\infty \leq \frac{\varepsilon}{2}$ and $|g'(x)| > n$ for all those $x \in [0,1]$ for which g' exists.

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- Thus, $\text{Int}(Y) = \emptyset$.

Proof Continued...

- Since $C[0,1]$ is complete, by Baire's category theorem, $C[0,1] \setminus Y \neq \emptyset$.

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- Since $C[0,1]$ is complete, by Baire's category theorem, $C[0,1] \setminus Y \neq \emptyset$.
- This non-empty collection is, precisely, the collection of all nowhere differentiable continuous functions on $[0,1]$.
- In fact, we have proved that for any $f \in Y$ and $\varepsilon > 0$, there is a $g \in C[0,1]$ which is nowhere differentiable such that $\|f - g\|_\infty \leq \varepsilon$ or, more particularly, any continuous function which is differentiable, at least, at one point is a uniform limit of a sequence of nowhere differentiable continuous functions.

Definition

Let V denote a vector space over a field $\mathbb{F}$. If U is a subset of V, we define the *span* of U, denoted as $[U]$, to be the set of all finite linear combinations of elements of U. Equivalently,

$$[U] := \left\{ \sum_{i=1}^{n} \lambda_i x_i \mid x_i \in U, \lambda_i \in \mathbb{F}, \text{ and } \forall n \in \mathbb{N} \right\}.$$

Span and Linear Independence

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Definition

We say a subset U of V is linearly independent if for any finite set of elements $\{x_i\}_1^n \subset U$, $\sum_{i=1}^{n} \lambda_i x_i = 0$ implies that $\lambda_i = 0$ for all $1 \leq i \leq n$. A subset which is not linearly independent is said to be linearly dependent.

Hamel Basis

Definition

A subset $U \subset V$ is said to be a Hamel basis of V if $[U] = V$ and U is linearly independent.

Every element of V can be written as a finite linear combination of elements from Hamel basis and the elements of Hamel basis are linearly independent.

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Exercise

Let $\mathbb{R}[x]$ denote the set of all polynomials (finite degree) with real coefficients in one variable. Show that $\mathbb{R}[x]$ is a vector space over $\mathbb{R}$. Further, show that the subset

$$U := \{1, x, x^2, \dots\}$$

is a Hamel basis of $\mathbb{R}[x]$.

Exercise

Let $\mathbb{R}[x_1, x_2, \ldots, x_n]$ denote the set of all polynomials (finite degree) with real coefficients in n-variable. Show that $\mathbb{R}[x_1, x_2, \ldots, x_n]$ is a vector space over $\mathbb{R}$. Further, show that the subset

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A natural question to ask is: Does every vector space V have a basis?

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A natural question to ask is: Does every vector space V have a basis? Obviously, if $V = \{0\}$ then V has no basis because the only subsets of V are $\emptyset$ and $\{0\}$. Both do not form basis because $\{0\}$ is not linearly independent and $[\emptyset] \neq V$.

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- Otherwise, we have a chain $\mathcal{C}$ of linearly independent subsets of V under the binary relation $\subseteq$.
- Thus, $\mathcal{C}$ is a chain in the partially ordered set $\mathcal{A}$ consisting of all linearly independent subsets of V.

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- Thus $[U] = V$.

Remark

The linear combination of a vector $x \in V$, in terms of Hamel basis, is unique. For instance, if $x = \sum_{i \in J_1} \alpha_i e_i$ and $x = \sum_{i \in J_2} \beta_i e_i$ then

$$0 = \sum_{i \in J_1 \cap J_2} (\alpha_i - \beta_i) e_i + \sum_{i \in J_1 \setminus J_2} \alpha_i e_i + \sum_{i \in J_2 \setminus J_1} \beta_i e_i.$$

By the linear independence of $\{e_i\}$, we get $\alpha_i = \beta_i$ for all $i \in J_1 \cap J_2$, $\alpha_i = 0$ in $J_1 \setminus J_2$ and $\beta_i = 0$ in $J_2 \setminus J_1$.

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Exercise

If V_0 is a subspace of V and U_0 is a basis for V_0, then there exists a basis U of V such that $U_0 \subset U$.

Exercise (Refer N. Jacobson, Basic Algebra for proof)

There is a bijective map between any two bases of a vector space.

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The above theorem motivates following definition.

Definition

We say V is finite dimensional if its basis set contains finite number of elements and the dimension of V is the cardinality of U. If V is not a finite dimensional, then V is said to be infinite dimensional.

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The vector space $\mathbb{R}$ over $\mathbb{Q}$ is infinite dimensional!

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- Consider the set $\{\ln p\}$ where p runs over all primes numbers. The set is infinite because there are infinitely many primes.
- For some finite index set I, if $\sum_{i \in I} \alpha_i \ln p_i = 0$ then

$$0 = \sum_{i \in I} \alpha_i \ln p_i = \ln \left(\prod_{i \in I} p_i^{\alpha_i} \right),$$

i.e., $\prod_{i \in I} p_i^{\alpha_i} = 1$.

Proof Continued...

- Note that some α_i could be negative. If $J \subset I$ is the collection such that $\alpha_i < 0$ then
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This is a contradiction by the unique prime factorization theorem. Thus, all $\alpha_i = 0$ for all $i \in I$.

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- **Aliter:** We know there are transcendental real numbers, viz., e, π etc.
- Take a transcendental real number τ and consider the infinite set
$$\{\tau, \tau^2, \ldots, \tau^k, \ldots\}.$$
- This set is linearly independent over $\mathbb{Q}$. If not we have finite collection of non-zero $\{\alpha_i\} \subset \mathbb{Q}$ such that $\sum_i \alpha_i \tau^i = 0$ implying that τ is solution to a polynomial with rational coefficients contradicting the fact that it is transcendental.

- Recall that every vector space has a Hamel basis (cf. Theorem 43).

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- Thus, any normed space also has a Hamel basis. If the vector space is finite dimensional there are finite number of basis elements.
- We shall now show that an infinite dimensional *Banach* space cannot have a countable/denumerable Hamel basis.

Theorem

An infinite dimensional Banach space always has a uncountable Hamel basis.

Non-existence of Countably Infinite Hamel Basis

Proof.

- Suppose that a Banach space X has a countably infinite Hamel basis, say, $\{x_1, x_2, \ldots\}$.

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- Therefore, since X is complete, $\cap_{m=1}^{\infty} Z_m$ is dense in X, by Baire's category theorem.
- Therefore, $\cup_{m=1}^{\infty} Y_m$ has empty interior which contradicts our assumption that $[x_1, x_2, \ldots] = X$.

$\square$

Non-Completeness of Space of Polynomials

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- Thus, a Banach space is either finite-dimensional or has an uncountable Hamel basis.
- In fact, one can show that a infinite dimensional separable Banach space has a Hamel basis which is in one-to-one correspondence with the set of real numbers.
- The concept of Hamel basis has to be relaxed in an infinite dimensional Banach space called the *Schauder basis*.

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- We have the system of k first order ODEs $\mathbf{u}' = \mathbf{f}(x, \mathbf{u})$ where $f_i(x, \mathbf{u}) = u_{i+1}$ for $1 \le i \le k-1$ and $f_k(x, \mathbf{u}) = f(x, u_1, u_2, \ldots, u_{(k-1)})$.

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- Thus, the existence and uniqueness queries for the above k-th order ODE can be reduced to similar queries for a first order system of ODE.

Interpretation of Solution as a Fixed Point

- If u is a solution of

$$\begin{cases} u'(x) &= f(x,u) \quad x \in (a,b) \\ u(x_0) &= u_0, \end{cases} \tag{9.1}$$

where $x_0 \in (a,b)$, on some interval $I \subset (a,b)$ containing x_0 then the graph of u lies in the strip $I \times (-\infty, \infty)$ passing through (x_0, u_0).

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- Suppose that f is continuous on the closure of this rectangle, then f is Riemann integrable because f is bounded on the closure of the rectangle.

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- Now, integrating both sides of (9.1), we get the integral equation

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- If the integral is well-defined then the solution u of (9.1) is a fixed point for the operator $T : C(I) \to C(I)$ defined as

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- We have observe that $u \in C(I)$ is a fixed point of the operator T, as defined in (9.2), then $u \in C^1(I)$ and solves (9.1). Conversely, if $u \in C^1(I)$ solves (9.1) then u is a fixed point of T.

Contraction Maps

Definition

Let X be a metric space with metric d. An operator $f : X \to X$ is said to be a contraction if for some $0 \leq \alpha < 1$,

$$d\left(f(x), f(y)\right) \leq \alpha d(x, y), \quad \forall x, y \in X.$$

If $\alpha = 1$, the map f is called non-expansive. If $0 \leq \alpha < +\infty$, the map f is called Lipschitz continuous.

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Exercise

Every contraction operator is Lipschitz and every Lipschitz map is continuous.

Contraction Mapping Theorem

Theorem (Contraction Mapping)

Let X be a complete metric space and $f : X \to X$ be a contraction mapping. Then there exists a unique fixed point of f, i.e., there exists a unique $x \in X$ such that $f(x) = x$.

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$$\begin{aligned} d(x_n, x_{n+1}) = d(f(x_{n-1}), f(x_n)) &\leq \alpha d(x_{n-1}, x_n) \\ &\leq \alpha^2 d(x_{n-2}, x_{n-1}) \\ &\leq \ldots \leq \alpha^n d(x_0, x_1). \end{aligned}$$

Proof Continued...

By triangle inequality, we have

$$\begin{aligned} d(x_n, x_{n+m}) &\leq d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2}) + \ldots + d(x_{n+m-1}, x_{n+m}) \\ &\leq (\alpha^n + \alpha^{n+1} + \ldots + \alpha^{n+m-1}) d(x_0, x_1) \\ &= \alpha^n (1 + \alpha + \ldots + \alpha^{m-1}) d(x_0, x_1) \\ &\leq \alpha^n \Sigma_{i=0}^{\infty} \alpha^i d(x_0, x_1) \\ &\leq \alpha^n (1-\alpha)^{-1} d(x_0, x_1). \end{aligned}$$

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Since $\alpha < 1$, for a given $\varepsilon > 0$, one can choose a $n_0 \in \mathbb{N}$ such that

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Therefore, the sequence $\{x_n\}$ is Cauchy. Since X is a complete space $x_n \to x$ for some $x \in X$.

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Remark

The above theorem is generally not true when f is non-expansive. For instance, a translation of a vector space in to itself does not admit a fixed point, *i.e.*, define $f(x) = x + a$ for any fixed vector $a \in X$.

Corollary

Let X be a complete metric space and $f : X \to X$ be a mapping such that $f^n : X \to X$ is contraction for some positive integer n. Then there exists a unique fixed point of f, i.e., there exists a unique $x \in X$ such that $f(x) = x$.

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Similarly, $f^{n-1}(y^*) = f^{n-2}(y^*)$. Thus, $f^n(y^*) = f(y^*)$ and $f^n(y^*) = y^*$. Hence $y^* = x^*$.

Banach Fixed Point Theorem

Theorem (Banach Fixed Point Theorem)

Let I be any closed interval of $\mathbb{R}$. Fix a $g \in C(I)$ and $r > 0$. Let $B := \{f \in C(I) \mid \|f - g\| \leq r\}$ and $T : B \to B$ be an operator which is a contraction on B, i.e., for some $0 \leq \alpha < 1$

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Then T has a unique fixed point in B.

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Then T has a unique fixed point in B.

Since $C(I)$ is a Banach space and B is closed subspace of a complete space, B is complete. This result is a particular case of the more general result called the *contraction mapping principle* (cf. 45).

Cauchy-Lipschitz or Picard-Lindelöf

Theorem (Cauchy-Lipschitz)

Let $T > 0$ *and* $\mathbf{f} \in [C([0,T] \times \mathbb{R}^n)]^n$ *admits a* $\alpha > 0$ *such that*

$$|\mathbf{f}(t,\xi_1) - \mathbf{f}(t,\xi_2)| \leq \alpha|\xi_1 - \xi_2| \quad \forall t \in [0,T], \xi_1, \xi_2 \in \mathbb{R}^n.$$

Then, for a given vector $\mathbf{u}_0 \in \mathbb{R}^n$*, there is a unique solution* $\mathbf{u} \in (C^1[0,T])^n$ *of the system of ODE*

$$\begin{cases} \mathbf{u}'(t) &= \mathbf{f}(t,\mathbf{u}(t)) \quad t \in [0,T] \\ \mathbf{u}(0) &= \mathbf{u}_0. \end{cases} \tag{10.1}$$

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Proof: We define $T : (C[0, T])^n \to (C[0, T])^n$ as

$$T\mathbf{u}(t) := \mathbf{u}_0 + \int_0^t \mathbf{f}(s, \mathbf{u}(s))\, ds.$$

Proof Continued...

- If T has a fixed point $\mathbf{u}$ then we have already argued above that $\mathbf{u} \in (C^1[0, T])^n$ and solves (10.1).

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- In the case when one prefers to work with the usual sup norm, then one can prove the contraction of T^k, for some very large k, and proceed in a similar manner.

Proof Continued...

- Consider, for $0 \leq t \leq T$,

$$|(T\mathbf{v} - T\mathbf{w})(t)| = \int_0^t e^{\alpha s} e^{-\alpha s} \mathbf{f}(s, \mathbf{v}(s)) - \mathbf{f}(s, \mathbf{w}(s))\, ds$$

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&\leq \sup_{0 \leq s \leq T} \left(e^{-\alpha s} |\mathbf{f}(s, \mathbf{v}(s)) - \mathbf{f}(s, \mathbf{w}(s))| \right) \int_0^t e^{\alpha s}\, ds
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- Hence, T is contraction. By Theorem 45, there is a unique fixed point for T which is a solution for (10.1).

Linear System of ODE

Corollary (Linear System of ODE)

Let $T > 0$, A be a $n \times n$ matrix with entries in $C[0, T]$ and $\mathbf{b} \in (C[0, T])^n$. Then, for a given vector $\mathbf{u}_0 \in \mathbb{R}^n$, there is a unique solution $\mathbf{u} \in (C^1[0, T])^n$ of the system of linear ODE

$$\begin{cases} \mathbf{u}'(t) &= A(t)\mathbf{u}(t) + b(t) \quad t \in [0, T] \\ \mathbf{u}(0) &= \mathbf{u}_0. \end{cases}$$

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Proof.

Set $\mathbf{f}(t,\xi) := A(t)\xi + \mathbf{b}(t)$. Then

$$|\mathbf{f}(t,\xi_1) - \mathbf{f}(t,\xi_2)| = |A(t)||\xi_1 - \xi_2| \leq \alpha|\xi_1 - \xi_2|$$

where $\alpha = \sup_{0 \leq t \leq T} |A(t)|$. □

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Relaxing Hypothesis

Example

The relaxation on the assumptions on f may also lead to non-uniqueness of solution. For instance, consider

$$\begin{cases} u'(t) & = 3u^{3/2}(t) \quad t \in [0, \infty) \\ u(0) & = u_0. \end{cases}$$

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$$u(t) = \begin{cases} 0 & t \in [0, t_0] \\ (t - t_0)^3 & t \in [t_0, \infty). \end{cases}$$

Cauchy-Peano Theorem

Theorem (Cauchy-Peano (Local Existence))

Given $T > 0$*,* $r > 0$*,* $\mathbf{u}_0 \in \mathbb{R}^n$ *and* $\mathbf{f} \in C([0, T] \times \overline{B_r(\mathbf{u}_0)})^n$*. Then there exists a* $0 < h \leq T$ *and, at least, one solution* $\mathbf{u} \in (C^1[0, h])^n$ *of the system of ODE*

$$\begin{cases} \mathbf{u}'(t) &= \mathbf{f}(t, \mathbf{u}(t)) \quad t \in [0, h] \\ \mathbf{u}(0) &= \mathbf{u}_0. \end{cases} \tag{10.2}$$

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Proof: We shall choose h subsequently. We have already argued that, for $t \in [0,h]$, if

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has a fixed point $\mathbf{u}$ then $\mathbf{u} \in (C^1[0, h])^n$ and solves (10.2). Let us partition the interval $[0, h]$ in to m intervals of length h/m.

Using a finite difference approximation of the IVP, we define vectors $\mathbf{u}_i \in \mathbb{R}^n$, for $0 \leq i \leq m-1$, by

$$\frac{\mathbf{u}_{i+1} - \mathbf{u}_i}{\frac{h}{m}} = \mathbf{f}\left(\frac{ih}{m}, \mathbf{u}_i\right).$$

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where $M := \sup_{(t,\xi)\in[0,T]\times\overline{B_r(\mathbf{u}_0)}} |\mathbf{f}(t,\xi)|$ and $h := \min\{\frac{r}{M}, T\}$.

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$$|\mathbf{u}_2 - \mathbf{u}_0| \leq |\mathbf{u}_2 - \mathbf{u}_1| + |\mathbf{u}_1 - \mathbf{u}_0| \leq \frac{hM}{m} + \frac{hM}{m} = \frac{2hM}{m} \leq hM \leq r.$$

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Proceeding inductively, we have $\mathbf{u}_i$ well-defined for all $1 \leq i \leq m$ because

$$|\mathbf{u}_i - \mathbf{u}_0| \leq |\mathbf{u}_i - \mathbf{u}_{i-1}| + |\mathbf{u}_{i-1} - \mathbf{u}_0| \leq \frac{hM}{m} + \frac{(i-1)hM}{m} = \frac{ihM}{m} \leq hM \leq r.$$

Note that, for each $m \in \mathbb{N}$, we have $m+1$ distinct equi-distant points ih/m of $[0,h]$ and m distinct vectors $\mathbf{u}_i$, for $0 \leq i \leq m$.

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Note that $U_m \in (C[0,h])^n$, for all $m \in \mathbb{N}$. Now,

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The last equality is clear by the piecewise linear construction of U_m. Also, $|\mathbf{u}_i| \leq |\mathbf{u}_0| + |\mathbf{u}_i - \mathbf{u}_0| \leq |\mathbf{u}_0| + r$.

Note that, for each $m \in \mathbb{N}$, we have $m+1$ distinct equi-distant points ih/m of $[0,h]$ and m distinct vectors $\mathbf{u}_i$, for $0 \leq i \leq m$. We shall now define a continuous function $U_m : [0,h] \to \mathbb{R}^n$ such that $U_m(ih/m) = \mathbf{u}_i$ for $0 \leq i \leq m$. This is done by piecewise joining the line $(ih/m, \mathbf{u}_i)$ and $((i+1)h/m, \mathbf{u}_{i+1})$. Hence, for each $t \in [0,h]$ and all $0 \leq i \leq m-1$,

$$U_m(t) := \mathbf{u}_i + \frac{m}{h}\left(t - \frac{ih}{m}\right)(\mathbf{u}_{i+1} - \mathbf{u}_i) \text{ when } \frac{ih}{m} \leq t \leq \frac{(i+1)h}{m}.$$

Note that $U_m \in (C[0,h])^n$, for all $m \in \mathbb{N}$. Now,

$$\|U_m\|_\infty = \sup_{t \in [0,h]} |U_m(t)| = \sup_{0 \leq i \leq m} |\mathbf{u}_i|.$$

The last equality is clear by the piecewise linear construction of U_m. Also, $|\mathbf{u}_i| \leq |\mathbf{u}_0| + |\mathbf{u}_i - \mathbf{u}_0| \leq |\mathbf{u}_0| + r$. Thus, the sequence is uniformly bounded in $(C[0,h])^n$.

The sequence $\{U_m\}$ is also equicontinuous because, for each $0 \leq i \leq m-1$ and $ih/m \leq t \leq (i+1)h/m$,

$$|U_m(t) - U_m(ih/m)| = |U_m - \mathbf{u}_i| \leq (t - ih/m)|\mathbf{f}(ih/m, \mathbf{u}_i)| \leq (t - ih/m)M$$

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Using the recursive relation of u_i, we get

$$\mathbf{u}_{i+1} = \mathbf{u}_0 + \frac{h}{m}\left(\sum_{j=0}^{i} \mathbf{f}(jh/m, \mathbf{u}_j)\right) = \mathbf{u}_0 + \int_0^{ih/m} f_m(s)\,ds$$

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Note that $\mathbf{f}$ is uniformly continuous in both variables because it is a continuous function on a compact set and the uniform convergence of U_m to $\mathbf{u}$ implies that the above limit in RHS is zero. Thus $T\mathbf{u} = \mathbf{u}$.

Two Point Boundary Value Problem

Let $f \in C([0,1] \times \mathbb{R})$. For any two given constants $u_0, u_1 \in \mathbb{R}$, consider the second order nonlinear boundary value problem

$$\begin{cases} -u''(x) &= f(x, u(x)) \quad x \in (0,1) \\ u(0) &= u_0 \\ u(1) &= u_1. \end{cases} \tag{10.3}$$

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Proof: For any $x \in (0,1)$ and fixed $x_0 \in (0,1)$, integrate both sides of (10.3) in the range x_0 and x, then

$$-\int_{x_0}^{x} u''(t)\,dt = \int_{x_0}^{x} f(t, u(t))\,dt$$

Proof Continued

- or, equivalently,

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- By Mean value theorem, for each $0 < x < 1$, there exists a $c \in (0,x)$ such that

$$\frac{u(x) - u(0)}{x} = u'(c).$$

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- Arguing similarly, one can show that u is differentiable at 1 and $u'(1) = \lim_{c \to 1} u'(c)$. Hence $u \in C^1[0,1]$.
- It follows from the ODE that $u \in C^2[0,1]$ because the RHS f and u can be continuously extended to boundary.

Lemma

$u \in C^2[0,1]$ is a solution of (10.3) iff $u \in C[0,1]$ solves the integral equation

$$u(x) = u_0(1-x) + u_1 x + \int_0^1 G(x,s) f(s, u(s))\, ds \quad x \in [0,1] \qquad (10.4)$$

where the Green's function $G \in C([0,1] \times [0,1])$ is defined as

$$G(x,s) := \begin{cases} s(1-x) & 0 \le s \le x \le 1 \\ x(1-s) & 0 \le x < s \le 1. \end{cases}$$

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Proof: If $u \in C^2[0,1]$ is a solution of (10.3) then, for any fixed $x \in [0,1]$,

$$\begin{aligned} \int_0^1 G(x,s) f(s,u(s))\,ds &= -(1-x)\int_0^x s u''(s)\,ds - x\int_x^1 (1-s)u''(s)\,ds \\ &= u(x) - u_0(1-x) - u_1 x. \end{aligned}$$

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- and

$$-u''(x) = xf(x, u(x)) + (1-x)f(x, u(x)) = f(x, u(x)).$$

Thus, u is a solution to (10.3).

Existence of Solution

Theorem

Let $f \in C([0,1] \times \mathbb{R})$ admit a $0 \leq \alpha < 8$ such that, for all $x \in [0,1]$,

$$|f(x,r) - f(x,s)| \leq \alpha |r - s|.$$

For any two given constants $u_0, u_1 \in \mathbb{R}$ there is a unique solution $u \in C[0,1] \cap C^2(0,1)$ of (10.3).

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Proof Continued...

Consider

$$|(Tv - Tw)(x)| \leq \int_0^1 G(x,s)|f(s,v(s)) - f(s,w(s))|\,ds$$

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Note that $1/4$ is the maximum of $x - x^2$.

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Note that $1/4$ is the maximum of $x - x^2$. Since $\alpha < 8$, T is a contraction. Thus, by Lemma 17, the fixed point u of T is in $C^2[0,1]$ and solves (10.3).

Open Map

Definition

Let X and Y be topological spaces. We say a map $T : X \to Y$ is an open map if the image of every open subset of X under T is an open subset of Y.

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Let X be a Banach space and Y be a normed space. Let $T \in \mathcal{B}(X, Y)$ be such that $\overline{T(B_r^X(0))} \supset B_s^Y(0)$, then $T(B_r^X(0)) \supset B_s^Y(0)$.

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$$E := T(B_1^X(0)) \cap B_1^Y(0)$$

which is non-empty because $0 \in E$.

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Now, set $x = \sum_{n=1}^\infty \varepsilon^{n-1}x_n$ and, hence,

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Further, $Tx = \sum_{n=1}^\infty \varepsilon^{n-1}Tx_n = \sum_n (z_n - z_{n-1}) = z$. Therefore, $z \in (1-\varepsilon)^{-1}T(B_1^X(0))$ and $y \in T(B_1^X(0))$. Thus, $B_1^Y(0) \subset T(B_1^X(0))$.

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Let X and Y be Banach spaces and let $T \in \mathcal{B}(X, Y)$ be a surjective map, i.e., $T(X) = Y$. Then T is an open map.

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Corollary (Inverse Mapping)

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Equivalent Norms

Theorem

Let X be a vector space with two different norms $\|\cdot\|$ and $|||\cdot|||$ such that it is complete with respect to both the norms. If there exists a constant $C > 0$ such that $|||x||| \leq C\|x\|$, for all $x \in X$, then the two norms are equivalent.

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To observe this note that the identity map from $(X, \|\cdot\|)$ to $(X, |||\cdot|||)$, which is linear and bijective, is continuous, by the assumption.

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Stability of two-point Boundary Value Problem

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For given functions $a, b, c \in C[0,1]$, *let the boundary value problem*

$$\begin{cases} a(x)u''(x) + b(x)u'(x) + c(x)u(x) &= f(x) \quad \text{in } (0,1) \\ u(0) = u(1) &= 0 \end{cases}$$

admit a unique solution $u \in C^2[0,1]$ *for every given* $f \in C[0,1]$. *Then there exists a constant* $C > 0$ *such that*

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Proof: To see this consider $X := \{v \in C^2[0,1] \mid v(0) = v(1) = 0\}$ endowed with the norm $|||v||| := \|v\|_\infty + \|v'\|_\infty + \|v''\|_\infty$.

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$$Tv(x) := a(x)v''(x) + b(x)v'(x) + c(x)v(x).$$

Proof Continued...

Note that T is continuous (or bounded) because

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- To solve $f(x) = p$ when f is nonlinear, it is significant to note that f has a linear approximation as follows: $f(x) \approx f(a) + Df(a) \cdot (x - a)$.
- Thus, we expect f to admit a 'local' inverse if the linear aprroximation is invertible, i.e. $Df(a)$ is invertible. This is the Inverse Function Theorem.

Solutions in Finite Dimensions

- Every mathematical modelling reduces to the question of seeking solutions to equation of the form $f(x) = p$.
- If $f : \mathbb{R}^n \to \mathbb{R}^m$ and is linear then solving $f(x) = p$ is same as solving the associated matrix equation $Ax = p$. It has a unique solution if A is a invertible square matrix.
- Since $f_i(x) = \sum_{j=1}^{n} a_{ij}x_j$, if f admits first order partial derivatives then invertibility of A is same as the invertibility of the Jacobian of f, $D_j f_i := \frac{\partial f_i}{\partial x_j} = a_{ij}$.
- To solve $f(x) = p$ when f is nonlinear, it is significant to note that f has a linear approximation as follows: $f(x) \approx f(a) + Df(a) \cdot (x - a)$.
- Thus, we expect f to admit a 'local' inverse if the linear aprroximation is invertible, i.e. $Df(a)$ is invertible. This is the Inverse Function Theorem.
- The inverse function theorem gives the necessary condition for solving $f(x) = p$, locally, for a system of n nonlinear equations in n unknowns.

Properties of Non-zero Jacobian Matrix

Theorem (For Open Ball)

Let $B := B_r(a) \subset \mathbb{R}^n$ *be an open ball of radius* r *centred at* $a \in \mathbb{R}^n$, ∂B *denotes the boundary of* B, *i.e.,* $\partial B := \{x \in \mathbb{R}^n \mid |x - a| = r\}$ *and* $\overline{B}$ *be the closure of* B *in* $\mathbb{R}^n$. *Let*

1. $f : \overline{B} \to \mathbb{R}^n$ *be continuous,*
2. *all partial derivatives* $D_j f_i(x)$ *of* f *exists, for all* $x \in B$,
3. $f(x) \neq f(a)$ *for all* $x \in \partial B$,
4. $J_f(x) \neq 0$ *for all* $x \in B$.

Then $f(B)$ *contains an open ball centred at* $f(a)$.

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Proof: Define $g : \partial B \to (0, \infty)$ as $g(x) := |f(x) - f(a)|$.

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Proof Continued...

Let $y \in U$. We will show $y \in f(B)$, i.e., there is a point $c \in B$ such that $f(c) = y$.

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Thus, $h(x) \geq m/2$, for all $x \in \partial B$, and hence $c \in B$ and not in ∂B. Note that $c \in B$ is also a minimum of $h^2 : \overline{B} \to [0, \infty)$, where $h^2(x) = \sum_{i=1}^{n} (f_i(x) - y_i)^2$.

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$$\sum_{i=1}^{n}(f_i(c) - y_i) D_j f_i(c) = 0.$$

This is same as $[Df(c)](f(c) - y) = 0$.

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$$\sum_{i=1}^{n} (f_i(c) - y_i) D_j f_i(c) = 0.$$

This is same as $[Df(c)](f(c) - y) = 0$. Since $c \in B$, we have $J_f(c) \neq 0$. Therefore, $f(c) = y$ and $y \in f(B)$. Thus, $U \subseteq f(B)$.

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Theorem (For Open Set)

Let U be an open subset of $\mathbb{R}^n$ and

1. $f : U \to \mathbb{R}^n$ *be continuous,*
2. *all partial derivatives $D_j f_i(x)$ of f exists, for all $x \in U$,*
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Let U be an open subset of $\mathbb{R}^n$ and $f : U \to \mathbb{R}^n$ has continuous partial derivatives $D_j f_i$ on U. Also, $J_f(a) \neq 0$ for some $a \in U$. Then there exists an open ball B centred at a on which f is injective.

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Proof: For any choice of n points, $x_1, x_2, x_3, \ldots, x_n$ in U one can associate a point $z \in \mathbb{R}^{n^2}$, where $z := \{x_1; x_2; x_3; \ldots; x_n\}$ such that the first n components of z is same as that of x_1, the next n components are that of x_2 and so on.

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Proof Continued...

Thus, by continuity of h, there is an open ball Ω centered at $A \in \mathbb{R}^{n^2}$ such that $h(z) \neq 0$ for all $z \in \Omega$.

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Since B is convex (an open ball), we have $[x, y] \in B$ and hence $x_i \in B$ for all $i = 1, 2, \ldots, n$.

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Since B is convex (an open ball), we have $[x, y] \in B$ and hence $x_i \in B$ for all $i = 1, 2, \dots, n$. By the choice of x and y, LHS is zero and hence we have the system of linear equations

$$\sum_{j=1}^{n} D_j f_i(x_i)(y_j - x_j) = 0.$$

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Thus, by continuity of h, there is an open ball Ω centered at $A \in \mathbb{R}^{n^2}$ such that $h(z) \neq 0$ for all $z \in \Omega$. Therefore, $\det(D_j f_i(x_i)) \neq 0$ for all $x_i \in B$, where B is an open ball centred at a. We claim that f is injective on B. Suppose f is not injective on B, then for some $x, y \in B$ such that $x \neq y$ we have $f(x) = f(y)$.Let $[x, y]$ denote all the points on the line joining x and y. Now since f is differentiable on U, by Mean Value theorem, for each $i = 1, 2, \dots, n$, there is a $x_i \in [x, y]$ such that

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But $\det(D_j f_i(x_i)) \neq 0$, hence $y = x$, a contradiction. Hence f is injective on B.

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- If, in the above result, we replace $J_f(a) \neq 0$ for *some* $a \in U$ with $J_f(x) \neq 0$ for *all* $x \in U$ then we cannot conclude that f is injective on U.

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- For instance $f(z) = \exp(z)$ is not injective on $\mathbb{C}$. It is periodic with periodicity 2π. However, $J_f(z) = |f'(z)|^2 = |e^z|^2 = e^{2x} \neq 0$ for all $z \in \mathbb{C}$. The identification $J_f(z) = |f'(z)|^2$ is typical of holomorphic function due to Cauchy-Riemann equations.

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Theorem (Inverse Function Theorem)

Let $\Omega \subset \mathbb{R}^n$ be an open subset and $f : \Omega \to \mathbb{R}^n$ such that f has continuous partial derivatives in Ω. If, for some point $a \in \Omega$, $J_f(a) \neq 0$, then there are neighbourhoods U and V of a and $f(a)$, respectively, such that $f : U \to V$ is bijective, i.e., for all $p \in V$ the equation $f(x) = p$ has a unique solution in U. Further, the inverse of $f^{-1} : V \to U$ is in C^1.

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Proof: Since J_f is continuous (determinant map) on Ω and $J_f(a) \neq 0$, there is an open ball B_1 centred at a such that $J_f(x) \neq 0$ for all $x \in B_1$.

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It now only remains to show that f^{-1} is C^1 on V. As done in Theorem 55, for any choice of n points, $x_1, x_2, x_3, \ldots, x_n$ in Ω one can associate a point $z \in \mathbb{R}^{n^2}$, where $z := \{x_1; x_2; x_3; \ldots; x_n\}$ such that the first n components of z is same as that of x_1, the next n components are that of x_2 and so on.

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Thus, we have the system of equations

$$[D_k f_i(x_i)] \left[\frac{u' - u}{t} \right] = e_j.$$

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where A_ℓ is the matrix $[D_j f_i(x_i)]$ where the ℓ-th column is replaced by e_j.

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where $A_\ell(u)$ is the matrix $[D_j f_i(u)]$ where the ℓ-th column is replaced by e_j. Therefore, partial derivatives of g exists and is continuous because it is quotient of continuous functions.

Functions Locally as Graph

Recall that a curve in a plane is not always the graph of some function. For instance, the unit circle S^1 in a plane has the equation $x^2 + y^2 = 1$ and the form $y = \pm\sqrt{1 - x^2}$ is multi-valued.

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Functions Locally as Graph

Recall that a curve in a plane is not always the graph of some function. For instance, the unit circle S^1 in a plane has the equation $x^2 + y^2 = 1$ and the form $y = \pm\sqrt{1 - x^2}$ is multi-valued.

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The previous example suggests that one may have local explicit form at a point (x_0, y_0) provided $f_y(x_0, y_0) \neq 0$, a fact we shall prove in more generality in the implicit function theorem.

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- Consider the union of the axes in $\mathbb{R}^2$ given by the equation $f(x, y) = 0$ where $f(x, y) = xy$. Note that $f_y(x, y) = x$ and is non-zero for $x \neq 0$. Thus, for $x_0 \neq 0$, in a neighbourhood U of x_0 not containing 0, we may define $g(x) = 0$ mapping to any neighbourhood V of $y_0 = 0$. However, for $x_0 = 0$, there is no g, in any neighbourhood of x_0, such that $y_0 = g(x_0)$.

Implicit Function Theorem

Theorem (Implicit Function Theorem)

Let $\Omega \subset \mathbb{R}^m \times \mathbb{R}^n$ be an open subset and $f : \Omega \to \mathbb{R}^n$ such that f is continuously differentiable in Ω. Let $(x_0, y_0) \in \Omega$ be such that $f(x_0, y_0) = 0$ and the $n \times n$ matrix

$$D_y f(x_0, y_0) := \begin{pmatrix} \frac{\partial f_1}{\partial y_1}(x_0, y_0) & \cdots & \frac{\partial f_1}{\partial y_n}(x_0, y_0) \\ \vdots & \ddots & \vdots \\ \frac{\partial f_n}{\partial y_1}(x_0, y_0) & \cdots & \frac{\partial f_n}{\partial y_n}(x_0, y_0) \end{pmatrix}$$

is non-singular, then there is a neighbourhood $U \subset \mathbb{R}^m$ of x_0 and a unique function $g : U \to \mathbb{R}^n$ such that $g(x_0) = y_0$ and, for all $x \in U$, $f(x, g(x)) = 0$. Further g is continuously differentiable in U.

Proof

Let us define a function $F : \Omega \to \mathbb{R}^m \times \mathbb{R}^n$ as $F(x; y) := (Ix; f(x, y))$, where $I : \mathbb{R}^m \to \mathbb{R}^m$ is the identity map.

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$$J_F(x_0, y_0) = \begin{vmatrix} I & 0 \\ D_x f(x_0, y_0) & D_y f(x_0, y_0) \end{vmatrix}$$

is same as the determinant of the $n \times n$ matrix $D_y f(a)$.

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Proof Continued...

Let $U := \{x \in \mathbb{R}^m \mid (x, 0) \in V\}$ and is an open set containing x_0 and define $g : U \to \mathbb{R}^n$ as $g(x) := G_2(x; 0)$.

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For every $(v_1; v_2) \in V$ there is a unique $(w_1; w_2) \in W$ such that $F(w_1; w_2) = (v_1; v_2)$ because F is bijective from W to V.

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This implies that
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$(x; 0) = F(G_1(x; 0); G_2(x; 0)) = F(x; g(x)) = (x; f(x, g(x))$. Thus, $f(x, g(x)) = 0$. The uniqueness of g follows from the uniqueness of the inverse map G of F.